Visible Actions and Multiplicity-free Representations

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I. EXAMPLES OF MULTIPLICITY-FREE REPRESENTATIONS
II. INFINITE DIMENSIONAL EXAMPLES
III. PROPAGATION THEOREM OF MULTIPLICITY-FREE REPRESENTATIONS
IV. VISIBLE ACTIONS ON COMPLEX MANIFOLDS
V. APPLICATIONS OF VISIBLE ACTIONS TO MULTIPLICITY-FREE REPRESENTATIONS

Plan

I propose

a new method (based on "visible actions") to
prove/find/construct
multiplicity-free representations

for finite/infinite dimensional representations.

References (a) for Varna Lectures

- Overview
  Publ. Res. Inst. Math. Sci. (2005)

- Visible Actions — Classification Theory
  J. Math. Soc. Japan (2007) $\cdots$ GL_n case
  Transformation Groups (2007) $\cdots$ symmetric pairs
  A. Sasaki (IMRN (2009), IMRN (2011), Geometriae Dedicata (2010))
  Y. Tanaka (J. Algebra (2014), J. Math. Soc. Japan (2013), Tohoku J, B. Australian Math. Soc. (2013), etc)

- Multiplicity-free Theorem via Visible Actions
  Progr. Math. (2013) $\cdots$ general theory

References (b) for Varna Lectures

- Application to concrete examples
  Acta Appl. Math. (2004) $\cdots \otimes$ product, GL_n
  Progr. Math. (2008)

- Generalization of Kostant–Schmid formula
  Proc. Rep. Theory, Saga (1997)

- Multiplicity-free Theorems and Orbit Philosophy
  Adv. Math. Sci. (2003), AMS (with Nasrin)
  Nasrin (Geoemtriae Dedicata (2014))

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“Ah, they are not made by hammer and chisel. The eyebrows and nose are buried inside the tree. It is exactly the same as digging a rock out of the earth — there is no way to mistake.”

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Multiplicity-free property is 'rare' in general.

How to find such a structure systematically ?

New approach — “visible action”

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To give a new simple principle that explains the property MF for both finite and infinite dimensional reps

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(Strongly) Visible Action (K– 2004)

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Propagation of MF property from fiber to sections

$\Uparrow$ (Progr. Math 2013)

> Analysis on complex mfd with group action having ∞ many orbits

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§1 Multiplicity-free representations

Ex.1 (Eigenspace decomposition)

$\mathcal{H}$: Vector sp./$\mathbb{C}$, $\dim < \infty$

$A \in \mathrm{End}_{\mathbb{C}}(\mathcal{H})$

s.t. all eigenvalues are distinct. ①

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$$\det A \neq 0$$
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{MF representations}

Taylor series (MF rep.)

Ex.2 (Taylor expansion, Laurent expansion)

$$f(z_1, \ldots, z_n) = \sum_{\alpha=(\alpha_1,\ldots,\alpha_n)} a_\alpha z_1^{\alpha_1} \ldots z_n^{\alpha_n}$$

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Point (too obvious ♣ ref.)

$^{\exists 1} a_\alpha \in \mathbb{C}$ for each α

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$\Uparrow$

$\dim \mathrm{Hom}_{(\mathbb{C}^\times)^n}(\tau, \mathcal{O}(\{0\}) \leq 1$

(${}^\forall \tau = \tau_\alpha$: irred. rep. of $(\mathbb{C}^\times)^n$)

i.e. $(\mathbb{C}^\times)^n \curvearrowright \mathcal{O}(\{0\})$ is MF

Fourier series

Ex.3 (Fourier series expansion)

$$L^2(S^1) \simeq \sum_{n\in\mathbb{Z}}^{\oplus} \mathbb{C}e^{inx}$$

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$S^1 \curvearrowright L^2(S^1)$ is MF (multiplicity-free)

Peter–Weyl (MF rep.)

Ex.4 (Peter–Weyl)

G: compact (Lie) group

$$L^2(G) \simeq \sum_{\tau \in \widehat{G}}^{\oplus} \underline{\tau \boxtimes \tau^*}$$

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M: compact Riemannian manifold
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<u>Ex.5</u> $M = S^{n-1}$ (unit sphere)

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<u>Ex.5</u> $M = S^{n-1}$ (unit sphere)

$\Delta_{S^{n-1}} f = \lambda f$, $f \neq 0$

$\Rightarrow$

$\lambda = -l(l+n-2)$ for some $l \in \mathbb{N}$.

Fourier series $\Longrightarrow$ spherical harmonics

$O(n) \curvearrowright S^{n-1} \subset \mathbb{R}^n$

$\Delta_{S^{n-1}}$: Laplacian on S^{n-1}

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$O(n)$ acts on both sides; $\mathcal{H}_l$ irred.

$\Downarrow$

$O(n) \curvearrowright L^2(S^{n-1})$ is MF

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$O(n)$ (acting on $L^2(S^{n-1})$ and on each $\mathcal{H}_l$) irred.

$\Downarrow$

$O(n) \curvearrowright L^2(S^{n-1})$ is MF

$G \curvearrowright L^2(G/K)$ is MF (E. Cartan ('29)–I. M. Gelfand ('50))

$\otimes$-product rep.

$$SL_2(\mathbb{C}) \overset{\pi_k}{\underset{\text{irred.}}{\curvearrowright}} S^k(\mathbb{C}^2) \quad (k = 0, 1, 2, \dots)$$

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MF

Notation (finite dimensional reps)

$G = GL_n(\mathbb{C})$

Hightest weight

$$\lambda = (\lambda_1 \ldots, \lambda_n) \in \mathbb{Z}^n, \quad \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$$

$$\Downarrow$$

Irreducible rep.

$$\pi_\lambda^{GL_n} \equiv \pi_\lambda$$

<u>Ex.8</u>

$$\lambda = (k, 0, \ldots, 0) \quad \leftrightarrow \quad GL_n(\mathbb{C}) \curvearrowright S^k(\mathbb{C}^n)$$

$$\lambda = (\underbrace{1, \ldots, 1}_{k}, 0, \ldots, 0) \quad \leftrightarrow \quad GL_n(\mathbb{C}) \curvearrowright \Lambda^k(\mathbb{C}^n)$$

$\otimes$-product rep. (GL_n-case)

Ex.9 (Pieri's rule)

$$\pi_{(\lambda_1,\dots,\lambda_n)} \otimes \pi_{(k,0,\dots,0)} \simeq \bigoplus_{\substack{\mu_1 \geq \lambda_1 \geq \cdots \geq \mu_n \geq \lambda_n \\ \sum(\mu_i - \lambda_i) = k}} \pi_{(\mu_1,\dots,\mu_n)}$$

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MF as a GL_n-module.

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MF as a GL_n-module.

Ex.10 (counterexample)

$\pi_{(2,1,0)} \otimes \pi_{(2,1,0)}$ is NOT MF as a $GL_3(\mathbb{C})$-module.

$\otimes$-product for GL_3

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$$\pi_{(2,1,0)} \otimes \pi_{(2,1,0)}$$
$$\simeq \pi_{(4,2,0)} \oplus \pi_{(4,1,1)} \oplus \pi_{(2,2,0)}$$
$$\oplus \, \underline{2}\pi_{(3,2,1)} \oplus \pi_{(2,2,2)}$$

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Lecture 2. Various examples of MF representations

MF = multiplicity-free

Plan of Today

- finite-dimensional examples (continued)
- infinite-dimensional examples

Notation (finite dimensional reps)

$G = GL_n(\mathbb{C})$

Hightest weight

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When is $\pi_\lambda \otimes \pi_\nu$ MF?

$G = GL_n$

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(Necessary Condition)

If $\pi_\lambda \otimes \pi_\nu$ is MF

$\Rightarrow$?

When is $\pi_\lambda \otimes \pi_\nu$ MF?

$G = GL_n$
$\lambda = (\lambda_1, \ldots, \lambda_n), \quad \nu = (\nu_1, \ldots, \nu_n)$

(Necessary Condition)
If $\pi_\lambda \otimes \pi_\nu$ is MF
then at least one of λ or ν is of the form

$$(\underbrace{a, \ldots, a}_{p}, \underbrace{b, \ldots, b}_{n-p}),$$

for some $a \geq b$ and some p

$\otimes$-product rep. (continued)

$$\lambda = (\underbrace{a, \cdots, a}_{p}, \underbrace{b, \cdots, b}_{q}) \in \mathbb{Z}^n, \ a \geq b, \ p + q = n$$

Ex.11 (Stembridge 2001)

$\pi_\lambda \otimes \pi_\nu$ is MF as a $GL_n(\mathbb{C})$-module
iff one of the following holds

1) $\min(a-b, p, q) = 1$ (and ν is any),
2) $\min(a-b, p, q) = 2$ and
★ ν is of the form $\nu = (\underbrace{x \cdots x}_{n_1} \ \underbrace{y \cdots y}_{n_2} \ \underbrace{z \cdots z}_{n_3}) \ (x \geq y \geq z)$,
3) $\min(a-b, p, q) \geq 3$, ★ &
$\min(x-y, \ y-z, n_1, n_2, n_3) = 1$.

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Geometric interpretation (K–, 2004)

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Geometric interpretation (K–, 2004) $\cdots$ visible action

Restriction $(GL_n \downarrow G_{n_1} \times GL_{n_2} \times GL_{n_3})$

Ex.12 $(GL_n \downarrow (GL_p \times GL_q)) \quad n = p + q$

$\pi^{GL_n}_{(\underbrace{x,\dots,x}_{n_1},\underbrace{y,\dots,y}_{n_2},\underbrace{z,\dots,z}_{n_3})}|_{GL_p \times GL_q}$ is MF

if $\min(p, q) \le 2$ or

if $\min(n_1, n_2, n_3, x - y, y - z) \le 1$

(Kostant $n_3 = 0$; Krattenthaler 1998)

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Ex.13 $(GL_n \downarrow (GL_{n_1} \times GL_{n_2} \times GL_{n_3}))$ ([5])

$n = n_1 + n_2 + n_3$

$\pi^{GL_n}_{(\underbrace{a,\ldots,a}_{p},\underbrace{b,\ldots,b}_{q})}|_{GL_{n_1} \times GL_{n_2} \times GL_{n_3}}$ is MF

if $\min(n_1, n_2, n_3) \leq 1$ or

if $\min(p, q, a-b) \leq 2$

"Triunity"

MF results for

Ex.11 $\pi_\lambda \otimes \pi_\nu$

Ex.12 $GL_n \downarrow GL_p \times GL_q \ (p + q = n)$

Ex.13 $GL_n \downarrow GL_{n_1} \times GL_{n_2} \times GL_{n_3} \ (n_1 + n_2 + n_3 = n)$

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can be proved by combinatorial methods (e.g. Littlewood–Richardson rule) but
will be explained by “triunity” of visible actions on flag varieties:

$$\begin{cases} G \curvearrowright (G \times G)/(L \times H) & \text{(diagonal action)} \\ L \curvearrowright G/H \\ H \curvearrowright G/L \end{cases}$$

for $H \subset G \supset L$.

Restriction ($GL_n \downarrow GL_{n-1}$)

Finite dimensional rep.

Ex.14 ($GL_n \downarrow GL_{n-1}$)

$$\pi^{GL_n}_{(\lambda_1,\dots,\lambda_n)}|_{GL_{n-1}} \simeq \bigoplus_{\lambda_1 \geq \mu_1 \geq \cdots \geq \mu_{n-1} \geq \lambda_n} \pi^{GL_{n-1}}_{(\mu_1,\dots,\mu_{n-1})}$$

$\Longrightarrow$ restrictions is MF as a $GL_{n-1}(\mathbb{C})$-module

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$$GL_n \underset{\text{MF}}{\supset} GL_{n-1} \underset{\text{MF}}{\supset} GL_{n-2} \underset{\text{MF}}{\supset} \cdots \underset{\text{MF}}{\supset} GL_1$$

$\Longrightarrow$ Gelfand–Tsetlin basis

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Infinite dimensional version

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$\Longrightarrow$ restrictions is MF as a $GL_{n-1}(\mathbb{C})$-module

Infinite dimensional version

Ex.15 ♣ ref. ($U(p,q) \downarrow U(p-1,q)$)

${}^{\forall}\pi$: irred. unitary rep. of $U(p,q)$ with higest weight

$\Longrightarrow$ restriction $\pi|_{U(p-1,q)}$ is MF as a $U(p-1,q)$-module

GL–GL duality

$$N = mn$$

Ex.16 (GL–GL duality à la R. Howe)

$\Longrightarrow GL_m \times GL_n \curvearrowright S(\mathbb{C}^N) \simeq S(M(m,n;\mathbb{C}))$

This rep. is MF

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$\Downarrow$ generalization 1

Hidden symmetry $\Longleftrightarrow$ Broken symmetry

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Hidden symmetry $\Longleftrightarrow$ Broken symmetry

Ex.17 (Progress in Math. 2008)

Branching law of holomorphic discrete series rep. with respect to symmectric pair

Hua–Kostant–Schmid, K–

finite dim compact subgp

∞ dim non-compact subgp

$$U(m,n) \downarrow U(m) \times U(n)$$

$$U(m,n) \downarrow U(m_1,n_1) \times U(m_2,n_2)$$

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$\Downarrow$ generalization 2

MF space: $G \curvearrowright X \Longrightarrow G \curvearrowright \mathcal{O}(X)$

function on X

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Ex.18 (Kac's MF space '80)

$S(\mathbb{C}^N)$ is still MF as a $GL_{m-1} \times GL_n$ module

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$S(\mathbb{C}^N)$ is still MF as a $GL_{m-1} \times GL_n$ module

Ex.19 (counterexample)
$S(\mathbb{C}^N)$ is no more MF as a $GL_{m-1} \times GL_{n-1}$ module

MF for unitary rep (definition)

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Observation

$n \leq 1 \iff \mathrm{End}(\mathbb{C}^n)$ is commutative.

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$(\pi, \mathcal{H})$: unitary rep. of G $\Downarrow$ (Schur's lemma)

Def.

$(\pi, \mathcal{H})$ is MF if $\mathrm{End}_G(\mathcal{H})$ is commutative.

Def. $\mathrm{End}_{\mathbb{C}}(\mathcal{H}) = \{T : \mathcal{H} \to \mathcal{H}$ continuous linear maps$\}$

$\cup$

$\mathrm{End}_G(\mathcal{H}) := \{T \in \mathrm{End}_{\mathbb{C}}(\mathcal{H}) : T \circ \pi(g) = \pi(g) \circ T, \ ^{\forall} g \in G\}$

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Recall:

Def. (naive) $(\pi, \mathcal{H})$ is MF

multiplicity-free

if $\dim \mathrm{Hom}_G(\tau, \pi) \le 1$ ($\forall \tau$: irred. rep. of G).

MF for unitary rep (definition)

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<u>Def.</u>

$(\pi, \mathcal{H})$ is MF if $\mathrm{End}_G(\mathcal{H})$ is commutative.

$\Downarrow$

<u>Prop.</u> The irreducible decomp. of π is unique, and $m_\pi(\tau) \leq 1$ for almost every τ with respect to $d\mu$.

In particular, multiplicity for any discrete spectrum ≤ 1

$$\pi \simeq \int_{\widehat{G}} m_\pi(\tau)\, \tau \, d\mu(\tau) \quad \text{(direct integral)}$$

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$(\pi, \mathcal{H})$: unitary rep. of G $\Downarrow$ (Schur's lemma)

Def.
$(\pi, \mathcal{H})$ is MF if $\mathrm{End}_G(\mathcal{H})$ is commutative.

$(\pi, \mathcal{U})$: continuous rep.

Def. We say $(\pi, \mathcal{U})$ is (unitarily) MF
if, for any unitary rep. $(\varpi, \mathcal{H})$ s.t.
there exists an injective continuous G-map $\mathcal{H} \hookrightarrow \mathcal{U}$,
$(\varpi, \mathcal{H})$ is MF.

Fourier transform (MF rep.)

Ex.21 (Fourier transform)

$$L^2(\mathbb{R}) \simeq \int_{\mathbb{R}}^{\oplus} \mathbb{C} e^{i\zeta x} d\zeta$$

(direct integral of Hilbert spaces)

$$f(x) = \int_{\mathbb{R}} f(\zeta) e^{i\zeta x} d\zeta$$

Regular rep. of $\mathbb{R}$ on $L^2(\mathbb{R})$ by $f(*) \to f(* - c)$

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$\mathrm{End}_{\mathbb{R}}(L^2(\mathbb{R})) \simeq L^\infty(\mathbb{R})$ (ring of multiplier operators)

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continuous rep. $\mathbb{R} \curvearrowright \mathcal{S}'(\mathbb{R})$ is also MF

I. EXAMPLES OF MULTIPLICITY-FREE REPRESENTATIONS
II. INFINITE DIMENSIONAL EXAMPLES
III. PROPAGATION THEOREM OF MULTIPLICITY-FREE REPRESENTATIONS
IV. VISIBLE ACTIONS ON COMPLEX MANIFOLDS
V. APPLICATIONS OF VISIBLE ACTIONS TO MULTIPLICITY-FREE REPRESENTATIONS

Varna Lectures on visible actions and MF representations

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MF = multiplicity-free (definition)

$$\pi\colon \underset{\text{group}}{G} \to GL_{\mathbb{C}}(\mathcal{H})$$

Def. (naive) $(\pi, \mathcal{H})$ is MF (multiplicity-free)

if $\dim \mathrm{Hom}_G(\tau, \pi) \leq 1$ ($\forall \tau$: irred. rep. of G).

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$\Longrightarrow$ unitary rep. $\mathbb{R} \curvearrowright L^2(\mathbb{R})$ is MF

continuous spectrum

Plancherel formula for Riemannian symm. space G/K

Ex.22 (Harish-Chandra, Helgason) $G/K = SL(n,\mathbb{R})/SO(n)$

$$L^2(G/K) \simeq \underbrace{\int^{\oplus}_{\sum \lambda_i = 0,\ \lambda_1 \geq \cdots \geq \lambda_n} \mathcal{H}_\lambda \quad \underline{d\lambda}}_{\text{MF}}$$

cont. spec.

$\mathcal{H}_\lambda$: ∞-dim, irred. rep. of G

$\Longrightarrow$ The regular representation of G on $L^2(G/K)$ is MF

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cont. spec.

MF

$\mathcal{H}_\lambda$: ∞-dim, irred. rep. of G

$\Longrightarrow$ The regular representation of G on $L^2(G/K)$ is MF

$$\begin{aligned}\mathrm{End}_G(L^2(G/K)) &\simeq L^\infty((\mathbb{R}^n/\mathbb{R}))^{\mathcal{S}_n} \\ &\simeq L^\infty(\mathbb{R}^{n-1}/\mathcal{S}_n)\end{aligned}$$

(ring of multiplier operators)

Plancherel formula for Riemannian symm. space G/K

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cont. spec.

MF

$\mathcal{H}_\lambda$: ∞-dim, irred. rep. of G

$\Longrightarrow$ The regular representation of G on $L^2(G/K)$ is MF

MF still holds for vector bundle case of 'small' fibers,

$\mathcal{V} := G \times_K \wedge^k(\mathbb{C}^n) \to G/K \quad (0 \leq k \leq n)$,

associated to the $SO(n)$-representation on the exterior power $\Lambda^k(\mathbb{C}^n)$, but no other cases (Deitmar, K– 2005)

Plancherel formula for Riemannian symm. space G/K

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cont. spec.

$\mathcal{H}_\lambda$: ∞-dim, irred. rep. of G

$\Longrightarrow$ The regular representation of G on $L^2(G/K)$ is MF

MF still holds under certain deformation of G-regular representation of $L^2(G/K)$

deformation coming from hidden symmetry.
E.g. Gelfand–Vershik's canonical rep of $SL_2(\mathbb{R})$.

Plancherel formula for Riemannian symm. space G/K

Ex.22 (Harish-Chandra, Helgason) $G/K = SL(n,\mathbb{R})/SO(n)$

$$L^2(G/K) \simeq \int^{\oplus}_{\sum \lambda_i = 0,\ \lambda_1 \geq \cdots \geq \lambda_n} \mathcal{H}_\lambda \quad d\lambda$$

cont. spec.

MF

$\mathcal{H}_\lambda$: ∞-dim, irred. rep. of G

$\Longrightarrow$ The regular representation of G on $L^2(G/K)$ is MF

Other real forms of $SL(n,\mathbb{C})/SO(n,\mathbb{C})$:

Ex.23 (T. Oshima, Delorme) $G/H = SL(n,\mathbb{R})/SO(p,n-p)$

Multiplicity of most cont. spec. in $L^2(G/H)$

$$= \frac{n!}{p!(n-p)!} > 1 \text{ if } 0 < p < n.$$

$\Longrightarrow$ NOT MF

Broken symmetry and hidden symmetry

$$G \subset \widetilde{G} \qquad \underset{\text{unitary rep}}{\overset{\pi}{\longrightarrow}} \qquad GL_{\mathbb{C}}(\mathcal{H})$$

Broken symmetry and hidden symmetry

Hidden symmetry

$$G \subset \widetilde{G} \quad \xrightarrow[\text{unitary rep}]{\pi} \quad GL_{\mathbb{C}}(\mathcal{H})$$

Broken symmetry and hidden symmetry

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Broken symmetry and hidden symmetry

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Broken symmetry

Branching law
= description of broken symmetry

Deformation of $G \curvearrowright L^2(G/K)$

$$G \overset{\pi}{\curvearrowright} L^2(G/K)$$

Deformation of $G \curvearrowright L^2(G/K)$

$$
\begin{array}{ccc}
G & \overset{\pi}{\curvearrowright} & L^2(G/K) \\
\cap & \nearrow & \\
{}^{\exists}\widetilde{G} & & {}^{\exists}\widetilde{\pi}
\end{array}
$$

Deformation of $G \curvearrowright L^2(G/K)$

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<u>Prop</u> For any classical reductive G, there exist $\widetilde{G}$ ($\supsetneqq G$) and an irreducible unitary rep $\widetilde{\pi}$ of $\widetilde{G}$ s.t. $\widetilde{\pi}|_G = \pi$

Deformation of $G \curvearrowright L^2(G/K)$

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<u>E.g.</u> $G = GL(n, \mathbb{R}) \hookrightarrow \widetilde{G} = Sp(n, \mathbb{R})$

Deformation of $G \curvearrowright L^2(G/K)$

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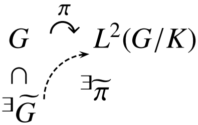
<u>E.g.</u> $G = GL(n, \mathbb{R}) \hookrightarrow \widetilde{G} = Sp(n, \mathbb{R})$

$\widetilde{\pi}$ lies in a continuous family $\{\widetilde{\pi}_\lambda\}$ of irred unitary reps of $\widetilde{G}$

$\Longrightarrow$ $\pi_\lambda := \widetilde{\pi}_\lambda|_G$ is a continuous family of (non-irreducible) representations of G

$\Longrightarrow$ deformation of $G \curvearrowright L^2(G/K)$

Deformation of $G \curvearrowright L^2(G/K)$



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$\Longrightarrow \pi_\lambda := \widetilde{\pi}_\lambda|_G$ is a continuous family of (non-irreducible) representations of G

$\Longrightarrow$ deformation of $G \curvearrowright L^2(G/K)$ (still <u>MF</u>)

Sometimes discrete spectrum may appear!

Known methods

Various techniques have been used in proving various MF results, in particular, for finite dim'l reps

For example, one may

1. look for an open orbit of a Borel subgroup.
2. apply Littlewood–Richardson rules and variants.
3. use computational combinatorics.
4. employ the Gelfand trick (the commutativity of the Hecke algebra).
5. apply Schur–Weyl duality and Howe duality.

New approach

Plan:

To give a new simple principle that explains the property MF for both finite and infinite dimensional reps

New approach

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Analysis on complex mfd with group action having ∞ many orbits

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Analysis on complex mfd with group action having ∞ many orbits

Theory of visible actions on complex manifolds

New approach

Plan:

To give a new simple principle that explains the property MF for both finite and infinite dimensional reps

Propagation of MF property from fiber to sections

$\Uparrow$

Analysis on complex mfd with group action having ∞ many orbits

Theory of visible actions on complex manifolds

Propagation Theorem

$$\boxed{\begin{array}{c}\mathcal{V}\\ \downarrow \\ D\end{array}}$$

H-equivariant holomorphic vector bundle

Propagation Theorem

$$\begin{array}{c}\mathcal{V}\\ \downarrow\\ D\end{array} \rightsquigarrow H \curvearrowright \mathcal{O}(D, \mathcal{V})$$

H-equivariant holomorphic vector bundle

Propagation Theorem

$$\begin{array}{ccc} \mathcal{V}_x & \subset & \mathcal{V} \\ \downarrow & & \downarrow \\ \{x\} & \subset & D \end{array} \quad \rightsquigarrow \quad \boxed{H \curvearrowright \mathcal{O}(D, \mathcal{V})}$$

H-equivariant holomorphic vector bundle

Propagation Theorem

$$\begin{array}{ccccc} \boxed{H_x \curvearrowright \mathcal{V}_x} & \subset & \mathcal{V} & & \\ \downarrow & & \downarrow & \rightsquigarrow & \boxed{H \curvearrowright \mathcal{O}(D, \mathcal{V})} \\ \{x\} & \subset & D & & \end{array}$$

H-equivariant holomorphic vector bundle

$$H_x = \{h \in H : h \cdot x = x\}$$

Propagation Theorem

$$\begin{array}{ccc} H_x \curvearrowright \mathcal{V}_x & \subset & \mathcal{V} \\ \downarrow & & \downarrow \\ \{x\} & \subset & D \end{array} \quad \rightsquigarrow \quad H \curvearrowright \mathcal{O}(D, \mathcal{V})$$

H-equivariant holomorphic vector bundle

Properties of $H_x \curvearrowright \mathcal{V}_x$	Propagation ? $\Longrightarrow$	Properties of $H \curvearrowright \mathcal{O}(D, \mathcal{V})$
"easy"		"difficult"

Propagation Theorem

$$\begin{array}{ccccc} H_x \curvearrowright \mathcal{V}_x & \subset & \mathcal{V} & & \\ \downarrow & & \downarrow & \rightsquigarrow & H \curvearrowright \mathcal{O}(D, \mathcal{V}) \\ \{x\} & \subset & D & & \end{array}$$

H-equivariant holomorphic vector bundle

Properties of $H_x \curvearrowright \mathcal{V}_x$ (fiber) $\xRightarrow{\text{Propagation ?}}$ Properties of $H \curvearrowright \mathcal{O}(D, \mathcal{V})$ (sections)

$\Uparrow$

Geometry of the base space D

Propagation Theorem

$$\begin{array}{ccccc} H_x \curvearrowright \mathcal{V}_x & \subset & \mathcal{V} & & \\ \downarrow & & \downarrow & \rightsquigarrow & H \curvearrowright \mathcal{O}(D, \mathcal{V}) \\ \{x\} & \subset & D & & \end{array}$$

H-equivariant holomorphic vector bundle

Theorem

Irreducibility $H_x \curvearrowright \mathcal{V}_x$ (fiber) $\xrightarrow{\text{Propagation}}$ Irreducibility $H \curvearrowright \mathcal{O}(D, \mathcal{V})$ (sections)

$\Uparrow$

H acts transitively on the base space D

Propagation Theorem

$$\begin{array}{ccccc} H_x \curvearrowright \mathcal{V}_x & \subset & \mathcal{V} & & \\ \downarrow & & \downarrow & \rightsquigarrow & H \curvearrowright \mathcal{O}(D, \mathcal{V}) \\ \{x\} & \subset & D & & \end{array}$$

H-equivariant holomorphic vector bundle

Theorem (Progress in Mathematics, 2013)

MF $H_x \curvearrowright \mathcal{V}_x$ (fiber) $\xrightarrow{\text{Propagation}}$ MF $H \curvearrowright \mathcal{O}(D, \mathcal{V})$ (sections)

$\Uparrow$

H acts strong visibly on the base space D

Automorphism of group action

$$H \curvearrowright D$$

Automorphism of group action

$$H \curvearrowright D$$

Lie group — manifold

<u>Def</u> $\sigma \in \mathrm{Aut}(H; D)$

$$\Longleftrightarrow \begin{cases} \sigma \curvearrowright H & \text{automorphism of Lie group} \\ \sigma \curvearrowright D & \text{diffeomorphism} \\ \sigma(g \cdot x) = \sigma(g) \cdot \sigma(x) & ({}^{\forall} g \in H,\ {}^{\forall} x \in D) \end{cases}$$

Automorphism of group action

$$\underset{\text{Lie group}}{H} \curvearrowright \underset{\text{manifold}}{D}$$

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Write simply $\sigma \curvearrowright D$ instead of $\sigma \in \mathrm{Aut}(H; D)$

Assumptions of MF theorem

$\mathcal{V} \to D$: H-equivariant

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Note: Assumption 2 is automatic for line bundles

Propagation of MF property

Progr. Math (2013)

H: Lie group

H-equiv. holo. vector b'dle:
$\mathcal{V} \to D$

$\Downarrow$

$H \curvearrowright \mathcal{O}(D, \mathcal{V}) = \{\text{holo. sections}\}$

Theorem (Propagation theorem)

$H_x \curvearrowright \mathcal{V}_x$ MF $(\forall x \in D)$

$\Longrightarrow H \curvearrowright \mathcal{O}(D, \mathcal{V})$ MF

if assumptions 1 & 2 hold.

Observations of MF theorem

Points: H has infinitely many orbits on D

- propagation of MF property

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fiber MF

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fiber MF ⟹ sections MF

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fiber MF — sections MF

$\Uparrow$

geometry of base space
. . . '(strongly) visible action'

Examples of MF theorem

Ex.20 $\quad H = U(m) \times U(n)$

$$D = M(m, n; \mathbb{C}) \simeq \mathbb{C}^{mn}$$

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$\Downarrow$ Propagation theorem

$H \curvearrowright \mathbf{Pol}(D)$ MF

$$GL_m \times GL_n \curvearrowright \mathbf{Pol}(M(m, n; \mathbb{C}))$$

Totally real submanifold

(D, J) complex manifold

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$\mathbb{C}^n \supset \mathbb{R}^k$ totally real

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Rem We do not request S to be of
maximal dimension (e.g. $S = \mathbb{R}^n$ in $\mathbb{C}^n$.)

§3 Visible actions

(D, J) complex mfd, connected

Def. A real submanifold S of D is totally real if $T_x S$ does not contain any complex subspace.

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$H \overset{\text{holomorphic}}{\curvearrowright} (D, J)$ complex mfd, connected

Def.(2003) A holomorphic action of H is visible w.r.t. a slice S if $^{\exists}D' \underset{\text{open}}{\subset} D$,

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$^\exists S \underset{\text{totally real}}{\subset} D'$ s.t.

$$\begin{cases} S \text{ meets every } H\text{-orbit} \\ J_x(T_xS) \subset T_x(H \cdot x) \ (x \in S) \end{cases}$$

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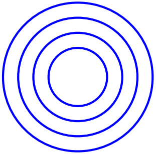

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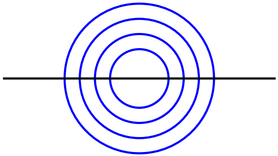

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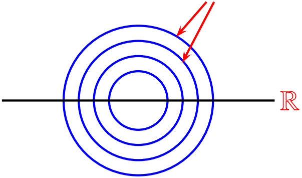

$$J_x(T_x S) \subset T_x(H \cdot x), \quad {}^\forall x \in \mathbb{R} \setminus \{0\} =: S$$

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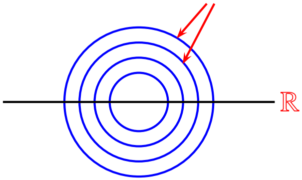

$\Longrightarrow$ $\mathbb{T}$-action on $\mathbb{C}$ is visible

Strongly visible actions

holomorphic
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s.t. $H \cdot D^{\sigma}$ contains a non-empty open set of D.

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Remark. Not necessarily $\sigma^2 = \mathrm{id}$

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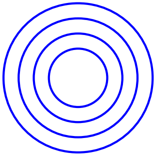

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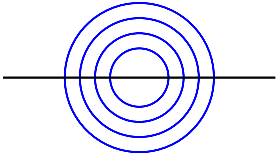

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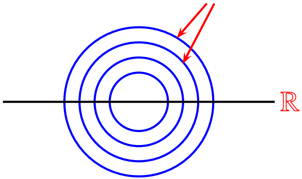

$\sigma(a) := \overline{a}, \ \sigma(z) := \overline{z}$ anti-holomorphic.

Then $\sigma(a \cdot z) = \sigma(a) \cdot \sigma(z)$ (compatibility), and $\sigma|_{\mathbb{R}} = \mathrm{id}$.

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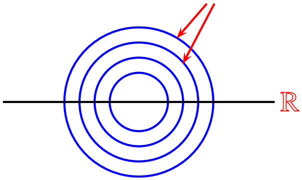

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$\sigma|_S = \mathrm{id}$

$H \cdot S$ contains a non-empty open set of D.

Remark. S is automatically totally real.

Point Try to find a smallest possible $S \subset D^{\sigma}$.

Strongly visible actions

Proposition Strongly visible $\Longrightarrow$ Visible

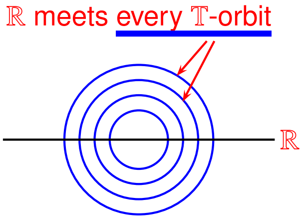
Strongly visible actions

Proposition Strongly visible $\Longrightarrow$ Visible

To be more precise,

strongly visible w.r.t. a slice S

$\Longrightarrow$ visible w.r.t. S' for some $S' \underset{\text{open dense}}{\subset} S$.



G/K Hermitian symm. space

Ex.22

$$G = SL(2, \mathbb{R})$$
$$K = SO(2)$$
$$H = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} : a > 0 \right\}$$
$$N = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in \mathbb{R} \right\}$$
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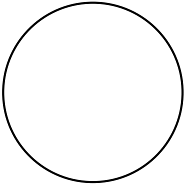

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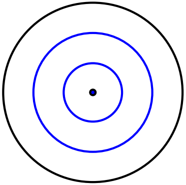

K-orbits

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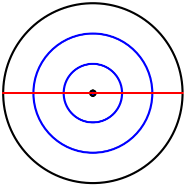
$K \curvearrowright G/K$ visible



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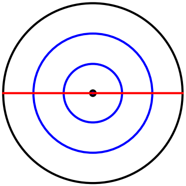

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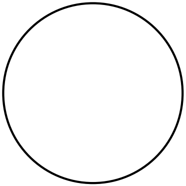

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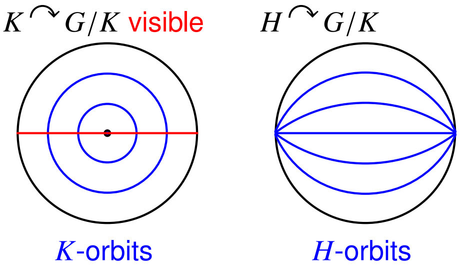

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Ex.22 $G = SL(2,\mathbb{R})$
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$H = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} : a > 0 \right\}$
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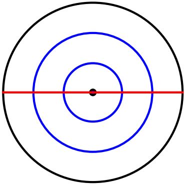

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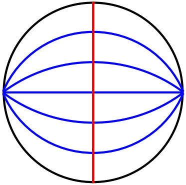

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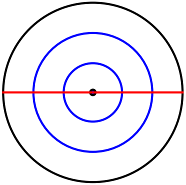

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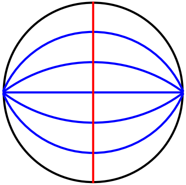

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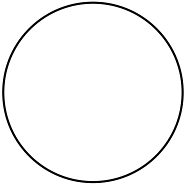

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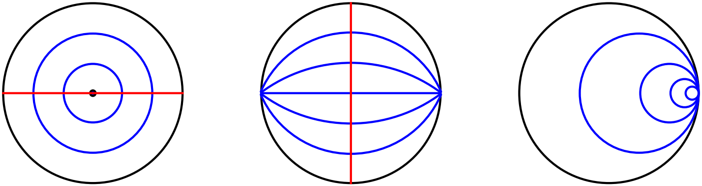

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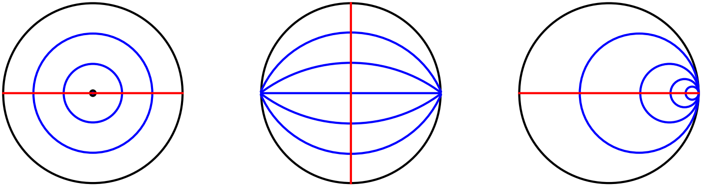

K-orbits $\qquad H$-orbits $\qquad N$-orbits

Visible actions on symmetric spaces

Theorem (Transf. Groups (2007))

Assume $\begin{cases} G/K & \text{Hermitian symm. sp.} \\ (G,H) & \text{symmetric pair} \end{cases}$

$\Longrightarrow H \curvearrowright G/K$ is (strongly) visible

§4 Compex / Riemannian / symplectic geometry

holomorphic

$H \curvearrowright (D, J)$ complex mfd, connected

Def. Action is visible if

$^{\exists}S \subset {}^{\exists}D' \subset D$ s.t.

totally real open

$\begin{cases} S \text{ meets every } H\text{-orbit in } D' \\ J_x(T_x S) \subset T_x(H \cdot x) \ (x \in S) \end{cases}$

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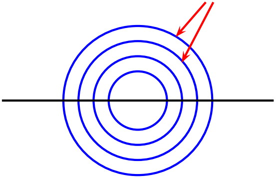
S meets every $\mathbb{T}$-orbit



$S = \mathbb{R}$

Complex / Riemannian / symplectic

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$H \overset{\text{symplectic}}{\curvearrowright} (D, \omega)$ symplectic mfd

Def. (Guillemin–Sternberg, Huckleberry–Wurzbacher)
Action is coisotropic (or multiplicity-free)
if $T_x(H \cdot x)^{\perp\omega} \subset T_x(H \cdot x)$ for principal orbits $H \cdot x$ in D

Three geometries

Complex geometry

Symplectic geometry

Riemannian geometry

Three geometries

Complex geometry

Visible action

K– (2004)

Symplectic geometry

Coisotropic action

Guillemin–Sternberg ('84)
Huckleberry–Wurzbacher ('90)

Riemannian geometry

Polar action

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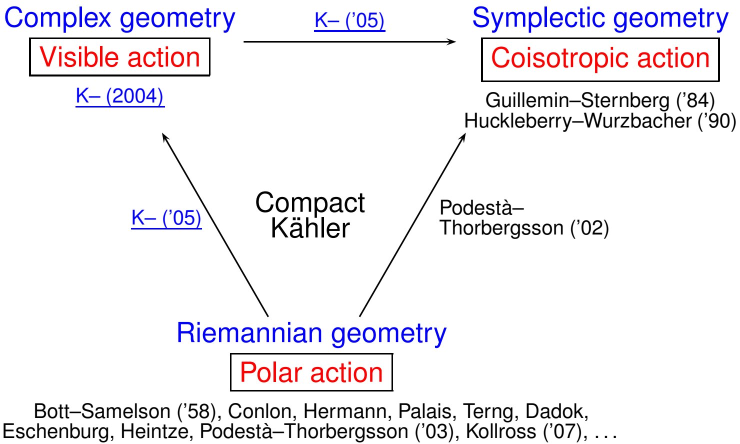

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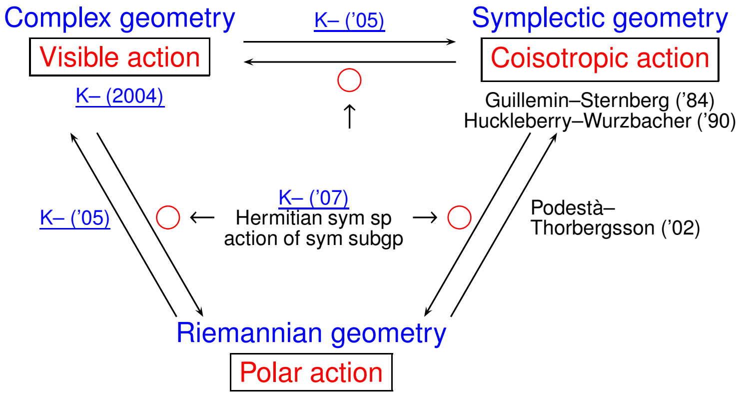

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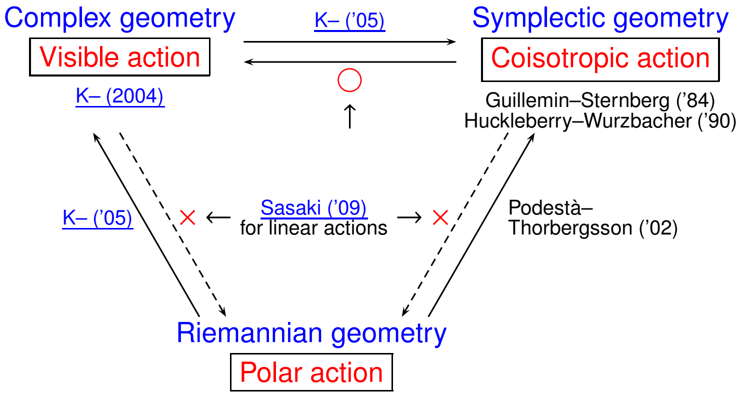

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$$\overline{H \cdot x} \underset{\text{compatibility}}{=} \overline{H} \cdot \overline{x} \underset{x \in M(m,n;\mathbb{R})}{=} H \cdot x$$

□

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In general,

Strongly visible

(i.e. $^{\exists}\sigma$ anti-holo s.t. $(H \cdot D^{\sigma})^{\circ} \neq \emptyset$)

$\Rightarrow$ Assumption 1 of Theorem

(i.e. $^{\exists}\sigma$ anti-holo s.t. σ preserves generic H-orbits)

Analysis on ∞-many orbits

$\mathcal{V} \to X$: H-equiv. holo vector bundle.

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	$\Uparrow$	
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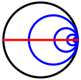

∞ many orbits

Classification theory of visible actions

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Want to find visible actions systematically

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Three involutions (special case)	$\longleftrightarrow$	visible action (2004–) (special case)

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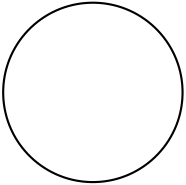

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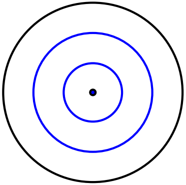

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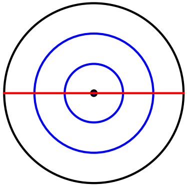

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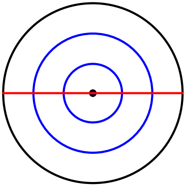

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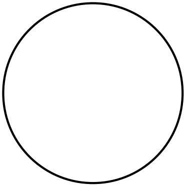

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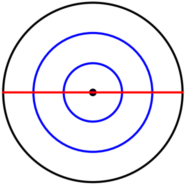

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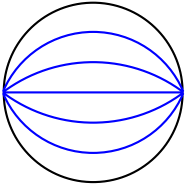

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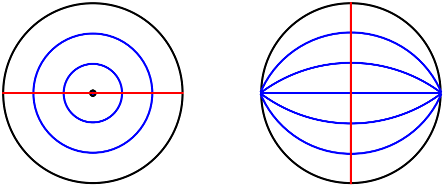

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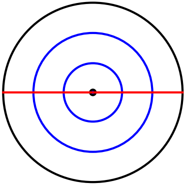

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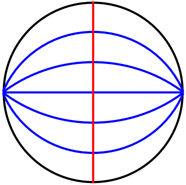

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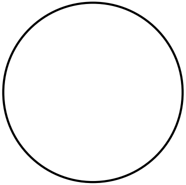

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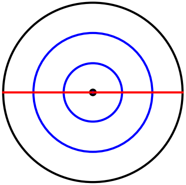

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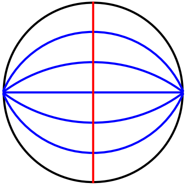

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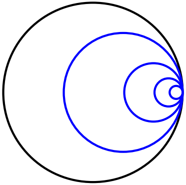

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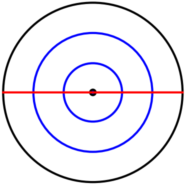
$$G/K \simeq \{ z \in \mathbb{C} : |z| < 1 \}$$

$K \curvearrowright G/K$ visible



K-orbits

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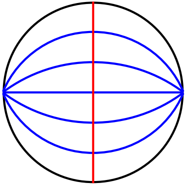

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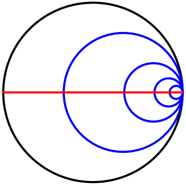

N-orbits

Visible actions on symmetric spaces

Theorem (Transf. Groups (2007))

Assume $\begin{cases} G/K & \text{Hermitian symm. sp.} \\ (G,H) & \text{symmetric pair} \end{cases}$

$\Longrightarrow H \curvearrowright G/K$ is (strongly) visible

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Ex.19 π_λ, π_μ highest wt. modules of scalar type

$\Longrightarrow \pi_\lambda \otimes \pi_\mu$ is MF

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Ex.20 $\begin{cases} \pi_\lambda & \text{highest wt. module of scalar type} \\ (G,H) & \text{symmetric pair} \end{cases}$

$\Longrightarrow \pi_\lambda|_H$ is MF

Visible actions on symmetric spaces

Theorem (geometry of three involutions '07)

Assume $\begin{cases} G/K & \text{Hermitian symm. sp.} \\ (G,H) & \text{any symmetric pair} \end{cases}$

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⇓ Propagation theorem

Thm $V_\lambda = U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} \mathbb{C}_\lambda$ (λ generic) is an algebraic MF direct sum of irreducible $\mathfrak{g}'$-modules if

- nilradical of $\mathfrak{p}_\mathbb{R}$ is abelian
- $G'_\mathbb{R} \cdot o$ is closed in $G_\mathbb{R}/P_\mathbb{R}$
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$G'_{\mathbb{R}}$	$\subset$	$G_{\mathbb{R}}$	$\supset$	$P_{\mathbb{R}}$
subgp		real reductive		real parabolic

Finite dimensional case

Also, for finite dimensional case

$\Downarrow$ Propagation theorem

Eg.23 (Okada, '98, rectangular shaped rep)

$\mathfrak{g}_{\mathbb{C}} = \mathfrak{gl}(n, \mathbb{C})$

$$\lambda = (\underbrace{a, \cdots, a}_{p}, \underbrace{b, \cdots, b}_{n-p}) \in \mathbb{Z}^n, \ a \geq b$$

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$$\mathfrak{h}_{\mathbb{C}} = \begin{cases} \mathfrak{gl}(k, \mathbb{C}) + \mathfrak{gl}(n-k, \mathbb{C}) & (1 \leq k \leq n) \\ \mathfrak{o}(n, \mathbb{C}) & \\ \mathfrak{sp}(\frac{n}{2}, \mathbb{C}) & (n : \text{even}) \end{cases}$$

G/K Hermitian symm. space

<u>Ex.22</u>

$$G = SL(2,\mathbb{R})$$
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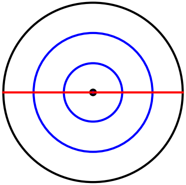

K-orbits

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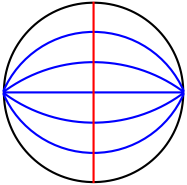
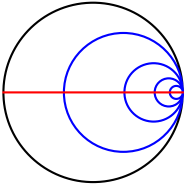


H-orbits

$N \curvearrowright G/K$ visible



N-orbits

Non-reductive example

<u>Theorem</u> $N \subset G \supset K$

Assume $\begin{cases} G/K & \text{Hermitian symm. of non-cpt. type} \\ N & \text{max. unipotent subgp.} \end{cases}$

$\Longrightarrow N \curvearrowright G/K$ (strongly) visible

$\Downarrow$ Propagation theorem

<u>Ex.24</u> π_λ: highest wt. module of scalar type

$\Longrightarrow \pi_\lambda|_N$ is MF

Classification theory of visible actions

Methods to find visible action

Want to find visible actions systematically

- Structure theory
  - geometry of three involutions (symmetric case)
  - generalized Cartan decomposition (non-symmetric case)
- Make new from old
  - make ‘large’ from ‘small’
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(Generalized) Cartan involutions

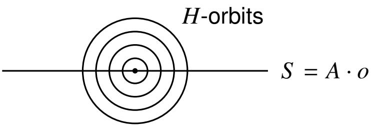
<u>Observation</u>

$$D = G/K$$

Suppose we have a decomposition

$$G = H\ A\ K$$

Set $S := A \cdot o \subset D$



$\Longrightarrow S$ is a candidate of ‘slice’ for (strongly) visible action

Classification theory of visible actions

Grassmannian $U(n)/(U(p) \times U(q)) \simeq Gr_p(\mathbb{C}^n) \quad (n = p + q)$

Ex.(symmetric case) $\quad n_1 + n_2 = p + q = n$

$\Longrightarrow$ $U(n_1) \times U(n_2)$ acts on $Gr_p(\mathbb{C}^n)$

in a strongly visible fashion

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Ex.34 (JMSJ 2007)

$n_1 + n_2 + n_3 = p + q = n$

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$\Longleftrightarrow \min(n_1 + 1, n_2 + 1, n_3 + 1, p, q) \leq 2$

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For type B, C, D and exceptional groups (Y. Tanaka, Tohoku J. (2013), J. Math. Soc. Japan (2013), B. Austrian Math Soc. (2013), J. Algebra (2014))

$\Downarrow$ Propagation theorem

MF property of the following

- $GL_m \times GL_n \curvearrowright S(\mathbb{C}^{mn})$ Ex.16
- $GL_{m-1} \times GL_n \curvearrowright S(\mathbb{C}^{mn})$ Ex.18 (Kac)
- the Stembridge list of $\pi_\lambda \otimes \pi_\nu$ Ex.11
- $GL_n \downarrow (GL_p \times GL_q)$ Ex.12
- $GL_n \downarrow (GL_{n_1} \times GL_{n_2} \times GL_{n_3})$ Ex.13
- ∞-dimensional versions

  ……

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Make ‘large’ from ‘small’

Idea: induced action preserving visibility

Make ‘large’ from ‘small’

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$$H \subset G$$

Make ‘large’ from ‘small’

Idea: induced action preserving visibility

$$H \subset G$$

$H \curvearrowright Y$ visible w.r.t. S

$\rightsquigarrow$ $\Leftarrow$ certain assumption

$G \curvearrowright X := G \times_H Y$ visible w.r.t. $S \simeq [\{e\}, S]$

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Ex. $H = U(p) \times U(q)$, $Y = M(p,q;\mathbb{C})$ $(p \geq q)$

$G = U(p+q)$, $X = T^*(G/H) = T^*(Gr_p(\mathbb{C}^{p+q}))$

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⇝ momentum map: nilpotent orbit for $GL(p+q, \mathbb{C})$ for partition $(2^q, 1^{p-q})$ is spherical (Panyushev)

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Triunity of visible actions

$$\begin{pmatrix} H & & L \\ & \rotatebox{45}{$\subset$} \; \rotatebox{-45}{$\supset$} & \\ & G & \\ & \cup & \\ & G^{\sigma} & \end{pmatrix} := \begin{pmatrix} \mathbb{T}^n & & U(1) \times U(n-1) \\ & \rotatebox{45}{$\subset$} \; \rotatebox{-45}{$\supset$} & \\ & U(n) & \\ & \cup & \\ & O(n) & \end{pmatrix}$$

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(visible actions)

$\mathbb{P}^{n-1}\mathbb{R}$ meets every $\mathbb{T}^n$-orbit on $\mathbb{P}^{n-1}\mathbb{C}$

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(visible actions)

$\boxed{\mathbb{P}^{n-1}\mathbb{R} \text{ meets every } \mathbb{T}^n\text{-orbit on } \mathbb{P}^{n-1}\mathbb{C}}$

$\updownarrow$ $\mathbb{P}^{n-1}\mathbb{R} = G^\sigma/G^\sigma \cap L$, $\mathbb{T}^n = H$, $\mathbb{P}^{n-1}\mathbb{C} = G/L$

Group

$\boxed{G = HG^\sigma L} \Rightarrow H \curvearrowright G/L$

Triunity of visible actions

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Group $\boxed{G = HG^\sigma L} \Rightarrow H \curvearrowright G/L$

$\updownarrow$

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Geometry (visible actions)

$\mathbb{P}^{n-1}\mathbb{R}$ meets every $\mathbb{T}^n$-orbit on $\mathbb{P}^{n-1}\mathbb{C}$

$\mathbb{P}^{n-1}\mathbb{R} = G^\sigma/G^\sigma \cap L$, $\mathbb{T}^n = H$, $\mathbb{P}^{n-1}\mathbb{C} = G/L$

$\updownarrow$

Group $G = HG^\sigma L \Rightarrow H \curvearrowright G/L$

$\updownarrow$

Group $G = LG^\sigma H \Rightarrow L \curvearrowright G/H$

$\updownarrow$

Group $(G\times G) = \text{diag}(G)(G^\sigma \times G^\sigma)(H\times L) \Rightarrow$ diag.

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$\Downarrow$ Propagation theorem

Three kinds of MF results:

- (Taylor series) $\mathbb{T}^n \curvearrowright \mathcal{O}(\mathbb{C}^n)$ Ex.2
- ($GL_n \downarrow GL_{n-1}$) Restriction $\pi|_{GL_{n-1}}$ Ex.14
- (Pieri) $\pi \otimes S^k(\mathbb{C}^n)$ Ex.9

$\otimes$-product rep.

$$\lambda = (\underbrace{a, \cdots, a}_{p}, \underbrace{b, \cdots, b}_{q}) \in \mathbb{Z}^n, \ a \geq b$$

Ex.11 (Stembridge 2001, K– 2002)

$\pi_\lambda \otimes \pi_\nu$ is MF as a $GL_n(\mathbb{C})$-module if

1) $\min(a-b, p, q) = 1$ (and ν is any),

or

2) $\min(a-b, p, q) = 2$ and

★ ν is of the form $\nu = (\underbrace{x \cdots x}_{n_1} \ \underbrace{y \cdots y}_{n_2} \ \underbrace{z \cdots z}_{n_3}) \ (x \geq y \geq z)$,

or

3) $\min(a-b, p, q) \geq 3$, ★ &

$\min(x-y, \ y-z, n_1, n_2, n_3) = 1.$

Restriction $(GL_n \downarrow G_{n_1} \times GL_{n_2} \times GL_{n_3})$

Ex.12 $(GL_n \downarrow (GL_p \times GL_q)) \quad n = p + q$

$$\pi^{GL_n}_{(\underbrace{x,\ldots,x}_{n_1},\underbrace{y,\ldots,y}_{n_2},\underbrace{z,\ldots,z}_{n_3})}|_{GL_p \times GL_q} \text{ is MF}$$

if $\min(p, q) \leq 2$ or

if $\min(n_1, n_2, n_3, x - y, y - z) \leq 1$

(Kostant $n_3 = 0$; Krattenthaler 1998)

Ex.13 $(GL_n \downarrow (GL_{n_1} \times GL_{n_2} \times GL_{n_3}))$ ([5])

$n = n_1 + n_2 + n_3$

$$\pi^{GL_n}_{(\underbrace{a,\ldots,a}_{p},\underbrace{b,\ldots,b}_{q})}|_{GL_{n_1} \times GL_{n_2} \times GL_{n_3}} \text{ is MF}$$

if $\min(n_1, n_2, n_3) \leq 1$ or

if $\min(p, q, a - b) \leq 2$

Analysis on ∞-many orbits

$\mathcal{V} \to X$: H-equiv. holo vector bundle.

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Theorem (Propagation thm of MF property)

Sections

$\Longrightarrow$

$H \curvearrowright \mathcal{O}(X, \mathcal{V})$
multiplicity-free

∞ many orbits

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Fiber		Sections
$H_x \curvearrowright \mathcal{V}_x$ multiplicity-free	$\Longrightarrow$	$H \curvearrowright \mathcal{O}(X, \mathcal{V})$ multiplicity-free

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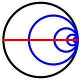
Fiber		Sections
$H_x \curvearrowright \mathcal{V}_x$ multiplicity-free	$\Longrightarrow$	$H \curvearrowright \mathcal{O}(X, \mathcal{V})$ multiplicity-free

$\Uparrow$

$H \curvearrowright X$
(strongly) visible

Base sp.



∞ many orbits

Heuristic idea of Theorem

Setting $\mathcal{V} \to D$: G-equiv. holomorphic vector bundle

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<u>Setting</u> $\mathcal{V} \to D$: G-equiv. holomorphic vector bundle

$$\rightsquigarrow \quad \mathcal{R}_x := \mathrm{End}_{G_x}(\mathcal{V}_x) \underset{\text{subring}}{\subset} \mathrm{End}_{\mathbb{C}}(\mathcal{V}_x)$$

$$\mathcal{R} := \coprod_{x \in D} \mathcal{R}_x \subset \mathcal{E}nd(\mathcal{V})$$

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$$\mathcal{R} := \coprod_{x \in D} \mathcal{R}_x \subset \mathcal{E}nd(\mathcal{V})$$

<u>Assumption</u> $G \curvearrowright D$ strongly visible w.r.t. S $(\subset D)$

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Reproducing kernel

Prototype (Scalar valued) holomorphic functions

$$\mathbb{C}^n \qquad \supset \qquad D \qquad \text{complex domain}$$

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Definition (reproducing kernel)

Let $\{\varphi_l\}$ be an orthonormal basis of $\mathcal{H}$.

$$K_{\mathcal{H}}(z,w) := \sum_l \varphi_l(z)\overline{\varphi_l(w)}$$

is independent of the choice of the basis.

Examples of reproducing kernels

$$\{\varphi_j\} \quad \subset \quad \mathcal{H} \quad \subset \quad \mathcal{O}(D)$$

orthonormal basis   Hilbert space

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<u>Example 1</u> (weighted Bergman space)

$$D := \{z \in \mathbb{C} : |z| < 1\}$$

Fix $\lambda > 1$.

$$\mathcal{H} := \{f \in \mathcal{O}(D) : \|f\|_\lambda < \infty\}$$

$$\|f\|_\lambda := \left(\int_D |f(x+iy)|^2(1-x^2-y^2)^{\lambda-2}dxdy\right)^{\frac{1}{2}}$$

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Properties of reproducing kernel

$$\begin{array}{c} \mathcal{O}(D) \\ \cup \\ \mathcal{H} \end{array} \quad \rightsquigarrow K_{\mathcal{H}}(z,w) = \sum_j \varphi(x)\overline{\varphi(w)}$$

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Then G acts on $\mathcal{H}$ as a unitary representation if and only if

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$$(\star) \iff K_{\mathcal{H}}(gz, gz) = K_{\mathcal{K}}(z,z) \quad {}^{\forall}g \in G, {}^{\forall}z \in D$$

Scalar-valued reproducing kernel

$\underset{\text{Hilbert space}}{\mathcal{H}} \subset \mathcal{O}(D)$

Assume that for each $x \in D$,

$$\begin{array}{rccc} \mathrm{ev}_x : & \mathcal{H} & \to & \mathbb{C} \\ & \Psi & & \Psi \\ & f & \mapsto & f(x) \end{array} \quad \text{is continuous.}$$

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$$\begin{aligned} K_{\mathcal{H}}(z, w) &= \mathrm{ev}_w \circ \mathrm{ev}_z^* \\ &= \sum_j \varphi_j(z)\varphi_j(w) \end{aligned}$$

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$\mathcal{V} \to D$: holomorphic vector bundle

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$$\mathcal{V} = \coprod_{x \in D} \mathcal{V}_x \longrightarrow D$$

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$\mathcal{H} \hookrightarrow \mathcal{O}(D, \mathcal{V})$
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$\Updownarrow$ one to one

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$\mathcal{H}$: unitarity, irreducibility, MF, $\ldots \Longleftrightarrow$ Properties on $K_{\mathcal{H}}$

‘Visible’ approach to multiplicity-free theorems

Theorem

$$\text{fiber} \xrightarrow{\text{visible action}} \text{sections}$$

'Visible' approach to multiplicity-free theorems

Thm (K– '08) $\pi|_H$ is multiplicity-free if
π: highest wt. rep. of scalar type
(G, H): semisimple symmetric pair
(Hua, Kostant, Schmid, K– : explicit formula)

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Multiplicity-free space
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Stembridge's list (2001) of multiplicity-free $\otimes$ product of finite dim'l reps (GL_n)

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Hermitian symm sp. (K– '07) ↖ Theorem ↗ Crown domain

Vector sp. (Sasaki '09) ↙ Theorem ↘ Grassmann mfd. (K– '07)

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fiber $\xrightarrow{\text{visible action}}$ sections

‘Visible’ approach

To give a simple principle that explains the property MF for both finite and infinite dimensional reps

MF (multiplicity-free) theorem

Propagation of MF property from fiber to sections

$\Uparrow$

Visible actions on complex mfds

Analysis of group action with infinitely many orbits

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Thank you !!

(Short story by Soseki, 1908)

"He uses the hammer and chisel without any forethought, and he can make the eyebrows and nose as live."

"Ah, they are not made by hammer and chisel. The eyebrows and nose are buried inside the tree. It is exactly the same as digging a rock out of the earth — there is no way to mistake."

$\overset{\exists 1}{\hookleftarrow}$