Visible Actions and Multiplicity-free Representations

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Varna Lectures on visible actions and MF representations

1. EXAMPLES OF MULTIPLICITY-FREE REPRESENTATIONS
2. INFINITE DIMENSIONAL EXAMPLES
3. PROPAGATION THEOREM OF MULTIPLICITY-FREE REPRESENTATIONS
4. VISIBLE ACTIONS ON COMPLEX MANIFOLDS
5. APPLICATIONS OF VISIBLE ACTIONS TO MULTIPLICITY-FREE REPRESENTATIONS

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- V. APPLICATIONS OF VISIBLE ACTIONS TO MULTIPLICITY-FREE REPRESENTATIONS

Plan

I propose

> *a new method (based on "visible actions") to*
> *prove/find/construct*
> *multiplicity-free representations*

for finite/infinite dimensional representations.

References (a) for Varna Lectures

- Overview
  Publ. Res. Inst. Math. Sci. (2005)

- Visible Actions — Classification Theory
  J. Math. Soc. Japan (2007) $\cdots$ GL_n case
  Transformation Groups (2007) $\cdots$ symmetric pairs
  A. Sasaki (IMRN (2009), IMRN (2011), Geometriae Dedicata (2010))
  Y. Tanaka (J. Algebra (2014), J. Math. Soc. Japan (2013), Tohoku J, B. Australian Math. Soc. (2013), etc)

- Multiplicity-free Theorem via Visible Actions
  Progr. Math. (2013) $\cdots$ general theory

References (b) for Varna Lectures

- Application to concrete examples
  Acta Appl. Math. (2004) $\cdots \otimes$ product, GL_n
  Progr. Math. (2008)

- Generalization of Kostant–Schmid formula
  Proc. Rep. Theory, Saga (1997)

- Multiplicity-free Theorems and Orbit Philosophy
  Adv. Math. Sci. (2003), AMS (with Nasrin)
  Nasrin (Geoemtriae Dedicata (2014))

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Multiplicity-free property is ‘rare’ in general.

How to find such a structure systematically ?

New approach — “visible action”

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To give a new simple principle that explains the property MF for both finite and infinite dimensional reps

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(Strongly) Visible Action (K– 2004)

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Propagation of MF property from fiber to sections

$\Uparrow$ (Progr. Math 2013)

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§1 Multiplicity-free representations

Ex.1 (Eigenspace decomposition)

$\mathcal{H}$: Vector sp./$\mathbb{C}$, $\dim < \infty$

$A \in \mathrm{End}_{\mathbb{C}}(\mathcal{H})$

s.t. all eigenvalues are distinct. ①

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Taylor series (MF rep.)

Ex.2 (Taylor expansion, Laurent expansion)

$$f(z_1, \ldots, z_n) = \sum_{\alpha=(\alpha_1,\ldots,\alpha_n)} a_\alpha z_1^{\alpha_1} \ldots z_n^{\alpha_n}$$

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$$\Uparrow$$

$\dim \mathrm{Hom}_{(\mathbb{C}^\times)^n}(\tau, \mathcal{O}(\{0\}) \leq 1$

($^\forall \tau = \tau_\alpha$: irred. rep. of $(\mathbb{C}^\times)^n$)

i.e. $(\mathbb{C}^\times)^n \curvearrowright \mathcal{O}(\{0\})$ is MF

Fourier series

Ex.3 (Fourier series expansion)

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$S^1 \curvearrowright L^2(S^1)$ is MF (multiplicity-free)

Peter–Weyl (MF rep.)

Ex.4 (Peter–Weyl)

G: compact (Lie) group

$$L^2(G) \simeq \sum_{\tau \in \widehat{G}}^{\oplus} \underline{\tau \boxtimes \tau^*}$$

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Translation $\quad f(\cdot) \mapsto f(g_1^{-1} \cdot g_2)$

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$\Longrightarrow$ $G \times G \curvearrowright L^2(G)$ is MF

Spherical harmonics

M: compact Riemannian manifold
Δ_M: Laplace–Beltrami operator on M
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<u>Ex.5</u> $M = S^{n-1}$ (unit sphere)

$\Delta_{S^{n-1}} f = \lambda f$, $f \neq 0$

$\Rightarrow$

$\lambda = -l(l+n-2)$ for some $l \in \mathbb{N}$.

Fourier series $\Longrightarrow$ spherical harmonics

$O(n) \curvearrowright S^{n-1} \subset \mathbb{R}^n$

$\Delta_{S^{n-1}}$: Laplacian on S^{n-1}

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$O(n) \curvearrowright L^2(S^{n-1})$ is MF

$G \curvearrowright L^2(G/K)$ is MF (E. Cartan ('29)–I. M. Gelfand ('50))

$\otimes$-product rep.

$$\boxed{SL_2(\mathbb{C}) \underset{\text{irred.}}{\overset{\pi_k}{\curvearrowright}} S^k(\mathbb{C}^2) \quad (k = 0, 1, 2, \dots)}$$

$\otimes$-product rep.

$$SL_2(\mathbb{C}) \underset{\text{irred.}}{\overset{\pi_k}{\curvearrowright}} S^k(\mathbb{C}^2) \quad (k = 0, 1, 2, \dots)$$

Ex.7 (Clebsch–Gordan)

$$\pi_k \otimes \pi_l \simeq \pi_{k+l} \oplus \pi_{k+l-2} \oplus \cdots \oplus \pi_{|k-l|}$$

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$$\underbrace{\pi_k \otimes \pi_l}_{\text{MF}} \simeq \pi_{k+l} \oplus \pi_{k+l-2} \oplus \cdots \oplus \pi_{|k-l|}$$

Notation (finite dimensional reps)

$G = GL_n(\mathbb{C})$

Hightest weight

$$\lambda = (\lambda_1 \ldots, \lambda_n) \in \mathbb{Z}^n, \quad \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$$

$$\Downarrow$$

Irreducible rep.

$$\pi_\lambda^{GL_n} \equiv \pi_\lambda$$

Ex.8

$$\lambda = (k, 0, \ldots, 0) \quad \leftrightarrow \quad GL_n(\mathbb{C}) \curvearrowright S^k(\mathbb{C}^n)$$

$$\lambda = (\underbrace{1, \ldots, 1}_{k}, 0, \ldots, 0) \quad \leftrightarrow \quad GL_n(\mathbb{C}) \curvearrowright \Lambda^k(\mathbb{C}^n)$$

$\otimes$-product rep. (GL_n-case)

Ex.9 (Pieri's rule)

$$\pi_{(\lambda_1,\dots,\lambda_n)} \otimes \pi_{(k,0,\dots,0)} \simeq \bigoplus_{\substack{\mu_1 \geq \lambda_1 \geq \cdots \geq \mu_n \geq \lambda_n \\ \sum(\mu_i - \lambda_i) = k}} \pi_{(\mu_1,\dots,\mu_n)}$$

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MF as a GL_n-module.

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MF as a GL_n-module.

Ex.10 (counterexample)

$\pi_{(2,1,0)} \otimes \pi_{(2,1,0)}$ is NOT MF as a $GL_3(\mathbb{C})$-module.

$\otimes$-product for GL_3

$\pi_{\{2,1,0\}} \simeq$ Adjoint reprsentation
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$$\begin{aligned} &\pi_{(2,1,0)} \otimes \pi_{(2,1,0)} \\ \simeq\; &\pi_{(4,2,0)} \oplus \pi_{(4,1,1)} \oplus \pi_{(2,2,0)} \\ &\oplus \underline{2}\pi_{(3,2,1)} \oplus \pi_{(2,2,2)} \end{aligned}$$

When is $\pi_\lambda \otimes \pi_\nu$ MF?

$G = GL_n$

$\lambda = (\lambda_1, \ldots, \lambda_n), \quad \nu = (\nu_1, \ldots, \nu_n)$

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(Necessary Condition)
If $\pi_\lambda \otimes \pi_\nu$ is MF

$\Rightarrow$?

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$G = GL_n$

$\lambda = (\lambda_1, \dots, \lambda_n), \quad \nu = (\nu_1, \dots, \nu_n)$

(Necessary Condition)

If $\pi_\lambda \otimes \pi_\nu$ is MF

then at least one of λ or ν is of the form

$$(\underbrace{a, \dots, a}_{p}, \underbrace{b, \dots, b}_{n-p}),$$

for some $a \geq b$ and some p

$\otimes$-product rep. (continued)

$$\lambda = (\underbrace{a, \cdots, a}_{p}, \underbrace{b, \cdots, b}_{q}) \in \mathbb{Z}^n,\ a \geq b,\ p + q = n$$

Ex.11 (Stembridge 2001)

$\pi_\lambda \otimes \pi_\nu$ is MF as a $GL_n(\mathbb{C})$-module

iff one of the following holds

1) $\min(a - b, p, q) = 1$ (and ν is any),

2) $\min(a - b, p, q) = 2$ and

★ $\boxed{\nu \text{ is of the form } \nu = (\underbrace{x \cdots x}_{n_1}\ \underbrace{y \cdots y}_{n_2}\ \underbrace{z \cdots z}_{n_3})\ (x \geq y \geq z)}$,

3) $\min(a - b, p, q) \geq 3$, ★ &

$\min(x - y,\ y - z, n_1, n_2, n_3) = 1.$

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Geometric interpretation (K–, 2004)

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Geometric interpretation (K–, 2004) $\cdots$ visible action

Restriction $(GL_n \downarrow G_{n_1} \times GL_{n_2} \times GL_{n_3})$

Ex.12 $(GL_n \downarrow (GL_p \times GL_q)) \quad n = p + q$

$\pi^{GL_n}_{(\underbrace{x, \ldots, x}_{n_1}, \underbrace{y, \ldots, y}_{n_2}, \underbrace{z, \ldots, z}_{n_3})}|_{GL_p \times GL_q}$ is MF

if $\min(p, q) \leq 2$ or

if $\min(n_1, n_2, n_3, x - y, y - z) \leq 1$

(Kostant $n_3 = 0$; Krattenthaler 1998)

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(Kostant $n_3 = 0$; Krattenthaler 1998)

Ex.13 $(GL_n \downarrow (GL_{n_1} \times GL_{n_2} \times GL_{n_3}))$ ([5])

$n = n_1 + n_2 + n_3$

$\pi^{GL_n}_{(\underbrace{a,\ldots,a}_{p},\underbrace{b,\ldots,b}_{q})}|_{GL_{n_1} \times GL_{n_2} \times GL_{n_3}}$ is MF

if $\min(n_1, n_2, n_3) \leq 1$ or

if $\min(p, q, a - b) \leq 2$

“Triunity”

MF results for

Ex.11 $\pi_\lambda \otimes \pi_\nu$

Ex.12 $GL_n \downarrow GL_p \times GL_q \ (p + q = n)$

Ex.13 $GL_n \downarrow GL_{n_1} \times GL_{n_2} \times GL_{n_3} \ (n_1 + n_2 + n_3 = n)$

“Triunity”

MF results for

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Ex.13 $GL_n \downarrow GL_{n_1} \times GL_{n_2} \times GL_{n_3} \ (n_1+n_2+n_3=n)$

can be proved by combinatorial methods (e.g. Littlewood–Richardson rule) but
will be explained by “triunity” of visible actions on flag varieties:

$$\begin{cases} G \curvearrowright (G \times G)/(L \times H) & \text{(diagonal action)} \\ L \curvearrowright G/H \\ H \curvearrowright G/L \end{cases}$$

for $H \subset G \supset L$.

Restriction ($GL_n \downarrow GL_{n-1}$)

Ex.14 ($GL_n \downarrow GL_{n-1}$)

$$\pi^{GL_n}_{(\lambda_1,\dots,\lambda_n)}|_{GL_{n-1}} \simeq \bigoplus_{\lambda_1 \geq \mu_1 \geq \cdots \geq \mu_{n-1} \geq \lambda_n} \pi^{GL_{n-1}}_{(\mu_1,\dots,\mu_{n-1})}$$

$\Longrightarrow$ restrictions is MF as a $GL_{n-1}(\mathbb{C})$-module

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$$GL_n \underset{\text{MF}}{\underset{\curvearrowright}{\supset}} GL_{n-1} \underset{\text{MF}}{\underset{\curvearrowright}{\supset}} GL_{n-2} \underset{\text{MF}}{\underset{\curvearrowright}{\supset}} \dots \underset{\text{MF}}{\underset{\curvearrowright}{\supset}} GL(1)$$

$\Longrightarrow$ Gelfand–Tsetlin basis

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Restriction ($GL_n \downarrow GL_{n-1}$)

Finite dimensional rep.

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Infinite dimensional version

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Infinite dimensional version

Ex.15 ♣ ref. ($U(p,q) \downarrow U(p-1,q)$)

${}^\forall\pi$: irred. unitary rep. of $U(p,q)$ with higest weight

$\Longrightarrow$ restriction $\pi|_{U(p-1,q)}$ is MF as a $U(p-1,q)$-module

GL–GL duality

$$N = mn$$

Ex.16 (GL–GL duality)

$\Longrightarrow GL_m \times GL_n \curvearrowright S(\mathbb{C}^N) \simeq S(M(m, n; \mathbb{C}))$

This rep. is MF

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$\Downarrow$ generalization 1

Hidden symmetry $\Longleftrightarrow$ Broken symmetry

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$\Downarrow$ generalization 1

Hidden symmetry $\Longleftrightarrow$ Broken symmetry

Ex.17 (Progress in Math. 2008)

Branching law of holomorphic discrete series rep. with respect to symmectric pair

$$\underbrace{\text{Hua–Kostant–Schmid}}_{\substack{\text{finite dim} \\ \text{compact subgp}}}, \qquad \underbrace{\text{K–}}_{\substack{\infty\ \text{dim} \\ \text{non-compact subgp}}}$$

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$\Downarrow$ generalization 2

MF space: $G \curvearrowright X \Longrightarrow G \curvearrowright \mathcal{O}(X)$

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$S(\mathbb{C}^N)$ is still MF as a $GL_{m-1} \times GL_n$ module

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Ex.19 (counterexample)

$S(\mathbb{C}^N)$ is no more MF as a $GL_{m-1} \times GL_{n-1}$ module

MF for unitary rep (definition)

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Observation

$n \leq 1 \iff \mathrm{End}(\mathbb{C}^n)$ is commutative.

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$(\pi, \mathcal{H})$: unitary rep. of G $\Downarrow$ (Schur's lemma)

Def.

$(\pi, \mathcal{H})$ is MF if $\mathrm{End}_G(\mathcal{H})$ is commutative.

Def. $\mathrm{End}_{\mathbb{C}}(\mathcal{H}) = \{T : \mathcal{H} \to \mathcal{H}$ continuous linear maps$\}$

$$\cup$$

$$\mathrm{End}_G(\mathcal{H}) := \{T \in \mathrm{End}_{\mathbb{C}}(\mathcal{H}) : T \circ \pi(g) = \pi(g) \circ T, \ {}^{\forall} g \in G\}$$

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Recall:

Def. (naive) $(\pi, \mathcal{H})$ is MF
multiplicity-free
if $\dim \mathrm{Hom}_G(\tau, \pi) \leq 1$ ($\forall\tau$: irred. rep. of G).

MF for unitary rep (definition)

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Def.

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$\Downarrow$

Prop. The irreducible decomp. of π is unique, and $m_\pi(\tau) \leq 1$ for almost every τ with respect to $d\mu$.

In particular, multiplicity for any discrete spectrum ≤ 1

$$\pi \simeq \int_{\widehat{G}} m_\pi(\tau)\, \tau \, d\mu(\tau) \quad \text{(direct integral)}$$

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$(\pi, \mathcal{U})$: continuous rep.

Def. We say $(\pi, \mathcal{U})$ is (unitarily) MF
if, for any unitary rep. $(\varpi, \mathcal{H})$ s.t.
there exists an injective continuous G-map $\mathcal{H} \hookrightarrow \mathcal{U}$,
$(\varpi, \mathcal{H})$ is MF.

Fourier transform (MF rep.)

Ex.21 (Fourier transform)

$$L^2(\mathbb{R}) \simeq \int_{\mathbb{R}}^{\oplus} \mathbb{C}e^{i\zeta x} d\zeta$$

(direct integral of Hilbert spaces)

$$f(x) = \int_{\mathbb{R}} f(\zeta) e^{i\zeta x} d\zeta$$

Regular rep. of $\mathbb{R}$ on $L^2(\mathbb{R})$ by $f(*) \to f(* - c)$

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$\mathrm{End}_{\mathbb{R}}(L^2(\mathbb{R})) \simeq L^\infty(\mathbb{R})$ (ring of multiplier operators)

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$\Longrightarrow$ unitary rep. $\mathbb{R} \curvearrowright L^2(\mathbb{R})$ is MF

continuous spectrum

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Regular rep. of $\mathbb{R}$ on $L^2(\mathbb{R})$ by $f(*) \to f(* - c)$

$\mathrm{End}_{\mathbb{R}}(L^2(\mathbb{R})) \simeq L^\infty(\mathbb{R})$ (ring of multiplier operators)

continuous rep. $\mathbb{R} \curvearrowright \mathcal{S}'(\mathbb{R})$ is also MF

Plancherel formula for G/K

Ex.22 (Harish-Chandra, Helgason) $G/K = SL(n, \mathbb{R})/SO(n)$

$$L^2(G/K) \simeq \underbrace{\int^{\oplus}_{\sum \lambda_i=0,\ \lambda_1 \geq \cdots \geq \lambda_n} \mathcal{H}_\lambda \qquad \underbrace{d\lambda}_{\text{cont. spec.}}}_{\text{MF}}$$

$\mathcal{H}_\lambda$: ∞-dim, irred. rep. of G

$\Longrightarrow$ MF

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$\mathcal{H}_\lambda$: ∞-dim, irred. rep. of G

$\Longrightarrow$ MF

$$\begin{aligned}\mathrm{End}_G(L^2(G/K)) &\simeq L^\infty((\mathbb{R}^n/\mathbb{R}))^{\mathcal{S}_n} \\ &\simeq L^\infty(\mathbb{R}^{n-1}/\mathcal{S}_n)\end{aligned}$$

(ring of multiplier operators)

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$\mathcal{H}_\lambda$: ∞-dim, irred. rep. of G

$\Longrightarrow$ MF

MF still holds for vector bundle case of 'small' fibers,

$$\mathcal{V} := G \times_K \wedge^k(\mathbb{C}^n) \to G/K \qquad (0 \leq k \leq n),$$

associated to the $SO(n)$-representation on the exterior power $\Lambda^k(\mathbb{C}^n)$, but no other cases (Deitmar, K– 2005)

Plancherel formula for G/K

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$\mathcal{H}_\lambda$: ∞-dim, irred. rep. of G

$\Longrightarrow$ MF

MF still holds under certain deformation of G-regular representation of $L^2(G/K)$

deformation coming from hidden symmetry.
E.g. Gelfand–Vershik canonical rep of $SL_2(\mathbb{R})$.

Plancherel formula for G/K

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$\mathcal{H}_\lambda$: ∞-dim, irred. rep. of G

$\Longrightarrow$ MF

Other real forms of $SL(n,\mathbb{C})/SO(n,\mathbb{C})$:

Ex.23 (T. Oshima, Delorme) $G/H = SL(n,\mathbb{R})/SO(p,n-p)$

Multiplicity of most cont. spec. in $L^2(G/H)$

$$= \frac{n!}{p!(n-p)!} > 1 \text{ if } 0 < p < n.$$

$\Longrightarrow$ NOT MF

Broken symmetry and hidden symmetry

$$\boxed{H \subset G \quad \underset{\text{unitary rep}}{\xrightarrow{\pi}} \quad GL_{\mathbb{C}}(\mathcal{H})}$$

Broken symmetry and hidden symmetry

Hidden symmetry

$$\begin{tikzcd} H \arrow[r, phantom, "\subset"] \arrow[r, bend left] & G \arrow[r, "\pi", "\text{unitary rep}"'] & GL_{\mathbb{C}}(\mathcal{H}) \end{tikzcd}$$

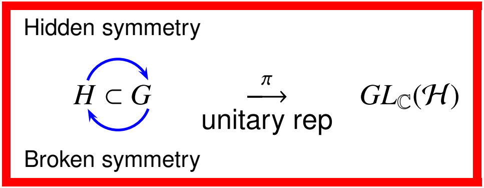
Broken symmetry and hidden symmetry

Hidden symmetry

$$H \subset G \xrightarrow[\text{unitary rep}]{\pi} GL_{\mathbb{C}}(\mathcal{H})$$

Broken symmetry

Broken symmetry and hidden symmetry



Hidden symmetry

$$H \subset G \xrightarrow[\text{unitary rep}]{\pi} GL_{\mathbb{C}}(\mathcal{H})$$

Broken symmetry

Branching law
= description of broken symmetry

Deformation of $G \curvearrowright L^2(G/K)$

$$G \overset{\pi}{\curvearrowright} L^2(G/K)$$

Deformation of $G \curvearrowright L^2(G/K)$

$$\begin{tikzcd}
G \arrow[r, phantom, "\overset{\pi}{\curvearrowright}"] & L^2(G/K) \\
\cap & \\
{}^{\exists}\widetilde{G} \arrow[ruu, dashed, bend left, "{}^{\exists}\widetilde{\pi}"'] &
\end{tikzcd}$$

Deformation of $G \curvearrowright L^2(G/K)$

$$\begin{tikzcd} G \arrow[r, phantom, "\overset{\pi}{\curvearrowright}"] \arrow[d, phantom, "\cap" description] & L^2(G/K) \\ {}^{\exists}\widetilde{G} \arrow[ur, dashed, bend left, "{}^{\exists}\widetilde{\pi}"'] & \end{tikzcd}$$

Prop For any classical reductive G, there exist $\widetilde{G}$ ($\supsetneqq G$) and an irreducible unitary rep $\widetilde{\pi}$ of $\widetilde{G}$ s.t. $\widetilde{\pi}|_G = \pi$

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<u>E.g.</u> $G = GL(n, \mathbb{R}) \hookrightarrow \widetilde{G} = Sp(n, \mathbb{R})$

Deformation of $G \curvearrowright L^2(G/K)$

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<u>E.g.</u> $G = GL(n,\mathbb{R}) \hookrightarrow \widetilde{G} = Sp(n,\mathbb{R})$

$\widetilde{\pi}$ lies in a continuous family $\{\widetilde{\pi}_\lambda\}$ of irred unitary reps of $\widetilde{G}$

$\Longrightarrow \pi_\lambda = \widetilde{\pi}_\lambda|_G$

$\Longrightarrow$ deformation of $G \curvearrowright L^2(G/K)$

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$\Longrightarrow \pi_\lambda = \widetilde{\pi}_\lambda|_G$

$\Longrightarrow$ deformation of $G \curvearrowright L^2(G/K)$ (still MF)

Sometimes discrete spectrum may appear!

Known methods

Various techniques have been used in proving various MF results, in particular, for finite dim'l reps

For example, one may

1. look for an open orbit of a Borel subgroup.
2. apply Littlewood–Richardson rules and variants.
3. use computational combinatorics.
4. employ the Gelfand trick (the commutativity of the Hecke algebra).
5. apply Schur–Weyl duality and Howe duality.

New approach

Plan:

To give a new simple principle that explains the property MF for both finite and infinite dimensional reps

New approach

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To give a new simple principle that explains the property $\underline{\text{MF}}$ for both finite and infinite dimensional reps

Analysis on complex mfd with group action having ∞ many orbits

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Analysis on complex mfd with group action having ∞ many orbits

Theory of visible actions on complex manifolds

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Propagation of MF property from fiber to sections

$\Uparrow$

Analysis on complex mfd with group action having ∞ many orbits

Theory of visible actions on complex manifolds

Propagation Theorem

$$\boxed{\begin{array}{c}\mathcal{V}\\ \downarrow\\ D\end{array}}$$

H-equivariant holomorphic vector bundle

Propagation Theorem

$$\boxed{\begin{matrix}\mathcal{V}\\ \downarrow \\ D\end{matrix}} \quad \rightsquigarrow \boxed{H \curvearrowright \mathcal{O}(D, \mathcal{V})}$$

H-equivariant holomorphic vector bundle

Propagation Theorem

$$\begin{tikzcd} \mathcal{V}_x \arrow[r, phantom, "\subset"] \arrow[d] & \mathcal{V} \arrow[d] \\ \{x\} \arrow[r, phantom, "\subset"] & D \end{tikzcd} \rightsquigarrow \boxed{H \curvearrowright \mathcal{O}(D, \mathcal{V})}$$

H-equivariant holomorphic vector bundle

Propagation Theorem

$$\begin{tikzcd}
\boxed{H_x \curvearrowright \mathcal{V}_x} \arrow[dd] & \subset & \mathcal{V} \arrow[dd] & & \\
 & & & \rightsquigarrow & \boxed{H \curvearrowright \mathcal{O}(D, \mathcal{V})} \\
\{x\} & \subset & D & &
\end{tikzcd}$$

H-equivariant holomorphic vector bundle

$$H_x = \{h \in H : h \cdot x = x\}$$

Propagation Theorem

$$\begin{tikzcd}
\boxed{H_x \curvearrowright \mathcal{V}_x} \arrow[d] \arrow[r, phantom, "\subset"] & \mathcal{V} \arrow[d] \arrow[r, squiggly] & \boxed{H \curvearrowright \mathcal{O}(D, \mathcal{V})} \\
\{x\} \arrow[r, phantom, "\subset"] & D &
\end{tikzcd}$$

H-equivariant holomorphic vector bundle

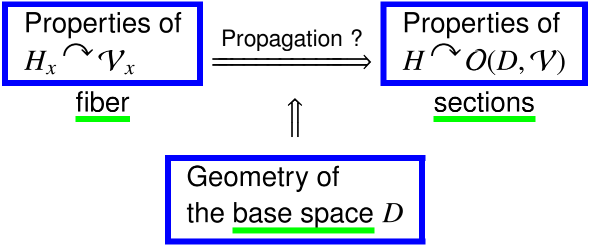
$$\begin{tikzcd}
\boxed{\text{Properties of } H_x \curvearrowright \mathcal{V}_x} \arrow[r, Rightarrow, "\text{Propagation ?}"] & \boxed{\text{Properties of } H \curvearrowright \mathcal{O}(D, \mathcal{V})}
\end{tikzcd}$$

“easy” “difficult”

Propagation Theorem

$$\begin{tikzcd}
\boxed{H_x \curvearrowright \mathcal{V}_x} \arrow[d] & \subset & \mathcal{V} \arrow[d] & \\
\{x\} & \subset & D \arrow[r, squiggly] & \boxed{H \curvearrowright \mathcal{O}(D, \mathcal{V})}
\end{tikzcd}$$

H-equivariant holomorphic vector bundle



Properties of $H_x \curvearrowright \mathcal{V}_x$

fiber

Propagation ? $\Longrightarrow$

Properties of $H \curvearrowright \mathcal{O}(D, \mathcal{V})$

sections

$\Uparrow$

Geometry of the base space D

Propagation Theorem

$$\begin{tikzcd}
\boxed{H_x \curvearrowright \mathcal{V}_x} \arrow[d] & \subset & \boxed{\mathcal{V}} \arrow[d] \arrow[r, rightsquigarrow] & \boxed{H \curvearrowright \mathcal{O}(D, \mathcal{V})} \\
\{x\} & \subset & D &
\end{tikzcd}$$

H-equivariant holomorphic vector bundle

Theorem

$$\boxed{\begin{array}{l}\text{Irreducibility}\\ H_x \curvearrowright \mathcal{V}_x\end{array}} \xRightarrow{\text{Propagation}} \boxed{\begin{array}{l}\text{Irreducibility}\\ H \curvearrowright \mathcal{O}(D, \mathcal{V})\end{array}}$$

fiber

sections

$\Uparrow$

$\boxed{\begin{array}{l}H \text{ acts transitively on}\\ \text{the base space } D\end{array}}$

Propagation Theorem

$$\begin{tikzcd} \boxed{H_x \curvearrowright \mathcal{V}_x} \arrow[d] & \subset & \boxed{\begin{array}{c}\mathcal{V}\\ \downarrow\\ D\end{array}} \\ \{x\} & \subset & \end{tikzcd} \rightsquigarrow \boxed{H \curvearrowright \mathcal{O}(D, \mathcal{V})}$$

H-equivariant holomorphic vector bundle

Theorem

$$\begin{array}{ccc} \boxed{\begin{array}{l}\text{MF}\\ H_x \curvearrowright \mathcal{V}_x\end{array}} & \xRightarrow{\text{Propagation}} & \boxed{\begin{array}{l}\text{MF}\\ H \curvearrowright \mathcal{O}(D, \mathcal{V})\end{array}} \\ \text{fiber} & & \text{sections} \\ & \Uparrow & \\ & \boxed{\begin{array}{l} H \text{ acts strong visibly on} \\ \text{the base space } D \end{array}} & \end{array}$$

§2 Propagation of MF property

Progr. Math (2013)

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H-equiv. holo. vector b'dle:

$$\mathcal{V} \to D$$

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Progr. Math (2013)

H: Lie group

$$\boxed{\text{H-equiv. holo. vector b'dle: } \mathcal{V} \to D}$$

$$\Downarrow$$

$\underline{H \curvearrowright \mathcal{O}(D, \mathcal{V})}$= {holo. sections}

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Progr. Math (2013)

H: Lie group

$$\boxed{\begin{array}{c}H\text{-equiv. holo. vector b'dle:}\\ \mathcal{V} \to D\end{array}}$$

$$\Downarrow$$

$\underline{H \curvearrowright \mathcal{O}(D, \mathcal{V})}$= {holo. sections}

$$D \ni x \quad \rightsquigarrow \quad \underline{H_x \curvearrowright \mathcal{V}_x}$$

$$H_x := \{g \in H : g \cdot x = x\} \subset H$$

$\mathcal{V}_x \subset \mathcal{V}$ (fiber)

§2 Propagation of MF property

Progr. Math (2013)

H: Lie group

H-equiv. holo. vector b'dle:
$\mathcal{V} \to D$

$$\Downarrow$$

$\underline{H \curvearrowright \mathcal{O}(D, \mathcal{V})}$= {holo. sections}

$$D \ni x \quad \rightsquigarrow \quad \underline{H_x \curvearrowright \mathcal{V}_x}$$

$$H_x := \{g \in H : g \cdot x = x\} \subset H$$

$\mathcal{V}_x \subset \mathcal{V}$ (fiber)

$$\begin{tikzcd}
\underline{H_x \curvearrowright \mathcal{V}_x} \arrow[d] \arrow[r, phantom, "\subset"] & \mathcal{V} \arrow[d] & \Rightarrow \underline{H \curvearrowright \mathcal{O}(D, \mathcal{V})} \\
\{x\} \arrow[r, phantom, "\in"] & D &
\end{tikzcd}$$

§2 Propagation of MF property

Progr. Math (2013)

H: Lie group

H-equiv. holo. vector b'dle:
$\mathcal{V} \to D$

$\Downarrow$

$H \curvearrowright \mathcal{O}(D, \mathcal{V})$= {holo. sections}

Theorem (Propagation theorem)

$H_x \curvearrowright \mathcal{V}_x$ MF $(\forall x \in D)$

$\Longrightarrow H \curvearrowright \mathcal{O}(D, \mathcal{V})$ MF

if assumptions 1 & 2 hold.

Automorphism of group action

$$H \curvearrowright D$$

Automorphism of group action

$$\underset{\text{Lie group}}{H} \curvearrowright \underset{\text{manifold}}{D}$$

Def $\sigma \in \mathrm{Aut}(H;D)$

$$\iff \begin{cases} \sigma \curvearrowright H & \text{automorphism of Lie group} \\ \sigma \curvearrowright D & \text{diffeomorphism} \\ \sigma(g \cdot x) = \sigma(g) \cdot \sigma(x) & ({}^{\forall} g \in H,\ {}^{\forall} x \in D) \end{cases}$$

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Write simply $\sigma \curvearrowright D$ instead of $\sigma \in \mathrm{Aut}(H; D)$

Assumptions of MF theorem

$\mathcal{V} \to D$: H-equivariant

Assumption 1 $\quad {}^{\exists}\sigma \curvearrowright D$ anti-holo. s.t.
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$$D \ni x \rightsquigarrow H_x \subset H, \mathcal{V}_x \subset \mathcal{V}$$

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Note: Assumption 2 is automatic for line bundles

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Points: H has infinitely many orbits on D

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fiber MF

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fiber MF sections MF

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fiber MF   sections MF

$$\Uparrow$$

geometry of base space
. . . '(strongly) visible action'

Examples of MF theorem

<u>Ex.20</u> $H = U(m) \times U(n)$

$D = M(m, n; \mathbb{C}) \simeq \mathbb{C}^{mn}$

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⇓ Propagation theorem

$H \curvearrowright$ **Pol**(D) <u>MF</u>

$$GL_m \times GL_n \curvearrowright \textbf{Pol}(M(m, n; \mathbb{C}))$$

§3 Visible actions

(D, J) complex mfd, connected

Def. A real submanifold S of D is totally real if $T_x S$ does not contain any complex subspace.

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$H \overset{\text{holomorphic}}{\curvearrowright} (D, J)$ complex mfd, connected

Def.(K– '03) A holomorphic action of H is visible w.r.t. S if
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$^{\exists}D' \underset{\text{open}}{\subset} D$,

$^{\exists}S \underset{\text{totally real}}{\subset} D'$ s.t.

$$\begin{cases} S \text{ meets every } H\text{-orbit} \\ J_x(T_x S) \subset T_x(H \cdot x) \ (x \in S) \end{cases}$$

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Visible actions on symmetric spaces

Theorem (Transf. Groups (2007))

Assume $\begin{cases} G/K & \text{Hermitian symm. sp.} \\ (G,H) & \text{symmetric pair} \end{cases}$

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Example of visible actions

$$\mathbb{T} = \{a \in \mathbb{C} : |a| = 1\} \quad (\simeq S^1)$$

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<u>Ex.21</u>

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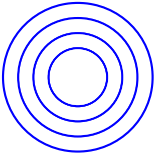

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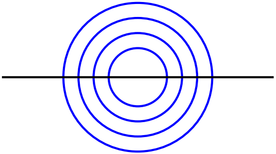

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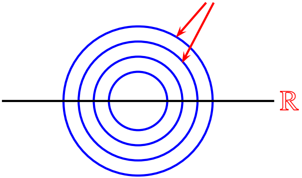
$\mathbb{R}$ meets every $\mathbb{T}$-orbit



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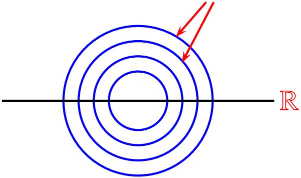

$\Longrightarrow$ $\mathbb{T}$-action on $\mathbb{C}$ is visible

Strongly visible actions

$H \overset{\text{holomorphic}}{\curvearrowright} D$ complex mfd, connected

Strongly visible actions

holomorphic

$H \curvearrowright D$ complex mfd, connected

Def. ♣ ref. A holomorphic action is strongly visible if

$\exists \sigma$ $\circlearrowleft$ H as Lie group auto

$\curvearrowright D$ as anti-holomorphic diffeo

in a compatible way

s.t. $(H \cdot D^{\sigma})^{\circ} \neq \emptyset$.

(generic H-orbits meets the fixed point set D^{σ})

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Remark. Not necessarily $\sigma^2 = \mathrm{id}$

Strongly visible actions

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$H \overset{\text{holomorphic}}{\curvearrowright} D$ complex mfd, connected

<u>Def.</u> ♣ ref. A holomorphic action is strongly visible w.r.t. S

if

$^{\exists}\sigma$ $\curvearrowright H$

$\curvearrowright D$ anti-holomorphic

in a compatible way

s.t.

$$\sigma|_S = \mathrm{id}$$

$$(H \cdot S)^{\circ} \neq \varnothing$$

Remark. S is automatically totally real.

Strongly visible actions

Proposition   Strongly visible $\Longrightarrow$ Visible

Strongly visible actions

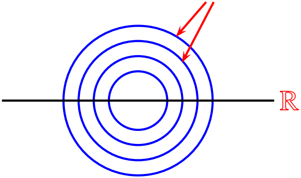
Proposition Strongly visible $\Longrightarrow$ Visible

To be more precise,

strongly visible w.r.t. S

$\Longrightarrow$ visible w.r.t. S' for some $S' \underset{\text{open dense}}{\subset} S$.

$\mathbb{R}$ meets every $\mathbb{T}$-orbit



§4 Visible action

$H \overset{\text{holomorphic}}{\curvearrowright} (D, J)$ complex mfd, connected

<u>Def.</u> Action is visible if

$${}^{\exists}S \underset{\text{totally real}}{\subset} {}^{\exists}D' \underset{\text{open}}{\subset} D \text{ s.t.}$$

$$\begin{cases} S \text{ meets every } H\text{-orbit in } D' \\ J_x(T_x S) \subset T_x(H \cdot x) \ (x \in S) \end{cases}$$

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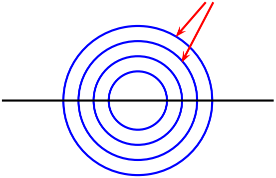
S meets every $\mathbb{T}$-orbit



$S = \mathbb{R}$

Complex / Riemannian / symplectic

$H \overset{\text{isometric}}{\curvearrowright} (D, g)$ Riemannian mfd

Def. Action is polar if $^{\exists}S \underset{\text{closed submfd}}{\subset} D$ s.t.

$$\begin{cases} S \text{ meets every } H\text{-orbit} \\ T_xS \perp T_x(H \cdot x) \quad (x \in S) \end{cases}$$

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$H \overset{\text{symplectic}}{\curvearrowright} (D, \omega)$ symplectic mfd

Def. (Guillemin–Sternberg, Huckleberry–Wurzbacher)
Action is coisotropic (or multiplicity-free)
if $T_x(H \cdot x)^{\perp\omega} \subset T_x(H \cdot x)$ for principal orbits $H \cdot x$ in D

Three geometries

Complex geometry

Symplectic geometry

Riemannian geometry

Three geometries

Complex geometry

Visible action

K– (2004)

Symplectic geometry

Coisotropic action

Guillemin–Sternberg ('84)
Huckleberry–Wurzbacher ('90)

Riemannian geometry

Polar action

Bott–Samelson ('58), Conlon, Hermann, Palais, Terng, Dadok,
Eschenburg, Heintze, Podestà–Thorbergsson ('03), Kollross ('07), . . .

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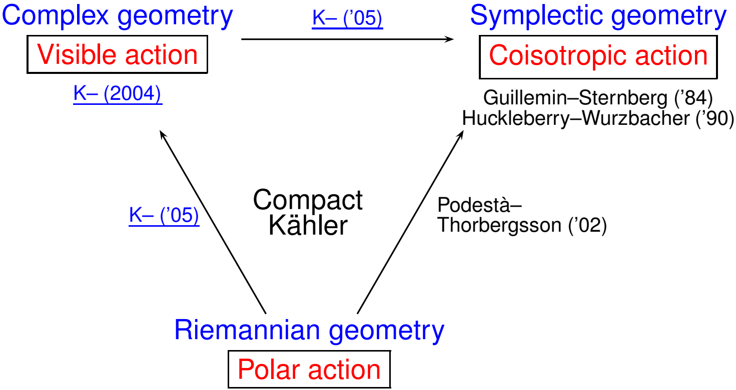

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K– (2004)

K– ('05)

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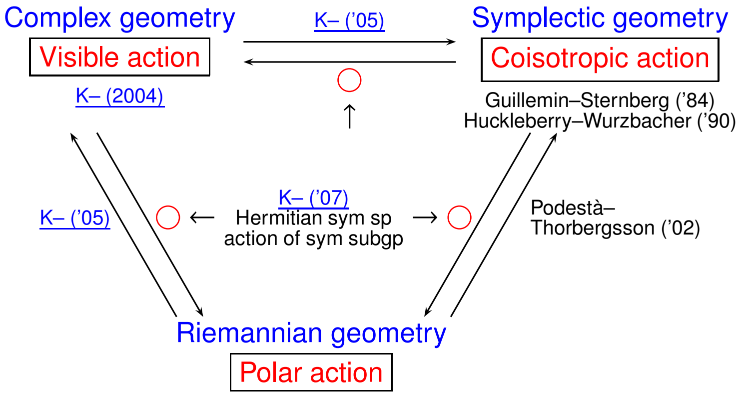

Complex geometry

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K– (2004)

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○
↑

K– ('05)

○ ← K– ('07)
Hermitian sym sp
action of sym subgp → ○

Podestà–
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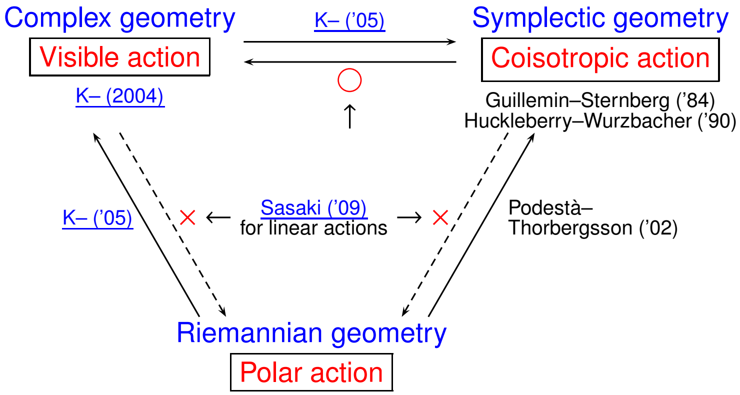

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○

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× ← Sasaki ('09) → ×
for linear actions

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§5 Making examples of visible actions

Ex.20 $H = U(m) \times U(n)$

$D = M(m, n; \mathbb{C})$

$\Rightarrow$ Every H-orbit is preserved by $z \mapsto \overline{z}$

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Proof Let $m \leq n$. Set

$$S := \left\{ \begin{pmatrix} a_1 & & 0 & \\ & \ddots & & 0 \\ 0 & & a_m & \end{pmatrix} : a_1, \ldots, a_m \in \mathbb{R} \right\}$$

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⇒ Every H-orbit meets S, i.e. $H \cdot S = D$

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$\Rightarrow$ Every H-orbit meets S, i.e. $H \cdot S = D$

$\Rightarrow$ Any H-orbit is of the form $H \cdot x$ ($^{\exists} x \in S$)

$$\overline{H \cdot x} \underset{\text{compatibility}}{=} \overline{H} \cdot \overline{x} \underset{x \in M(m,n;\mathbb{R})}{=} H \cdot x$$

□

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In general,

Strongly visible

(i.e. ${}^{\exists}\sigma$ anti-holo s.t. $(H \cdot D^{\sigma})^{\circ} \neq \emptyset$)

$\Rightarrow$ Assumption 1 of Theorem

(i.e. ${}^{\exists}\sigma$ anti-holo s.t. σ preserves generic H-orbits)

Analysis on ∞-many orbits

$\mathcal{V} \to X$: H-equiv. holo vector bundle.

Analysis on ∞-many orbits

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<u>Theorem</u> (Propagation thm of MF property)

Sections

$\Longrightarrow$ $H \curvearrowright \mathcal{O}(X, \mathcal{V})$
multiplicity-free

∞ many orbits

Analysis on ∞-many orbits

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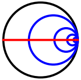
$\Uparrow$

$H \curvearrowright X$
(strongly) visible

Base sp.



∞ many orbits

Classification theory of visible actions

Methods to find visible action

Want to find visible actions systematically

Classification theory of visible actions

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	$\cdots$	classified: M. Berger (1957)
Three involutions (special case)	$\longleftrightarrow$	visible action (2004–) (special case)

G/K Hermitian symm. space

<ins>Ex.22</ins>

$$\begin{aligned} G &= SL(2,\mathbb{R}) \\ K &= SO(2) \\ H &= \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} : a > 0 \right\} \\ N &= \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in \mathbb{R} \right\} \\ G/K &\simeq \{ z \in \mathbb{C} : |z| < 1 \} \end{aligned}$$

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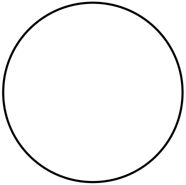

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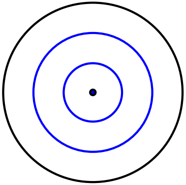

K-orbits

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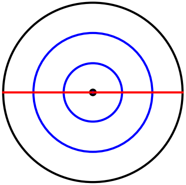

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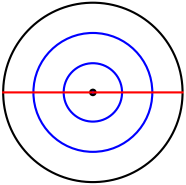

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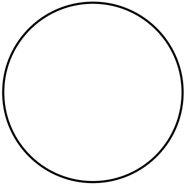

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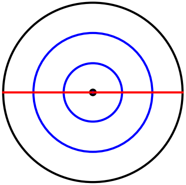

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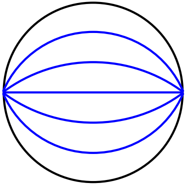

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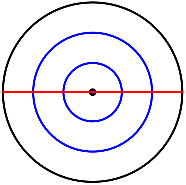
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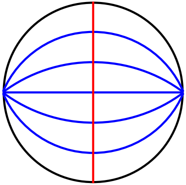
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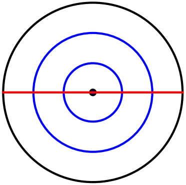

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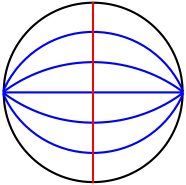

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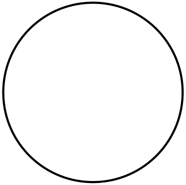

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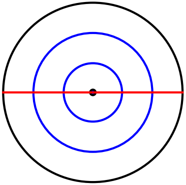

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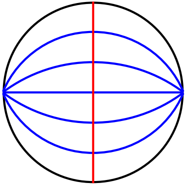

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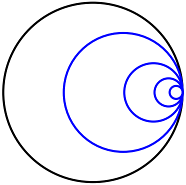

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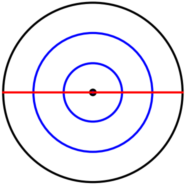

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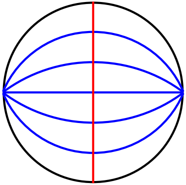

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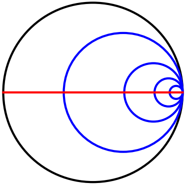

N-orbits

Visible actions on symmetric spaces

Theorem (geometry of three involutions '07)

Assume $\begin{cases} G/K & \text{Hermitian symm. sp.} \\ (G,H) & \text{any symmetric pair} \end{cases}$

$\Longrightarrow H \curvearrowright G/K$ is (strongly) visible

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$\Downarrow$ Propagation theorem

Thm $V_\lambda = U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} \mathbb{C}_\lambda$ (λ generic) is an algebraic MF direct sum of irreducible $\mathfrak{g}'$-modules if

- nilradical of $\mathfrak{p}_{\mathbb{R}}$ is abelian
- $G'_{\mathbb{R}} \cdot o$ is closed in $G_{\mathbb{R}}/P_{\mathbb{R}}$
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$G'_\mathbb{R}$	$\subset$	$G_\mathbb{R}$	$\supset$	$P_\mathbb{R}$
subgp		real reductive		real parabolic

Finite dimensional case

Also, for finite dimensional case

⇓ Propagation theorem

Eg.23 (Okada, '98, rectangular shaped rep)

$\mathfrak{g}_{\mathbb{C}} = \mathfrak{gl}(n, \mathbb{C})$

$$\lambda = (\underbrace{a, \cdots, a}_{p}, \underbrace{b, \cdots, b}_{n-p}) \in \mathbb{Z}^n, \; a \geq b$$

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$$\mathfrak{h}_{\mathbb{C}} = \begin{cases} \mathfrak{gl}(k, \mathbb{C}) + \mathfrak{gl}(n-k, \mathbb{C}) & (1 \leq k \leq n) \\ \mathfrak{o}(n, \mathbb{C}) & \\ \mathfrak{sp}(\frac{n}{2}, \mathbb{C}) & (n : \text{even}) \end{cases}$$

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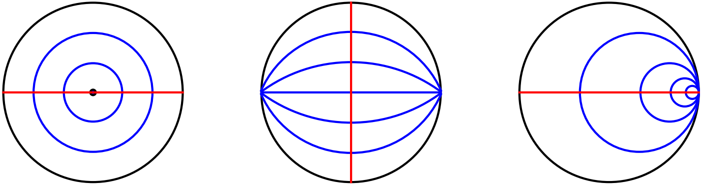

K-orbits

H-orbits

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Non-reductive example

<u>Theorem</u> $N \subset G \supset K$

Assume $\begin{cases} G/K & \text{Hermitian symm. of non-cpt. type} \\ N & \text{max. unipotent subgp.} \end{cases}$

$\Longrightarrow N \curvearrowright G/K$ (strongly) visible

$\Downarrow$ Propagation theorem

<u>Ex.24</u> π_λ: highest wt. module of scalar type

$\Longrightarrow \pi_\lambda|_N$ is MF

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(Generalized) Cartan involutions

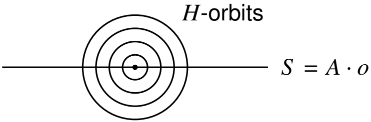
Observation

$$D = G/K$$

Suppose we have a decomposition

$$G = H\ A\ K$$

Set $S := A \cdot o \subset D$



$\Longrightarrow S$ is a candidate of ‘slice’ for (strongly) visible action

Classification theory of visible actions

Grassmannian $U(n)/(U(p) \times U(q)) \simeq Gr_p(\mathbb{C}^n) \quad (n = p + q)$

Ex.(symmetric case) $\quad n_1 + n_2 = p + q = n$

$\Longrightarrow$ $U(n_1) \times U(n_2)$ acts on $Gr_p(\mathbb{C}^n)$

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Ex.34 (JMSJ 2007)

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$\Longleftrightarrow \min(n_1 + 1, n_2 + 1, n_3 + 1, p, q) \leq 2$

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For type B, C, D and exceptional groups (Y. Tanaka, Tohoku J. (2013), J. Math. Soc. Japan (2013), B. Austrian Math Soc. (2013), J. Algebra (2014))

$\Downarrow$ Propagation theorem

MF property of the following

- $GL_m \times GL_n \curvearrowright S(\mathbb{C}^{mn})$ Ex.16
- $GL_{m-1} \times GL_n \curvearrowright S(\mathbb{C}^{mn})$ Ex.18 (Kac)
- the Stembridge list of $\pi_\lambda \otimes \pi_\nu$ Ex.11
- $GL_n \downarrow (GL_p \times GL_q)$ Ex.12
- $GL_n \downarrow (GL_{n_1} \times GL_{n_2} \times GL_{n_3})$ Ex.13
- ∞-dimensional versions

  ……

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Make ‘large’ from ‘small’

Idea: induced action preserving visibility

Make 'large' from 'small'

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$$H \subset G$$

Make ‘large’ from ‘small’

Idea: induced action preserving visibility

$$H \subset G$$

$H \curvearrowright Y$ visible w.r.t. S

$\rotatebox[origin=c]{-90}{$\rightsquigarrow$}$ $\Leftarrow$ certain assumption

$G \curvearrowright X := G \times_H Y$ visible w.r.t. $S \simeq [\{e\}, S\,]$

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$G \curvearrowright X := G \times_H Y$ visible w.r.t. $S \simeq [\{e\}, S\,]$

Ex. $H = U(p) \times U(q)$, $Y = M(p, q; \mathbb{C})$ $(p \geq q)$

$G = U(p + q)$, $X = T^*(G/H) = T^*(Gr_p(\mathbb{C}^{p+q}))$

Make ‘large’ from ‘small’

Idea: induced action preserving visibility

$$H \subset G$$

$H \curvearrowright Y$ visible w.r.t. S

$\rotatebox[origin=c]{-90}{$\rightsquigarrow$}$ $\Leftarrow$ certain assumption

$G \curvearrowright X := G \times_H Y$ visible w.r.t. $S \simeq [\{e\}, S\,]$

Ex. $H = U(p) \times U(q)$, $Y = M(p, q; \mathbb{C})$ $(p \geq q)$
$G = U(p+q)$, $X = T^*(G/H) = T^*(Gr_p(\mathbb{C}^{p+q}))$

$\underset{\text{momentum map}}{\rightsquigarrow}$ nilpotent orbit for $GL(p+q, \mathbb{C})$
for partition $(2^q, 1^{p-q})$ is spherical (Panyushev)

Classification theory of visible actions

Methods to find visible action

Want to find visible actions systematically

- Structure theory
  - geometry of three involutions
  - generalized Cartan decomposition
- Make new from old
  - make ‘large’ from ‘small’
  - make ‘three’ from ‘one’ (triunity)

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$$\left(\begin{tikzcd}[column sep=small] H \arrow[dr, phantom, sloped, "\subset"] & & L \arrow[dl, phantom, sloped, "\supset"] \\ & G \arrow[d, phantom, "\cup"] & \\ & G^{\sigma} & \end{tikzcd}\right) := \left(\begin{tikzcd}[column sep=small] \mathbb{T}^n \arrow[dr, phantom, sloped, "\subset"] & & U(1) \times U(n-1) \arrow[dl, phantom, sloped, "\supset"] \\ & U(n) \arrow[d, phantom, "\cup"] & \\ & O(n) & \end{tikzcd}\right)$$

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$\mathbb{P}^{n-1}\mathbb{R}$ meets every $\mathbb{T}^n$-orbit on $\mathbb{P}^{n-1}\mathbb{C}$

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Group $\boxed{G = HG^{\sigma}L} \Rightarrow H \curvearrowright G/L$

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Group $\boxed{(G\times G) = \mathrm{diag}(G)(G^{\sigma}\times G^{\sigma})(H\times L)} \Rightarrow$ diag.

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$\Downarrow$ Propagation theorem

Three kinds of MF results:

- (Taylor series) $\mathbb{T}^n \curvearrowright \mathcal{O}(\mathbb{C}^n)$ Ex.2
- $(GL_n \downarrow GL_{n-1})$ Restriction $\pi|_{GL_{n-1}}$ Ex.14
- (Pieri) $\pi \otimes S^k(\mathbb{C}^n)$ Ex.9

$\otimes$-product rep.

$$\lambda = (\underbrace{a, \cdots, a}_{p}, \underbrace{b, \cdots, b}_{q}) \in \mathbb{Z}^n, \ a \geq b$$

Ex.11 (Stembridge 2001, K– 2002)

$\pi_\lambda \otimes \pi_\nu$ is MF as a $GL_n(\mathbb{C})$-module if

1) $\min(a - b, p, q) = 1$ (and ν is any),

or

2) $\min(a - b, p, q) = 2$ and

★ ν is of the form $\nu = (\underbrace{x \cdots x}_{n_1} \ \underbrace{y \cdots y}_{n_2} \ \underbrace{z \cdots z}_{n_3}) \ (x \geq y \geq z)$,

or

3) $\min(a - b, p, q) \geq 3$, ★ &

$\min(x - y, \ y - z, n_1, n_2, n_3) = 1.$

Restriction $(GL_n \downarrow G_{n_1} \times GL_{n_2} \times GL_{n_3})$

Ex.12 $(GL_n \downarrow (GL_p \times GL_q)) \quad n = p + q$

$\pi^{GL_n}_{(\underbrace{x,\ldots,x}_{n_1},\underbrace{y,\ldots,y}_{n_2},\underbrace{z,\ldots,z}_{n_3})}|_{GL_p \times GL_q}$ is MF

if $\min(p,q) \leq 2$ or

if $\min(n_1, n_2, n_3, x-y, y-z) \leq 1$

(Kostant $n_3 = 0$; Krattenthaler 1998)

Ex.13 $(GL_n \downarrow (GL_{n_1} \times GL_{n_2} \times GL_{n_3}))$ ([5])

$n = n_1 + n_2 + n_3$

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Analysis on ∞-many orbits

$\mathcal{V} \to X$: H-equiv. holo vector bundle.

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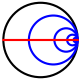
$\Uparrow$

$H \curvearrowright X$ (strongly) visible

Base sp.



∞ many orbits

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Setting $\mathcal{V} \to D$: G-equiv. holomorphic vector bundle

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Reproducing kernel

Prototype (Scalar valued) holomorphic functions

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Definition (reproducing kernel)

Let $\{\varphi_l\}$ be an orthonormal basis of $\mathcal{H}$.

$$K_{\mathcal{H}}(z,w) := \sum_l \varphi_l(z)\overline{\varphi_l(w)}$$

is independent of the choice of the basis.

Examples of reproducing kernels

$$\{\varphi_j\} \quad \subset \quad \mathcal{H} \quad \subset \quad \mathcal{O}(D)$$

orthonormal basis   Hilbert space

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Example 1 (weighted Bergman space)

$$D := \{z \in \mathbb{C} : |z| < 1\}$$

Fix $\lambda > 1$.

$$\mathcal{H} := \{f \in \mathcal{O}(D) : \|f\|_\lambda < \infty\}$$

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$$= \frac{\lambda-1}{\pi}(1-z\overline{w})^{-\lambda}$$

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$$D = \mathbb{C}$$

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$$= \frac{1}{\pi} e^{z\overline{w}}$$

Properties of reproducing kernel

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<ins>Corollary</ins> Suppose a group G acts on D as biholomorphic transformations. Then G acts on $\mathcal{H}$ as a unitary representation if and only if

$$K_{\mathcal{H}}(gz, gw) = K_{\mathcal{H}}(z,w) \quad {}^{\forall}g \in G, {}^{\forall}z, {}^{\forall}w \in D. \qquad (\star)$$

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$$(\star) \Longleftrightarrow K_{\mathcal{H}}(gz,gz) = K_{\mathcal{K}}(z,z) \quad {}^{\forall}g \in G, {}^{\forall}z \in D$$

Scalar-valued reproducing kernel

$$\underset{\text{Hilbert space}}{\mathcal{H}} \subset \mathcal{O}(D)$$

Assume that for each $x \in D$,

$$\begin{array}{cccccl} \mathrm{ev}_x : & \mathcal{H} & \to & \mathbb{C} & & \text{is continuous.} \\ & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} & & \\ & f & \mapsto & f(x) & & \end{array}$$

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$\mathrm{ev}_x^* : \mathbb{C} \to \mathcal{H}$ adjoint

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$$\underset{\text{Hilbert space}}{\mathcal{H}} \subset \mathcal{O}(D)$$

Assume that for each $x \in D$,

$$\begin{array}{ccccc} \mathrm{ev}_x : & \mathcal{H} & \to & \mathbb{C} & \text{is continuous.} \\ & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} & \\ & f & \mapsto & f(x) & \end{array}$$

$\mathrm{ev}_x^* : \mathbb{C} \to \mathcal{H}$ adjoint

$$\begin{aligned} K_{\mathcal{H}}(z, w) &= \mathrm{ev}_w \circ \mathrm{ev}_z^* \\ &= \sum_j \varphi_j(z) \varphi_j(w) \end{aligned}$$

Operator-valued reproducing kernel

$\mathcal{V} \to D$: holomorphic vector bundle

$\mathcal{H} \underset{\text{Hilbert space}}{\hookrightarrow} \mathcal{O}(D, \mathcal{V})$

Assume that for each $x \in D$,

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$$K_{\mathcal{H}}(x, y) := \mathrm{ev}_y \circ \mathrm{ev}_x^* \in \mathrm{Hom}_{\mathbb{C}}(\mathcal{V}_x^*, \mathcal{V}_y)$$

operator-valued reproducing kernel

Operator-valued reproducing kernel

$$\mathcal{V} = \coprod_{x \in D} \mathcal{V}_x \longrightarrow D$$

$$\mathcal{H}om(\mathcal{V}^*, \mathcal{V}) = \coprod_{x,y} \mathrm{Hom}(\mathcal{V}_x^*, \mathcal{V}_y) \longrightarrow D \times D$$

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$\Updownarrow$ one to one

$K_{\mathcal{H}} \in \mathcal{O}(D \times D, \mathcal{H}om(\mathcal{V}^*, \mathcal{V}))$

positive definite operator-valued reproducing kernel

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$\Updownarrow$ one to one

$K_{\mathcal{H}} \in \mathcal{O}(D \times D, \mathcal{H}om(\mathcal{V}^*, \mathcal{V}))$
positive definite operator-valued reproducing kernel

$\mathcal{H}$: unitarity, irreducibility, MF, . . . $\Longleftrightarrow$ Properties on $K_{\mathcal{H}}$

‘Visible’ approach to multiplicity-free theorems

Theorem

$$\boxed{\text{fiber} \xrightarrow{\text{visible action}} \text{sections}}$$

‘Visible’ approach to multiplicity-free theorems

Thm (K– ’08) $\pi|_H$ is multiplicity-free if
π: highest wt. rep. of scalar type
(G, H): semisimple symmetric pair
(Hua, Kostant, Schmid, K– : explicit formula)

Fact (É. Cartan ’29, I. M. Gelfand ’50)
$L^2(G/K)$ is multiplicity-free

Theorem

Multiplicity-free space
Kac ’80, Benson–Ratcliff ’91
Leahy ’98

Stembridge’s list (2001) of multiplicity-free $\otimes$ product of finite dim’l reps (GL_n)

fiber $\xrightarrow{\text{visible action}}$ sections

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Hermitian symm sp. (K– ’07) ↖

↗ Crown domain

Theorem

Vector sp. (Sasaki ’09) ↙

↘ Grassmann mfd. (K– ’07)

Multiplicity-free space
Kac ’80, Benson–Ratcliff ’91
Leahy ’98

Stembridge’s list (2001) of multiplicity-free ⊗ product of finite dim’l reps (GL_n)

$\text{fiber} \xrightarrow{\text{visible action}} \text{sections}$

‘Visible’ approach

To give a simple principle that explains the property MF for both finite and infinite dimensional reps

MF (multiplicity-free) theorem

Propagation of MF property
from fiber to sections

⇑

Visible actions on complex mfds

Analysis of group action with infinitely many orbits

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Thank you !!

(Short story by Soseki, 1908)

“He uses the hammer and chisel without any forethought, and he can make the eyebrows and nose as live.”

“Ah, they are not made by hammer and chisel. The eyebrows and nose are buried inside the tree. It is exactly the same as digging a rock out of the earth — there is no way to mistake.”

$$\overset{\exists 1}{\hookleftarrow}$$