Visible Actions and Multiplicity-free Representations

Toshiyuki Kobayashi

Kavli IPMU, the University of Tokyo
http://www.ms.u-tokyo.ac.jp/~toshi/

XVIth International Conference on
Geometry, Integrability and Quantization
6–11 June 2014, Varna, Bulgaria

Varna Lectures on visible actions and MF representations

I. EXAMPLES OF MULTIPLICITY-FREE REPRESENTATIONS
II. INFINITE DIMENSIONAL EXAMPLES
III. PROPAGATION THEOREM OF MULTIPLICITY-FREE REPRESENTATIONS
IV. VISIBLE ACTIONS ON COMPLEX MANIFOLDS
V. APPLICATIONS OF VISIBLE ACTIONS TO MULTIPLICITY-FREE REPRESENTATIONS

Varna Lectures on visible actions and MF representations

I. EXAMPLES OF MULTIPLICITY-FREE REPRESENTATIONS
II. INFINITE DIMENSIONAL EXAMPLES
III. PROPAGATION THEOREM OF MULTIPLICITY-FREE REPRESENTATIONS
IV. VISIBLE ACTIONS ON COMPLEX MANIFOLDS
V. APPLICATIONS OF VISIBLE ACTIONS TO MULTIPLICITY-FREE REPRESENTATIONS

Plan

I propose

> *a new method (based on "visible actions") to*
> *prove/find/construct*
> *multiplicity-free representations*

for finite/infinite dimensional representations.

References (a) for Varna Lectures

- Overview
  Publ. Res. Inst. Math. Sci. (2005)

- Visible Actions — Classification Theory
  J. Math. Soc. Japan (2007) $\cdots$ GL_n case
  Transformation Groups (2007) $\cdots$ symmetric pairs
  A. Sasaki (IMRN (2009), IMRN (2011), Geometriae Dedicata (2010))
  Y. Tanaka (J. Algebra (2014), J. Math. Soc. Japan (2013), Tohoku J, B. Australian Math. Soc. (2013), etc)

- Multiplicity-free Theorem via Visible Actions
  Progr. Math. (2013) $\cdots$ general theory

References (b) for Varna Lectures

- Application to concrete examples
  Acta Appl. Math. (2004) $\cdots \otimes$ product, GL_n
  Progr. Math. (2008)

- Generalization of Kostant–Schmid formula
  Proc. Rep. Theory, Saga (1997)

- Multiplicity-free Theorems and Orbit Philosophy
  Adv. Math. Sci. (2003), AMS (with Nasrin)
  Nasrin (Geoemtriae Dedicata (2014))

Unkei (Sculptor, 1148?–1223)

(Short story by Soseki, 1908)

Unkei (Sculptor, 1148?–1223)

(Short story by Soseki, 1908)

“He uses the hammer and chisel without any forethought, and he can make the eyebrows and nose as live.”

“Ah, they are not made by hammer and chisel. The eyebrows and nose are buried inside the tree. It is exactly the same as digging a rock out of the earth — there is no way to mistake.”

Unkei (Sculptor, 1148?–1223)

(Short story by Soseki, 1908)

“He uses the hammer and chisel without any forethought, and he can make the eyebrows and nose as live.”

“Ah, they are not made by hammer and chisel. The eyebrows and nose are buried inside the tree. It is exactly the same as digging a rock out of the earth — there is no way to mistake.”

Unkei (Sculptor, 1148?–1223)

(Short story by Soseki, 1908)

“He uses the hammer and chisel without any forethought, and he can make the eyebrows and nose as live.”

“Ah, they are not made by hammer and chisel. The eyebrows and nose are buried inside the tree. It is exactly the same as digging a rock out of the earth — there is no way to mistake.”

$\overset{\exists}{\hookleftarrow}$

Unkei (Sculptor, 1148?–1223)

(Short story by Soseki, 1908)

“He uses the hammer and chisel without any forethought, and he can make the eyebrows and nose as live.”

“Ah, they are not made by hammer and chisel. The eyebrows and nose are buried inside the tree. It is exactly the same as digging a rock out of the earth — there is no way to mistake.”

$\overset{\exists 1}{\hookleftarrow}$

Multiplicity-free theorems

Multiplicity-freeness
=each building block is used no more than once

Multiplicity-free theorems

Multiplicity-freeness
=each building block is used no more than once

The structure of “multiplicity-free representations” may be hidden in classical mathematical methods.

Multiplicity-free theorems

Multiplicity-freeness
=each building block is used no more than once

The structure of “multiplicity-free representations” may be hidden in classical mathematical methods.

<u>E.g.</u> the Taylor series, the Fourier transform,
the expansion into spherical harmonics, etc.

Multiplicity-free theorems

Multiplicity-freeness
=each building block is used no more than once

The structure of “multiplicity-free representations” may be hidden in classical mathematical methods.

E.g. the Taylor series, the Fourier transform,
the expansion into spherical harmonics, etc.

Multiplicity-free property is ‘rare’ in general.

Multiplicity-free theorems

Multiplicity-freeness
=each building block is used no more than once

The structure of “multiplicity-free representations” may be hidden in classical mathematical methods.

E.g. the Taylor series, the Fourier transform,
the expansion into spherical harmonics, etc.

Multiplicity-free property is ‘rare’ in general.

How to find such a structure systematically ?

New approach — “visible action”

Aim . . .

To give a new simple principle that explains the property MF for both finite and infinite dimensional reps

New approach — “visible action”

Aim …

To give a new simple principle that explains the property MF for both finite and infinite dimensional reps

Analysis on complex mfd with group action having ∞ many orbits

New approach — “visible action”

Aim . . .

To give a new simple principle that explains the property MF for both finite and infinite dimensional reps

Analysis on complex mfd with group action having ∞ many orbits

(Strongly) Visible Action (K– 2004)

New approach — “visible action”

Aim . . .

To give a new simple principle that explains the property MF for both finite and infinite dimensional reps

Propagation of MF property from fiber to sections

$\Uparrow$ (Progr. Math 2013)

Analysis on complex mfd with group action having ∞ many orbits

(Strongly) Visible Action (K– 2004)

§1 Multiplicity-free representations

Ex.1 (Eigenspace decomposition)

$\mathcal{H}$: Vector sp./$\mathbb{C}$, $\dim < \infty$

$A \in \mathrm{End}_{\mathbb{C}}(\mathcal{H})$

s.t. all eigenvalues are distinct. ①

§1 Multiplicity-free representations

Ex.1 (Eigenspace decomposition)

$\mathcal{H}$: Vector sp./$\mathbb{C}$, $\dim < \infty$

$A \in \mathrm{End}_{\mathbb{C}}(\mathcal{H})$

s.t. $\boxed{\text{all eigenvalues are distinct.}}$ ①

$\Longrightarrow \mathcal{H} = \mathbb{C}e_1 \oplus \cdots \oplus \mathbb{C}e_n \simeq \mathbb{C}^n$ (canonical)

§1 Multiplicity-free representations

Ex.1 (Eigenspace decomposition)

$\mathcal{H}$: Vector sp./$\mathbb{C}$, $\dim < \infty$

$A \in \mathrm{End}_{\mathbb{C}}(\mathcal{H})$

s.t. $\boxed{\text{all eigenvalues are distinct.}}$ ①

$\Longrightarrow \mathcal{H} = \mathbb{C}e_1 \oplus \cdots \oplus \mathbb{C}e_n \simeq \mathbb{C}^n$ (canonical)

$$\det A \neq 0$$

$$\Downarrow$$

$$\begin{array}{rcccc} \pi_A : & \mathbb{Z} & \longrightarrow & GL_{\mathbb{C}}(\mathcal{H}) \\ & \Cup & & \Cup \\ & n & \longmapsto & A^n \end{array}$$

§1 Multiplicity-free representations

Ex.1 (Eigenspace decomposition)

$\mathcal{H}$: Vector sp./$\mathbb{C}$, $\dim < \infty$

$A \in \mathrm{End}_{\mathbb{C}}(\mathcal{H})$

s.t. $\boxed{\text{all eigenvalues are distinct.}}$ ①

$\Longrightarrow \mathcal{H} = \mathbb{C}e_1 \oplus \cdots \oplus \mathbb{C}e_n \simeq \mathbb{C}^n$ (canonical)

① & $\det A \neq 0$

$$\Downarrow$$

$$\begin{array}{cccc} \pi_A : & \mathbb{Z} & \longrightarrow & GL_{\mathbb{C}}(\mathcal{H}) \\ & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ & n & \longmapsto & A^n \end{array}$$

is MF (multiplicity-free)

MF = multiplicity-free (definition)

$$\pi\colon \underset{\text{group}}{G} \to GL_{\mathbb{C}}(\mathcal{H})$$

MF $=$ multiplicity-free (definition)

$$\pi\colon \underset{\text{group}}{G} \to GL_{\mathbb{C}}(\mathcal{H})$$

Def. (naive) $(\pi, \mathcal{H})$ is $\underset{\text{multiplicity-free}}{\underline{\text{MF}}}$

if $\dim \mathrm{Hom}_G(\tau, \pi) \leq 1$ ($\forall\tau$: irred. rep. of G).

MF = multiplicity-free (definition)

$$\pi\colon \underset{\text{group}}{G} \to GL_{\mathbb{C}}(\mathcal{H})$$

Def. (naive) $(\pi, \mathcal{H})$ is MF
multiplicity-free
if $\dim \operatorname{Hom}_G(\tau, \pi) \leq 1$ ($\forall\tau$: irred. rep. of G).

{irreducible representations}

MF = multiplicity-free (definition)

$$\pi\colon \underset{\text{group}}{G} \to GL_{\mathbb{C}}(\mathcal{H})$$

<u>Def.</u> (naive) $(\pi, \mathcal{H})$ is $\underset{\text{multiplicity-free}}{\underline{\text{MF}}}$

if $\dim \operatorname{Hom}_G(\tau, \pi) \leq 1$ ($\forall \tau$: irred. rep. of G).

{irreducible representations}

$\cap$

MF = multiplicity-free (definition)

$$\pi\colon \underset{\text{group}}{G} \to GL_{\mathbb{C}}(\mathcal{H})$$

Def. (naive) $(\pi, \mathcal{H})$ is MF
multiplicity-free
if $\dim \operatorname{Hom}_G(\tau, \pi) \leq 1$ ($\forall\tau$: irred. rep. of G).

{irreducible representations}
∩
{MF representations}

Taylor series (MF rep.)

Ex.2 (Taylor expansion, Laurent expansion)

$$f(z_1, \ldots, z_n) = \sum_{\alpha=(\alpha_1,\ldots,\alpha_n)} a_\alpha z_1^{\alpha_1} \ldots z_n^{\alpha_n}$$

Taylor series (MF rep.)

Ex.2 (Taylor expansion, Laurent expansion)

$$f(z_1, \dots, z_n) = \sum_{\alpha=(\alpha_1,\dots,\alpha_n)} a_\alpha z_1^{\alpha_1} \dots z_n^{\alpha_n}$$

Point (too obvious ♣ ref.)

${}^{\exists 1} a_\alpha \in \mathbb{C}$ for each α

Taylor series (MF rep.)

Ex.2 (Taylor expansion, Laurent expansion)

$$f(z_1, \ldots, z_n) = \sum_{\alpha=(\alpha_1,\ldots,\alpha_n)} a_\alpha z_1^{\alpha_1} \ldots z_n^{\alpha_n}$$

Point (too obvious ♣ ref.)

$^{\exists 1} a_\alpha \in \mathbb{C}$ for each α

$$\Uparrow$$

$\dim \mathrm{Hom}_{(\mathbb{C}^\times)^n}(\tau, \mathcal{O}(\{0\}) \leq 1$

($^\forall \tau = \tau_\alpha$: irred. rep. of $(\mathbb{C}^\times)^n$)

i.e. $(\mathbb{C}^\times)^n \curvearrowright \mathcal{O}(\{0\})$ is MF

Fourier series

<u>Ex.3</u> (Fourier series expansion)

$$L^2(S^1) \simeq \sum_{n\in\mathbb{Z}}^{\oplus} \mathbb{C}e^{inx}$$

$$f(x) = \sum_{n\in\mathbb{Z}} a_n e^{inx}$$

Fourier series

Ex.3 (Fourier series expansion)

$$L^2(S^1) \simeq \sum_{n\in\mathbb{Z}}^{\oplus} \mathbb{C}e^{inx}$$

$$f(x) = \sum_{n\in\mathbb{Z}} a_n e^{inx}$$

Translation ($\Longrightarrow$ rep. of the group S^1)

$$f(\cdot) \mapsto f(\cdot - c) \quad (c \in S^1 \simeq \mathbb{R}/2\pi\mathbb{Z})$$

Fourier series

Ex.3 (Fourier series expansion)

$$L^2(S^1) \simeq \sum_{n\in\mathbb{Z}}{}^{\oplus} \mathbb{C}e^{inx} \overset{\exists 1}{\hookleftarrow} e^{inx}$$

$$f(x) = \sum_{n\in\mathbb{Z}} a_n e^{inx}$$

Translation ($\Longrightarrow$ rep. of the group S^1)

$$f(\cdot) \mapsto f(\cdot - c) \quad (c \in S^1 \simeq \mathbb{R}/2\pi\mathbb{Z})$$

$S^1 \curvearrowright L^2(S^1)$ is MF (multiplicity-free)

Peter–Weyl (MF rep.)

Ex.4 (Peter–Weyl)

G: compact (Lie) group

$$L^2(G) \simeq \sum_{\tau \in \widehat{G}}^{\oplus} \underline{\tau \boxtimes \tau^*}$$

Peter–Weyl (MF rep.)

Ex.4 (Peter–Weyl)

G: compact (Lie) group

$$\underline{L^2(G)} \simeq \sum_{\tau \in \widehat{G}}^{\oplus} \underline{\tau \boxtimes \tau^*}$$

irred. rep. of $G \times G$

$\Uparrow$

Translation $f(\cdot) \mapsto f(g_1^{-1} \cdot g_2)$

Peter–Weyl (MF rep.)

Ex.4 (Peter–Weyl)

G: compact (Lie) group

$$\underline{L^2(G)} \simeq \sum_{\tau \in \widehat{G}}^{\oplus} \underline{\tau \boxtimes \tau^*}$$

irred. rep. of $G \times G$

$\Uparrow$

Translation $\quad f(\cdot) \mapsto f(g_1^{-1} \cdot g_2)$

$\Longrightarrow$ $G \times G \curvearrowright L^2(G)$ is MF

Spherical harmonics

M: compact Riemannian manifold
Δ_M: Laplace–Beltrami operator on M
$\Rightarrow$

Spherical harmonics

M: compact Riemannian manifold

Δ_M: Laplace–Beltrami operator on M

$\Rightarrow$

$$L^2(M) = \sum_{\lambda:\ \text{countable}}^{\oplus} \operatorname{Ker}(\Delta_M - \lambda)$$

(direct sum of eigenspaces)

Spherical harmonics

M: compact Riemannian manifold

Δ_M: Laplace–Beltrami operator on M

$\Rightarrow$

$$L^2(M) = \sum_{\lambda:\ \text{countable}}^{\oplus} \operatorname{Ker}(\Delta_M - \lambda)$$

(direct sum of eigenspaces)

Ex.5 $M = S^{n-1}$ (unit sphere)

Spherical harmonics

M: compact Riemannian manifold

Δ_M: Laplace–Beltrami operator on M

$$\Rightarrow$$

$$L^2(M) = \sum_{\lambda:\text{ countable}}^{\oplus} \operatorname{Ker}(\Delta_M - \lambda)$$

(direct sum of eigenspaces)

Ex.5 $M = S^{n-1}$ (unit sphere)

$\Delta_{S^{n-1}} f = \lambda f$, $f \neq 0$

$$\Rightarrow$$

$\lambda = -l(l+n-2)$ for some $l \in \mathbb{N}$.

Fourier series $\Longrightarrow$ spherical harmonics

$O(n) \curvearrowright S^{n-1} \subset \mathbb{R}^n$

$\Delta_{S^{n-1}}$: Laplacian on S^{n-1}

Fourier series $\Longrightarrow$ spherical harmonics

$O(n) \curvearrowright S^{n-1} \subset \mathbb{R}^n$

$\Delta_{S^{n-1}}$: Laplacian on S^{n-1}

<u>Ex.6</u> (Expansion into spherical harmonics)

$$\mathcal{H}_l := \left\{ f \in C^\infty(S^{n-1}) : \Delta_{S^{n-1}} f = -l(l+n-2)f \right\}$$

Fourier series $\Longrightarrow$ spherical harmonics

$O(n) \curvearrowright S^{n-1} \subset \mathbb{R}^n$

$\Delta_{S^{n-1}}$: Laplacian on S^{n-1}

Ex.6 (Expansion into spherical harmonics)

$$\mathcal{H}_l := \left\{ f \in C^\infty(S^{n-1}) : \Delta_{S^{n-1}} f = -l(l+n-2)f \right\}$$

$$L^2(S^{n-1}) \simeq \sum_{l=0}^{\infty}{}^{\oplus} \mathcal{H}_l$$

Fourier series $\Longrightarrow$ spherical harmonics

$O(n) \curvearrowright S^{n-1} \subset \mathbb{R}^n$

$\Delta_{S^{n-1}}$: Laplacian on S^{n-1}

Ex.6 (Expansion into spherical harmonics)

$$\mathcal{H}_l := \left\{ f \in C^\infty(S^{n-1}) : \Delta_{S^{n-1}} f = -l(l+n-2)f \right\}$$

$$L^2(S^{n-1}) \simeq \sum_{l=0}^{\infty}{}^{\oplus} \mathcal{H}_l$$

$O(n)$ irred.

Fourier series $\Longrightarrow$ spherical harmonics

$O(n) \curvearrowright S^{n-1} \subset \mathbb{R}^n$

$\Delta_{S^{n-1}}$: Laplacian on S^{n-1}

Ex.6 (Expansion into spherical harmonics)

$$\mathcal{H}_l := \left\{ f \in C^\infty(S^{n-1}) : \Delta_{S^{n-1}} f = -l(l+n-2)f \right\}$$

$$\begin{tikzcd} L^2(S^{n-1}) & \simeq \displaystyle\sum_{l=0}^{\infty}{}^{\oplus} & \mathcal{H}_l \\ & O(n) \arrow[ul, bend left] \arrow[ur, bend right, "\text{irred.}"'] & \end{tikzcd}$$

$\Downarrow$

$O(n) \curvearrowright L^2(S^{n-1})$ is MF

Fourier series $\Longrightarrow$ spherical harmonics

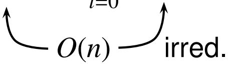
$O(n) \curvearrowright S^{n-1} \subset \mathbb{R}^n$

$\Delta_{S^{n-1}}$: Laplacian on S^{n-1}

<u>Ex.6</u> (Expansion into spherical harmonics)

$$\mathcal{H}_l := \left\{ f \in C^\infty(S^{n-1}) : \Delta_{S^{n-1}} f = -l(l+n-2)f \right\}$$

$$L^2(S^{n-1}) \simeq \sum_{l=0}^{\infty}{}^{\oplus} \mathcal{H}_l$$



$O(n)$ irred.

$$\Downarrow$$

$O(n) \curvearrowright L^2(S^{n-1})$ is <u>MF</u>

$G \curvearrowright L^2(G/K)$ is MF (E. Cartan ('29)–I. M. Gelfand ('50))

⊗-product rep.

$$\boxed{SL_2(\mathbb{C}) \underset{\text{irred.}}{\overset{\pi_k}{\curvearrowright}} S^k(\mathbb{C}^2) \quad (k = 0, 1, 2, \dots)}$$

$\otimes$-product rep.

$$SL_2(\mathbb{C}) \underset{\text{irred.}}{\overset{\pi_k}{\curvearrowright}} S^k(\mathbb{C}^2) \quad (k = 0, 1, 2, \dots)$$

Ex.7 (Clebsch–Gordan)

$$\pi_k \otimes \pi_l \simeq \pi_{k+l} \oplus \pi_{k+l-2} \oplus \cdots \oplus \pi_{|k-l|}$$

⊗-product rep.

$$SL_2(\mathbb{C}) \underset{\text{irred.}}{\overset{\pi_k}{\curvearrowright}} S^k(\mathbb{C}^2) \quad (k = 0, 1, 2, \dots)$$

<u>Ex.7</u> (Clebsch–Gordan)

$$\underset{\text{MF}}{\underline{\pi_k \otimes \pi_l}} \simeq \pi_{k+l} \oplus \pi_{k+l-2} \oplus \cdots \oplus \pi_{|k-l|}$$

Notation (finite dimensional reps)

$G = GL_n(\mathbb{C})$

Hightest weight

$$\lambda = (\lambda_1 \ldots, \lambda_n) \in \mathbb{Z}^n, \quad \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$$

$$\Downarrow$$

Irreducible rep.

$$\pi_\lambda^{GL_n} \equiv \pi_\lambda$$

Ex.8

$$\lambda = (k, 0, \ldots, 0) \quad \leftrightarrow \quad GL_n(\mathbb{C}) \curvearrowright S^k(\mathbb{C}^n)$$

$$\lambda = (\underbrace{1, \ldots, 1}_{k}, 0, \ldots, 0) \quad \leftrightarrow \quad GL_n(\mathbb{C}) \curvearrowright \Lambda^k(\mathbb{C}^n)$$

$\otimes$-product rep. (GL_n-case)

Ex.9 (Pieri's rule)

$$\pi_{(\lambda_1,\ldots,\lambda_n)} \otimes \pi_{(k,0,\ldots,0)} \simeq \bigoplus_{\substack{\mu_1\geq\lambda_1\geq\cdots\geq\mu_n\geq\lambda_n \\ \sum(\mu_i-\lambda_i)=k}} \pi_{(\mu_1,\ldots,\mu_n)}$$

$\otimes$-product rep. (GL_n-case)

Ex.9 (Pieri's rule)

$$\pi_{(\lambda_1,\dots,\lambda_n)} \otimes \pi_{(k,0,\dots,0)} \simeq \bigoplus_{\substack{\mu_1 \geq \lambda_1 \geq \cdots \geq \mu_n \geq \lambda_n \\ \sum(\mu_i - \lambda_i) = k}} \pi_{(\mu_1,\dots,\mu_n)}$$

$\Uparrow$

MF as a GL_n-module.

$\otimes$-product rep. (GL_n-case)

Ex.9 (Pieri's rule)

$$\pi_{(\lambda_1,\dots,\lambda_n)} \otimes \pi_{(k,0,\dots,0)} \simeq \bigoplus_{\substack{\mu_1 \geq \lambda_1 \geq \cdots \geq \mu_n \geq \lambda_n \\ \sum(\mu_i - \lambda_i) = k}} \pi_{(\mu_1,\dots,\mu_n)}$$

$\Uparrow$

MF as a GL_n-module.

Ex.10 (counterexample)

$\pi_{(2,1,0)} \otimes \pi_{(2,1,0)}$ is NOT MF as a $GL_3(\mathbb{C})$-module.

$\otimes$-product for GL_3

$\pi_{\{2,1,0\}} \simeq$ Adjoint reprsentation

(up to central character)

$\otimes$-product for GL_3

$\pi_{\{2,1,0\}} \simeq$ Adjoint reprsentation
(up to central character)

$$\pi_{(2,1,0)} \otimes \pi_{(2,1,0)}$$
$$\simeq$$

$\otimes$-product for GL_3

$\pi_{\{2,1,0\}} \simeq$ Adjoint reprsentation
(up to central character)

$$\begin{aligned} &\pi_{(2,1,0)} \otimes \pi_{(2,1,0)} \\ &\simeq \pi_{(4,2,0)} \oplus \pi_{(4,1,1)} \oplus \pi_{(2,2,0)} \\ &\quad \oplus \underline{2}\pi_{(3,2,1)} \oplus \pi_{(2,2,2)} \end{aligned}$$

Varna Lectures on visible actions and MF representations

1. EXAMPLES OF MULTIPLICITY-FREE REPRESENTATIONS
2. INFINITE DIMENSIONAL EXAMPLES
3. PROPAGATION THEOREM OF MULTIPLICITY-FREE REPRESENTATIONS
4. VISIBLE ACTIONS ON COMPLEX MANIFOLDS
5. APPLICATIONS OF VISIBLE ACTIONS TO MULTIPLICITY-FREE REPRESENTATIONS

Lecture 2. Various examples of MF representations

MF = multiplicity-free

Plan of Today

- finite-dimensional examples (continued)
- infinite-dimensional examples

Notation (finite dimensional reps)

$G = GL_n(\mathbb{C})$

Hightest weight

$$\lambda = (\lambda_1 \ldots, \lambda_n) \in \mathbb{Z}^n, \quad \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$$

$$\Downarrow$$

Irreducible rep.

$$\pi_\lambda^{GL_n} \equiv \pi_\lambda$$

<u>Ex.8</u>

$$\lambda = (k, 0, \ldots, 0) \quad \leftrightarrow \quad GL_n(\mathbb{C}) \curvearrowright S^k(\mathbb{C}^n)$$

$$\lambda = (\underbrace{1, \ldots, 1}_{k}, 0, \ldots, 0) \quad \leftrightarrow \quad GL_n(\mathbb{C}) \curvearrowright \Lambda^k(\mathbb{C}^n)$$

When is $\pi_\lambda \otimes \pi_\nu$ MF?

$G = GL_n$

$\lambda = (\lambda_1, \ldots, \lambda_n), \quad \nu = (\nu_1, \ldots, \nu_n)$

When is $\pi_\lambda \otimes \pi_\nu$ MF?

$G = GL_n$
$\lambda = (\lambda_1, \dots, \lambda_n), \quad \nu = (\nu_1, \dots, \nu_n)$

(Necessary Condition)
If $\pi_\lambda \otimes \pi_\nu$ is MF

$\Rightarrow$?

When is $\pi_\lambda \otimes \pi_\nu$ MF?

$G = GL_n$

$\lambda = (\lambda_1, \ldots, \lambda_n), \quad \nu = (\nu_1, \ldots, \nu_n)$

(Necessary Condition)

If $\pi_\lambda \otimes \pi_\nu$ is MF

then at least one of λ or ν is of the form

$$(\underbrace{a, \ldots, a}_{p}, \underbrace{b, \ldots, b}_{n-p}),$$

for some $a \geq b$ and some p

$\otimes$-product rep. (continued)

$$\lambda = (\underbrace{a, \cdots, a}_{p}, \underbrace{b, \cdots, b}_{q}) \in \mathbb{Z}^n, \ a \geq b, \ p + q = n$$

Ex.11 (Stembridge 2001)

$\pi_\lambda \otimes \pi_\nu$ is MF as a $GL_n(\mathbb{C})$-module

iff one of the following holds

1) $\min(a-b, p, q) = 1$ (and ν is any),

2) $\min(a-b, p, q) = 2$ and

★ $\boxed{\nu \text{ is of the form } \nu = (\underbrace{x \cdots x}_{n_1} \ \underbrace{y \cdots y}_{n_2} \ \underbrace{z \cdots z}_{n_3}) \ (x \geq y \geq z)}$,

3) $\min(a-b, p, q) \geq 3$, ★ &

$\min(x-y, \ y-z, n_1, n_2, n_3) = 1.$

$\otimes$-product rep. (continued)

$$\lambda = (\underbrace{a, \cdots, a}_{p}, \underbrace{b, \cdots, b}_{q}) \in \mathbb{Z}^n,\ a \geq b,\ p + q = n$$

Ex.11 (Stembridge 2001)

$\pi_\lambda \otimes \pi_\nu$ is MF as a $GL_n(\mathbb{C})$-module

iff one of the following holds

1) $\min(a-b, p, q) = 1$ (and ν is any),

2) $\min(a-b, p, q) = 2$ and

★ $\boxed{\nu \text{ is of the form } \nu = (\underbrace{x \cdots x}_{n_1}\ \underbrace{y \cdots y}_{n_2}\ \underbrace{z \cdots z}_{n_3})\ (x \geq y \geq z)}$,

3) $\min(a-b, p, q) \geq 3$, ★ &

$\min(x-y,\ y-z, n_1, n_2, n_3) = 1.$

Geometric interpretation (K–, 2004)

$\otimes$-product rep. (continued)

$$\lambda = (\underbrace{a, \cdots, a}_{p}, \underbrace{b, \cdots, b}_{q}) \in \mathbb{Z}^n,\ a \geq b,\ p + q = n$$

Ex.11 (Stembridge 2001)

$\pi_\lambda \otimes \pi_\nu$ is MF as a $GL_n(\mathbb{C})$-module

iff one of the following holds

1) $\min(a - b, p, q) = 1$ (and ν is any),

2) $\min(a - b, p, q) = 2$ and

$\star$ $\boxed{\nu \text{ is of the form } \nu = (\underbrace{x \cdots x}_{n_1}\ \underbrace{y \cdots y}_{n_2}\ \underbrace{z \cdots z}_{n_3})\ (x \geq y \geq z)}$,

3) $\min(a - b, p, q) \geq 3$, $\star$ &

$\min(x - y,\ y - z, n_1, n_2, n_3) = 1.$

Geometric interpretation (K–, 2004) $\cdots$ visible action

Restriction $(GL_n \downarrow G_{n_1} \times GL_{n_2} \times GL_{n_3})$

Ex.12 $(GL_n \downarrow (GL_p \times GL_q)) \quad n = p + q$

$\pi^{GL_n}_{(\underbrace{x,\dots,x}_{n_1},\underbrace{y,\dots,y}_{n_2},\underbrace{z,\dots,z}_{n_3})}|_{GL_p \times GL_q}$ is MF

if $\min(p,q) \leq 2$ or

if $\min(n_1, n_2, n_3, x-y, y-z) \leq 1$

(Kostant $n_3 = 0$; Krattenthaler 1998)

Restriction $(GL_n \downarrow G_{n_1} \times GL_{n_2} \times GL_{n_3})$

<u>Ex.12</u> $(GL_n \downarrow (GL_p \times GL_q)) \quad n = p + q$

$$\pi^{GL_n}_{(\underbrace{x,\dots,x}_{n_1},\underbrace{y,\dots,y}_{n_2},\underbrace{z,\dots,z}_{n_3})}|_{GL_p \times GL_q} \text{ is } \underline{\text{MF}}$$

if $\min(p, q) \leq 2$ or

if $\min(n_1, n_2, n_3, x - y, y - z) \leq 1$

(Kostant $n_3 = 0$; Krattenthaler 1998)

<u>Ex.13</u> $(GL_n \downarrow (GL_{n_1} \times GL_{n_2} \times GL_{n_3}))$ ([5])

$n = n_1 + n_2 + n_3$

$$\pi^{GL_n}_{(\underbrace{a,\dots,a}_{p},\underbrace{b,\dots,b}_{q})}|_{GL_{n_1} \times GL_{n_2} \times GL_{n_3}} \text{ is } \underline{\text{MF}}$$

if $\min(n_1, n_2, n_3) \leq 1$ or

if $\min(p, q, a - b) \leq 2$

“Triunity”

MF results for

Ex.11 $\pi_\lambda \otimes \pi_\nu$

Ex.12 $GL_n \downarrow GL_p \times GL_q \ (p + q = n)$

Ex.13 $GL_n \downarrow GL_{n_1} \times GL_{n_2} \times GL_{n_3} \ (n_1 + n_2 + n_3 = n)$

“Triunity”

MF results for

Ex.11 $\pi_\lambda \otimes \pi_\nu$

Ex.12 $GL_n \downarrow GL_p \times GL_q \ (p+q=n)$

Ex.13 $GL_n \downarrow GL_{n_1} \times GL_{n_2} \times GL_{n_3} \ (n_1+n_2+n_3=n)$

can be proved by combinatorial methods (e.g. Littlewood–Richardson rule) but
will be explained by “triunity” of visible actions on flag varieties:

$$\begin{cases} G \curvearrowright (G \times G)/(L \times H) & \text{(diagonal action)} \\ L \curvearrowright G/H \\ H \curvearrowright G/L \end{cases}$$

for $H \subset G \supset L$.

Restriction ($GL_n \downarrow GL_{n-1}$)

Finite dimensional rep.

Ex.14 ($GL_n \downarrow GL_{n-1}$)

$$\pi^{GL_n}_{(\lambda_1,\dots,\lambda_n)}|_{GL_{n-1}} \simeq \bigoplus_{\lambda_1 \geq \mu_1 \geq \cdots \geq \mu_{n-1} \geq \lambda_n} \pi^{GL_{n-1}}_{(\mu_1,\dots,\mu_{n-1})}$$

$\Longrightarrow$ restrictions is MF as a $GL_{n-1}(\mathbb{C})$-module

Restriction ($GL_n \downarrow GL_{n-1}$)

Finite dimensional rep.

Ex.14 ($GL_n \downarrow GL_{n-1}$)

$$\pi^{GL_n}_{(\lambda_1,\dots,\lambda_n)}|_{GL_{n-1}} \simeq \bigoplus_{\lambda_1 \geq \mu_1 \geq \cdots \geq \mu_{n-1} \geq \lambda_n} \pi^{GL_{n-1}}_{(\mu_1,\dots,\mu_{n-1})}$$

$\Longrightarrow$ restrictions is MF as a $GL_{n-1}(\mathbb{C})$-module

$$GL_n \underset{\substack{\curvearrowright \\ \text{MF}}}{\supset} GL_{n-1} \underset{\substack{\curvearrowright \\ \text{MF}}}{\supset} GL_{n-2} \underset{\substack{\curvearrowright \\ \text{MF}}}{\supset} \cdots \underset{\substack{\curvearrowright \\ \text{MF}}}{\supset} GL_1$$

$\Longrightarrow$ Gelfand–Tsetlin basis

Restriction ($GL_n \downarrow GL_{n-1}$)

Finite dimensional rep.

Ex.14 ($GL_n \downarrow GL_{n-1}$)

$$\pi^{GL_n}_{(\lambda_1,\ldots,\lambda_n)}|_{GL_{n-1}} \simeq \bigoplus_{\lambda_1 \geq \mu_1 \geq \cdots \geq \mu_{n-1} \geq \lambda_n} \pi^{GL_{n-1}}_{(\mu_1,\ldots,\mu_{n-1})}$$

$\Longrightarrow$ restrictions is MF as a $GL_{n-1}(\mathbb{C})$-module

Restriction ($GL_n \downarrow GL_{n-1}$)

Finite dimensional rep.

Ex.14 ($GL_n \downarrow GL_{n-1}$)

$$\pi^{GL_n}_{(\lambda_1,\dots,\lambda_n)}|_{GL_{n-1}} \simeq \bigoplus_{\lambda_1 \geq \mu_1 \geq \cdots \geq \mu_{n-1} \geq \lambda_n} \pi^{GL_{n-1}}_{(\mu_1,\dots,\mu_{n-1})}$$

$\Longrightarrow$ restrictions is MF as a $GL_{n-1}(\mathbb{C})$-module

Infinite dimensional version

Restriction ($GL_n \downarrow GL_{n-1}$)

Finite dimensional rep.

Ex.14 ($GL_n \downarrow GL_{n-1}$)

$$\pi^{GL_n}_{(\lambda_1,\ldots,\lambda_n)}|_{GL_{n-1}} \simeq \bigoplus_{\lambda_1\geq\mu_1\geq\cdots\geq\mu_{n-1}\geq\lambda_n} \pi^{GL_{n-1}}_{(\mu_1,\ldots,\mu_{n-1})}$$

$\Longrightarrow$ restrictions is MF as a $GL_{n-1}(\mathbb{C})$-module

Infinite dimensional version

Ex.15 ♣ ref. ($U(p,q) \downarrow U(p-1,q)$)

${}^{\forall}\pi$: irred. unitary rep. of $U(p,q)$ with higest weight

$\Longrightarrow$ restriction $\pi|_{U(p-1,q)}$ is MF as a $U(p-1,q)$-module

GL–GL duality

$$N = mn$$

Ex.16 (GL–GL duality à la R. Howe)

$\Longrightarrow GL_m \times GL_n \curvearrowright S(\mathbb{C}^N) \simeq S(M(m, n; \mathbb{C}))$

This rep. is MF

GL–GL **duality**

$$N = mn$$

<u>Ex.16</u> (GL–GL duality à la R. Howe)

$\Longrightarrow GL_m \times GL_n \curvearrowright S(\mathbb{C}^N) \simeq S(M(m,n;\mathbb{C}))$

This rep. is <u>MF</u>

⇓ generalization 1

Hidden symmetry $\Longleftrightarrow$ Broken symmetry

GL–GL duality

$$N = mn$$

Ex.16 (GL–GL duality à la R. Howe)

$\Longrightarrow GL_m \times GL_n \curvearrowright S(\mathbb{C}^N) \simeq S(M(m,n;\mathbb{C}))$

This rep. is MF

⇓ generalization 1

Hidden symmetry ⟺ Broken symmetry

Ex.17 (Progress in Math. 2008)

Branching law of holomorphic discrete series rep. with respect to symmectric pair

Hua–Kostant–Schmid, K–

finite dim
compact subgp

∞ dim
non-compact subgp

$$\underline{U(m,n) \downarrow U(m) \times U(n)}$$

$$\underline{U(m,n) \downarrow U(m_1,n_1) \times U(m_2,n_2)}$$

GL–GL **duality**

$N = mn$

<u>Ex.16</u> (GL–GL duality à la R. Howe)

$\Longrightarrow GL_m \times GL_n \curvearrowright S(\mathbb{C}^N) \simeq S(M(m,n;\mathbb{C}))$

This rep. is <u>MF</u>

$\Downarrow$ generalization 2

MF space: $G \curvearrowright X \Longrightarrow G \curvearrowright \mathcal{O}(X)$

function on X

GL–GL **duality**

$$N = mn$$

Ex.16 (GL–GL duality à la R. Howe)

$\Longrightarrow GL_m \times GL_n \curvearrowright S(\mathbb{C}^N) \simeq S(M(m,n;\mathbb{C}))$

This rep. is MF

⇓ generalization 2

MF space: $G \curvearrowright X \Longrightarrow G \curvearrowright \mathcal{O}(X)$

Ex.18 (Kac's MF space '80)

$S(\mathbb{C}^N)$ is still MF as a $GL_{m-1} \times GL_n$ module

GL–GL duality

$$N = mn$$

Ex.16 (GL–GL duality à la R. Howe)

$\Longrightarrow GL_m \times GL_n \curvearrowright S(\mathbb{C}^N) \simeq S(M(m,n;\mathbb{C}))$

This rep. is MF

$\Downarrow$ generalization 2

MF space: $G \curvearrowright X \Longrightarrow G \curvearrowright \mathcal{O}(X)$

Ex.18 (Kac's MF space '80)

$S(\mathbb{C}^N)$ is still MF as a $GL_{m-1} \times GL_n$ module

Ex.19 (counterexample)

$S(\mathbb{C}^N)$ is no more MF as a $GL_{m-1} \times GL_{n-1}$ module

MF for unitary rep (definition)

MF for unitary rep (definition)

Observation

$n \leq 1 \Longleftrightarrow \mathrm{End}(\mathbb{C}^n)$ is commutative.

MF for unitary rep (definition)

Observation

$n \leq 1 \iff \mathrm{End}(\mathbb{C}^n)$ is commutative.

$(\pi, \mathcal{H})$: unitary rep. of G $\Downarrow$ (Schur's lemma)

Def.

$(\pi, \mathcal{H})$ is MF if $\mathrm{End}_G(\mathcal{H})$ is commutative.

Def. $\mathrm{End}_{\mathbb{C}}(\mathcal{H}) = \{T : \mathcal{H} \to \mathcal{H}$ continuous linear maps$\}$

$$\cup$$

$$\mathrm{End}_G(\mathcal{H}) := \{T \in \mathrm{End}_{\mathbb{C}}(\mathcal{H}) : T \circ \pi(g) = \pi(g) \circ T, \ {}^{\forall} g \in G\}$$

MF for unitary rep (definition)

Observation

$n \leq 1 \iff \mathrm{End}(\mathbb{C}^n)$ is commutative.

$(\pi, \mathcal{H})$: unitary rep. of G $\Downarrow$ (Schur's lemma)

Def.

$(\pi, \mathcal{H})$ is MF if $\mathrm{End}_G(\mathcal{H})$ is commutative.

Recall:

Def. (naive) $(\pi, \mathcal{H})$ is MF

multiplicity-free

if $\dim \mathrm{Hom}_G(\tau, \pi) \leq 1$ ($\forall \tau$: irred. rep. of G).

MF for unitary rep (definition)

Observation
$n \leq 1 \Longleftrightarrow \mathrm{End}(\mathbb{C}^n)$ is commutative.

$(\pi, \mathcal{H})$: unitary rep. of G $\Downarrow$ (Schur's lemma)

Def.
$(\pi, \mathcal{H})$ is MF if $\mathrm{End}_G(\mathcal{H})$ is commutative.

$\Downarrow$

Prop. The irreducible decomp. of π is unique, and $m_\pi(\tau) \leq 1$ for almost every τ with respect to $d\mu$.

In particular, multiplicity for any discrete spectrum ≤ 1

$$\pi \simeq \int_{\widehat{G}} m_\pi(\tau)\, \tau \, d\mu(\tau) \quad \text{(direct integral)}$$

MF for unitary rep (definition)

Observation

$n \leq 1 \iff \mathrm{End}(\mathbb{C}^n)$ is commutative.

$(\pi, \mathcal{H})$: unitary rep. of G $\Downarrow$ (Schur's lemma)

Def.

$(\pi, \mathcal{H})$ is MF if $\mathrm{End}_G(\mathcal{H})$ is commutative.

$(\pi, \mathcal{U})$: continuous rep.

Def. We say $(\pi, \mathcal{U})$ is (unitarily) MF
if, for any unitary rep. $(\varpi, \mathcal{H})$ s.t.
there exists an injective continuous G-map $\mathcal{H} \hookrightarrow \mathcal{U}$,
$(\varpi, \mathcal{H})$ is MF.

Fourier transform (MF rep.)

<u>Ex.21</u> (Fourier transform)

$$L^2(\mathbb{R}) \simeq \int_{\mathbb{R}}^{\oplus} \mathbb{C}e^{i\zeta x} d\zeta$$

(direct integral of Hilbert spaces)

$$f(x) = \int_{\mathbb{R}} f(\zeta) e^{i\zeta x} d\zeta$$

Regular rep. of $\mathbb{R}$ on $L^2(\mathbb{R})$ by $f(*) \to f(* - c)$

Fourier transform (MF rep.)

Ex.21 (Fourier transform)

$$L^2(\mathbb{R}) \simeq \int_{\mathbb{R}}^{\oplus} \mathbb{C}e^{i\zeta x} d\zeta$$

(direct integral of Hilbert spaces)

$$f(x) = \int_{\mathbb{R}} f(\zeta) e^{i\zeta x} d\zeta$$

Regular rep. of $\mathbb{R}$ on $L^2(\mathbb{R})$ by $f(*) \to f(* - c)$

$\mathrm{End}_{\mathbb{R}}(L^2(\mathbb{R})) \simeq L^\infty(\mathbb{R})$ (ring of multiplier operators)

Fourier transform (MF rep.)

Ex.21 (Fourier transform)

$$L^2(\mathbb{R}) \simeq \int_{\mathbb{R}}^{\oplus} \mathbb{C}e^{i\zeta x} d\zeta$$

(direct integral of Hilbert spaces)

$$f(x) = \int_{\mathbb{R}} f(\zeta) e^{i\zeta x} d\zeta$$

Regular rep. of $\mathbb{R}$ on $L^2(\mathbb{R})$ by $f(*) \to f(* - c)$

$\mathrm{End}_{\mathbb{R}}(L^2(\mathbb{R})) \simeq L^\infty(\mathbb{R})$ (ring of multiplier operators)

$\Longrightarrow$ unitary rep. $\mathbb{R} \curvearrowright L^2(\mathbb{R})$ is MF

continuous spectrum

Fourier transform (MF rep.)

Ex.21 (Fourier transform)

$$L^2(\mathbb{R}) \simeq \int_{\mathbb{R}}^{\oplus} \mathbb{C}e^{i\zeta x} d\zeta$$

(direct integral of Hilbert spaces)

$$f(x) = \int_{\mathbb{R}} f(\zeta) e^{i\zeta x} d\zeta$$

Regular rep. of $\mathbb{R}$ on $L^2(\mathbb{R})$ by $f(*) \to f(* - c)$

$\mathrm{End}_{\mathbb{R}}(L^2(\mathbb{R})) \simeq L^\infty(\mathbb{R})$ (ring of multiplier operators)

continuous rep. $\mathbb{R} \curvearrowright \mathcal{S}'(\mathbb{R})$ is also MF

Varna Lectures on visible actions and MF representations

I. EXAMPLES OF MULTIPLICITY-FREE REPRESENTATIONS
II. INFINITE DIMENSIONAL EXAMPLES
III. PROPAGATION THEOREM OF MULTIPLICITY-FREE REPRESENTATIONS
IV. VISIBLE ACTIONS ON COMPLEX MANIFOLDS
V. APPLICATIONS OF VISIBLE ACTIONS TO MULTIPLICITY-FREE REPRESENTATIONS

Varna Lectures on visible actions and MF representations

I. EXAMPLES OF MULTIPLICITY-FREE REPRESENTATIONS
II. INFINITE DIMENSIONAL EXAMPLES
III. PROPAGATION THEOREM OF MULTIPLICITY-FREE REPRESENTATIONS
IV. VISIBLE ACTIONS ON COMPLEX MANIFOLDS
V. APPLICATIONS OF VISIBLE ACTIONS TO MULTIPLICITY-FREE REPRESENTATIONS

MF = multiplicity-free (definition)

$$\pi\colon \underset{\text{group}}{G} \to GL_{\mathbb{C}}(\mathcal{H})$$

Def. (naive) $(\pi, \mathcal{H})$ is $\underset{\text{multiplicity-free}}{\underline{\text{MF}}}$

if $\dim \operatorname{Hom}_G(\tau, \pi) \leq 1$ ($\forall \tau$: irred. rep. of G).

MF for unitary rep (definition)

Observation

$n \leq 1 \Longleftrightarrow \mathrm{End}(\mathbb{C}^n)$ is commutative.

$(\pi, \mathcal{H})$: unitary rep. of G $\Downarrow$ (Schur's lemma)

Def.

$(\pi, \mathcal{H})$ is MF if $\mathrm{End}_G(\mathcal{H})$ is commutative.

Def. $\mathrm{End}_{\mathbb{C}}(\mathcal{H}) = \{T : \mathcal{H} \to \mathcal{H}$ continuous linear maps$\}$

$$\cup$$

$$\mathrm{End}_G(\mathcal{H}) := \{T \in \mathrm{End}_{\mathbb{C}}(\mathcal{H}) : T \circ \pi(g) = \pi(g) \circ T,\ {}^{\forall} g \in G\}$$

Fourier transform (MF rep.)

Ex.21 (Fourier transform)

$$L^2(\mathbb{R}) \simeq \int_{\mathbb{R}}^{\oplus} \mathbb{C} e^{i\zeta x} d\zeta$$

(direct integral of Hilbert spaces)

$$f(x) = \int_{\mathbb{R}} f(\zeta) e^{i\zeta x} d\zeta$$

Regular rep. of $\mathbb{R}$ on $L^2(\mathbb{R})$ by $f(*) \to f(* - c)$

$\mathrm{End}_{\mathbb{R}}(L^2(\mathbb{R})) \simeq L^\infty(\mathbb{R})$ (ring of multiplier operators)

$\Longrightarrow$ unitary rep. $\mathbb{R} \curvearrowright L^2(\mathbb{R})$ is MF

continuous spectrum

Plancherel formula for Riemannian symm. space G/K

Ex.22 (Harish-Chandra, Helgason) $G/K = SL(n,\mathbb{R})/SO(n)$

$$L^2(G/K) \simeq \underbrace{\int^{\oplus}_{\sum \lambda_i=0,\ \lambda_1\geq\cdots\geq\lambda_n} \mathcal{H}_\lambda \quad \underbrace{d\lambda}_{\text{cont. spec.}}}_{\text{MF}}$$

$\mathcal{H}_\lambda$: ∞-dim, irred. rep. of G

$\Longrightarrow$ The regular representation of G on $L^2(G/K)$ is MF

Plancherel formula for Riemannian symm. space G/K

Ex.22 (Harish-Chandra, Helgason) $G/K = SL(n,\mathbb{R})/SO(n)$

$$L^2(G/K) \simeq \underbrace{\int^{\oplus}_{\sum \lambda_i=0,\ \lambda_1\geq\cdots\geq\lambda_n} \mathcal{H}_\lambda \quad \underbrace{d\lambda}_{\text{cont. spec.}}}_{\text{MF}}$$

$\mathcal{H}_\lambda$: ∞-dim, irred. rep. of G

$\Longrightarrow$ The regular representation of G on $L^2(G/K)$ is MF

$$\begin{aligned}\mathrm{End}_G(L^2(G/K)) &\simeq L^\infty((\mathbb{R}^n/\mathbb{R}))^{\mathcal{S}_n}\\ &\simeq L^\infty(\mathbb{R}^{n-1}/\mathcal{S}_n)\end{aligned}$$

(ring of multiplier operators)

Plancherel formula for Riemannian symm. space G/K

Ex.22 (Harish-Chandra, Helgason) $G/K = SL(n,\mathbb{R})/SO(n)$

$$L^2(G/K) \simeq \underbrace{\int^{\oplus}_{\sum \lambda_i=0,\ \lambda_1\geq\cdots\geq\lambda_n} \mathcal{H}_\lambda \quad \underbrace{d\lambda}_{\text{cont. spec.}}}_{\text{MF}}$$

$\mathcal{H}_\lambda$: ∞-dim, irred. rep. of G

$\Longrightarrow$ The regular representation of G on $L^2(G/K)$ is MF

MF still holds for vector bundle case of 'small' fibers,

$$\mathcal{V} := G \times_K \wedge^k(\mathbb{C}^n) \to G/K \qquad (0 \leq k \leq n),$$

associated to the $SO(n)$-representation on the exterior power $\Lambda^k(\mathbb{C}^n)$, but no other cases (Deitmar, K– 2005)

Plancherel formula for Riemannian symm. space G/K

Ex.22 (Harish-Chandra, Helgason) $G/K = SL(n,\mathbb{R})/SO(n)$

$$L^2(G/K) \simeq \underbrace{\int^{\oplus}_{\sum \lambda_i=0,\ \lambda_1\geq\cdots\geq\lambda_n} \mathcal{H}_\lambda \quad \underbrace{d\lambda}_{\text{cont. spec.}}}_{\text{MF}}$$

$\mathcal{H}_\lambda$: ∞-dim, irred. rep. of G

$\Longrightarrow$ The regular representation of G on $L^2(G/K)$ is MF

MF still holds under certain deformation of G-regular representation of $L^2(G/K)$

deformation coming from hidden symmetry.
E.g. Gelfand–Vershik's canonical rep of $SL_2(\mathbb{R})$.

Plancherel formula for Riemannian symm. space G/K

Ex.22 (Harish-Chandra, Helgason) $G/K = SL(n,\mathbb{R})/SO(n)$

$$L^2(G/K) \simeq \underset{\text{MF}}{\underline{\int^{\oplus}_{\sum \lambda_i=0,\ \lambda_1\geq\cdots\geq\lambda_n} \mathcal{H}_\lambda \quad \underset{\text{cont. spec.}}{\underline{d\lambda}}}}$$

$\mathcal{H}_\lambda$: ∞-dim, irred. rep. of G

$\Longrightarrow$ The regular representation of G on $L^2(G/K)$ is MF

Other real forms of $SL(n,\mathbb{C})/SO(n,\mathbb{C})$:

Ex.23 (T. Oshima, Delorme) $G/H = SL(n,\mathbb{R})/SO(p, n-p)$

Multiplicity of most cont. spec. in $L^2(G/H)$

$$= \frac{n!}{p!(n-p)!} > 1 \text{ if } 0 < p < n.$$

$\Longrightarrow$ NOT MF

Broken symmetry and hidden symmetry

$$\boxed{\quad G \subset \widetilde{G} \qquad \xrightarrow[\text{unitary rep}]{\pi} \qquad GL_{\mathbb{C}}(\mathcal{H}) \quad}$$

Broken symmetry and hidden symmetry

Hidden symmetry

$$\begin{tikzcd} G \arrow[rr, bend left] & \subset & \widetilde{G} \arrow[r, "\pi", "\text{unitary rep}"'] & GL_{\mathbb{C}}(\mathcal{H}) \end{tikzcd}$$

Broken symmetry and hidden symmetry

Hidden symmetry

$$\begin{tikzcd} G \arrow[r, phantom, "\subset"] \arrow[r, bend left] & \widetilde{G} \arrow[l, bend left] \arrow[r, "\pi", "\text{unitary rep}"'] & GL_{\mathbb{C}}(\mathcal{H}) \end{tikzcd}$$

Broken symmetry

Broken symmetry and hidden symmetry

Hidden symmetry

$$\begin{tikzcd} G \arrow[r, phantom, "\subset"] \arrow[r, bend left] & \widetilde{G} \arrow[l, bend left] \arrow[r, "\pi", "\text{unitary rep}"'] & GL_{\mathbb{C}}(\mathcal{H}) \end{tikzcd}$$

Broken symmetry

Branching law
= description of broken symmetry

Deformation of $G \curvearrowright L^2(G/K)$

$$G \overset{\pi}{\curvearrowright} L^2(G/K)$$

Deformation of $G \curvearrowright L^2(G/K)$

$$\begin{tikzcd}
G \arrow[r, phantom, "\overset{\pi}{\curvearrowright}"] \arrow[d, phantom, "\cap"] & L^2(G/K) \\
{}^{\exists}\widetilde{G} \arrow[ur, dashed, bend left, "{}^{\exists}\widetilde{\pi}"'] &
\end{tikzcd}$$

Deformation of $G \curvearrowright L^2(G/K)$

$$\begin{tikzcd}
G \arrow[r, phantom, "\overset{\pi}{\curvearrowright}"] \arrow[d, phantom, "\cap"] & L^2(G/K) \\
{}^{\exists}\widetilde{G} \arrow[ur, dashed, bend left, "{}^{\exists}\widetilde{\pi}"'] &
\end{tikzcd}$$

<u>Prop</u> For any classical reductive G, there exist $\widetilde{G}$ ($\supsetneqq G$) and an irreducible unitary rep $\widetilde{\pi}$ of $\widetilde{G}$ s.t. $\widetilde{\pi}|_G = \pi$

Deformation of $G \curvearrowright L^2(G/K)$

$$\begin{tikzcd}
G \arrow[r, phantom, "\overset{\pi}{\curvearrowright}"] \arrow[d, phantom, "\cap"] & L^2(G/K) \\
{}^{\exists}\widetilde{G} \arrow[ur, dashed, bend left, "{}^{\exists}\widetilde{\pi}"'] &
\end{tikzcd}$$

<u>Prop</u> For any classical reductive G, there exist $\widetilde{G}$ ($\supsetneqq G$) and an irreducible unitary rep $\widetilde{\pi}$ of $\widetilde{G}$ s.t. $\widetilde{\pi}|_G = \pi$

<u>E.g.</u> $G = GL(n, \mathbb{R}) \hookrightarrow \widetilde{G} = Sp(n, \mathbb{R})$

Deformation of $G \curvearrowright L^2(G/K)$

$$\begin{tikzcd}
G \arrow[r, phantom, "\overset{\pi}{\curvearrowright}"] \arrow[d, phantom, "\cap"] & L^2(G/K) \\
{}^{\exists}\widetilde{G} \arrow[ur, dashed, bend left, "{}^{\exists}\widetilde{\pi}"'] &
\end{tikzcd}$$

<u>Prop</u> For any classical reductive G, there exist $\widetilde{G}$ ($\supsetneqq G$) and an irreducible unitary rep $\widetilde{\pi}$ of $\widetilde{G}$ s.t. $\widetilde{\pi}|_G = \pi$

<u>E.g.</u> $G = GL(n, \mathbb{R}) \hookrightarrow \widetilde{G} = Sp(n, \mathbb{R})$

$\widetilde{\pi}$ lies in a continuous family $\{\widetilde{\pi}_\lambda\}$ of irred unitary reps of $\widetilde{G}$

$\Longrightarrow$ $\pi_\lambda := \widetilde{\pi}_\lambda|_G$ is a continuous family of (non-irreducible) representations of G

$\Longrightarrow$ deformation of $G \curvearrowright L^2(G/K)$

Deformation of $G \curvearrowright L^2(G/K)$

$$\begin{tikzcd}
G \arrow[r, phantom, "\overset{\pi}{\curvearrowright}" description] \arrow[d, phantom, "\cap" description] & L^2(G/K) \\
{}^{\exists}\widetilde{G} \arrow[ur, dashed, bend left, "{}^{\exists}\widetilde{\pi}"'] &
\end{tikzcd}$$

Prop For any classical reductive G, there exist $\widetilde{G}$ $(\supsetneqq G)$ and an irreducible unitary rep $\widetilde{\pi}$ of $\widetilde{G}$ s.t. $\widetilde{\pi}|_G = \pi$

E.g. $G = GL(n, \mathbb{R}) \hookrightarrow \widetilde{G} = Sp(n, \mathbb{R})$

$\widetilde{\pi}$ lies in a continuous family $\{\widetilde{\pi}_\lambda\}$ of irred unitary reps of $\widetilde{G}$

$\Longrightarrow$ $\pi_\lambda := \widetilde{\pi}_\lambda|_G$ is a continuous family of (non-irreducible) representations of G

$\Longrightarrow$ deformation of $G \curvearrowright L^2(G/K)$ (still MF)

Sometimes discrete spectrum may appear!

Plan

Lectures 1 and 2

Various examples of

MF representations

MF = multiplicity-free

Existing methods to prove MF

Various techniques have been used in proving various MF results, in particular, for finite dim’l reps

For example, one may

1. look for an open orbit of a Borel subgroup.
2. apply Littlewood–Richardson rules and variants.
3. use computational combinatorics.
4. employ the Gelfand trick (the commutativity of the Hecke algebra).
5. apply Schur–Weyl duality and Howe duality.

New approach to prove/construct MF reps

Plan:

> To give a new simple principle that explains the property MF for both finite and infinite dimensional reps

Plan of Lecture 4

Lectures 1 and 2
Various examples of
MF representations

Plan of Lecture 4

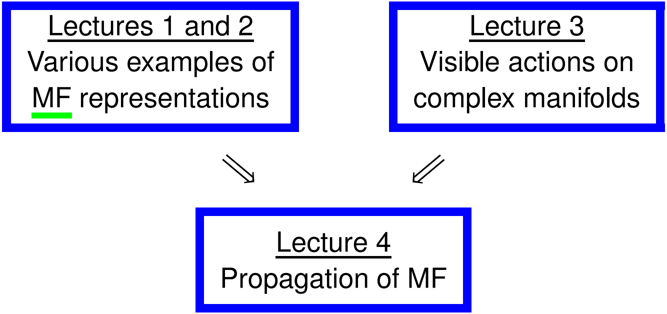
Lectures 1 and 2

Various examples of
MF representations

Lecture 3

Visible actions on
complex manifolds

Plan of Lecture 4



Lectures 1 and 2
Various examples of
MF representations

Lecture 3
Visible actions on
complex manifolds

⇘ ⇙

Lecture 4
Propagation of MF

Representation on holomorphic sections

$$\begin{tikzcd} \mathcal{L} \arrow[d, "p"] & \text{holomorphic line bundle} \\ D & \text{complex manifold} \end{tikzcd}$$

Representation on holomorphic sections

$$\begin{tikzcd} \mathcal{L} \arrow[d, "p"] & \text{holomorphic line bundle} \\ D & \text{complex manifold} \end{tikzcd}$$

$$\mathcal{O}(D,\mathcal{L}) := \{s : D \to \mathcal{L} \quad \text{holomorphic} : p \circ s = \mathrm{id}_D\}.$$

Representation on holomorphic sections

$H \curvearrowright \boxed{\begin{array}{c}\mathcal{L}\\ \downarrow p\\ D\end{array}}$ equivariant holomorphic line bundle

$$\mathcal{O}(D, \mathcal{L}) := \{s : D \to \mathcal{L} \quad \text{holomorphic} : p \circ s = \mathrm{id}_D\}.$$

Example $\quad \mathcal{L} = D \times \mathbb{C} \quad$ (trivial line bundle)

$\mathcal{O}(D, \mathcal{L}) \simeq \mathcal{O}(D) \quad$ (= {holomorphic functions})

Representation on holomorphic sections

$$H \curvearrowright \boxed{\begin{array}{c}\mathcal{L}\\ \downarrow p\\ D\end{array}}$$ equivariant holomorphic line bundle

$$\mathcal{O}(D, \mathcal{L}) := \{s : D \to \mathcal{L} \quad \text{holomorphic} : p \circ s = \mathrm{id}_D\}.$$

If $\mathcal{L} \to D$ is an H-equivariant holomorphic line bundle, then we get a rep of H on $\mathcal{O}(D, \mathcal{L})$ by

$$s \mapsto g \cdot s(g^{-1} \cdot) \quad (g \in H)$$

Geometry on base spaces and rep theory

Suppose $\mathcal{L} \to D$ is an H-equivariant holomorphic line bundle.

Geometry on base spaces and rep theory

Suppose $\mathcal{L} \to D$ is an H-equivariant holomorphic line bundle.

Theorem 1 If $H \curvearrowright D$ transitively, then
any unitary subrep of $\mathcal{O}(D, \mathcal{L})$ is irreducible.

Geometry on base spaces and rep theory

Suppose $\mathcal{L} \to D$ is an H-equivariant holomorphic line bundle.

Theorem 1 If $H \curvearrowright D$ transitively, then any unitary subrep of $\mathcal{O}(D, \mathcal{L})$ is irreducible.

Definition "unitary subrep"

$\iff \underset{\text{Hilbert space}}{\mathcal{H}} \subset \mathcal{O}(D, \mathcal{L})$ such that every $g \in H$ leaves $\mathcal{H}$ invariant and acts as a unitary operator of $\mathcal{H}$.

Geometry on base spaces and rep theory

Suppose $\mathcal{L} \to D$ is an H-equivariant holomorphic line bundle.

Theorem 1 If $H \curvearrowright D$ transitively, then
any unitary subrep of $\mathcal{O}(D, \mathcal{L})$ is irreducible.

Example (Borel–Weil)
D = flag variety H/T (T : maximal torus of compact H)
$\rightsquigarrow$ Any irred rep of H is given as $H \curvearrowright D$ on $\mathcal{O}(D, \mathcal{L})$
for some H-equivariant line bundle $\mathcal{L} \to D$.

Geometry on base spaces and rep theory

Suppose $\mathcal{L} \to D$ is an H-equivariant holomorphic line bundle.

Theorem 1 If $H \curvearrowright D$ transitively, then any unitary subrep of $\mathcal{O}(D, \mathcal{L})$ is irreducible.

Theorem 2 If $H \curvearrowright D$ strongly visibly, then any unitary subrep of $\mathcal{O}(D, \mathcal{L})$ is MF.

G/K Hermitian symm. space

Ex.22

$$G = SL(2, \mathbb{R})$$
$$K = SO(2)$$
$$H = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} : a > 0 \right\}$$
$$N = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in \mathbb{R} \right\}$$
$$G/K \simeq \{z \in \mathbb{C} : |z| < 1\}$$

G/K Hermitian symm. space

<u>Ex.22</u> $G = SL(2,\mathbb{R})$

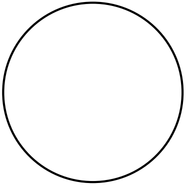
$K = SO(2)$

$H = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} : a > 0 \right\}$

$N = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in \mathbb{R} \right\}$

$G/K \simeq \{z \in \mathbb{C} : |z| < 1\}$

$K \curvearrowright G/K$



G/K Hermitian symm. space

<u>Ex.22</u> $G = SL(2,\mathbb{R})$

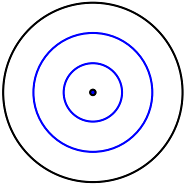
$K = SO(2)$

$H = \left\{\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} : a > 0\right\}$

$N = \left\{\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in \mathbb{R}\right\}$

$G/K \simeq \{z \in \mathbb{C} : |z| < 1\}$

$K \curvearrowright G/K$

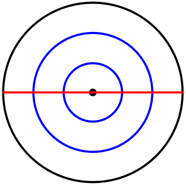


K-orbits

G/K Hermitian symm. space

Ex.22
$$
\begin{aligned}
G &= SL(2,\mathbb{R})\\
K &= SO(2)\\
H &= \left\{\begin{pmatrix} a & 0\\ 0 & a^{-1}\end{pmatrix} : a > 0\right\}\\
N &= \left\{\begin{pmatrix} 1 & x\\ 0 & 1\end{pmatrix} : x \in \mathbb{R}\right\}\\
G/K &\simeq \{z \in \mathbb{C} : |z| < 1\}
\end{aligned}
$$

$K \curvearrowright G/K$ visible

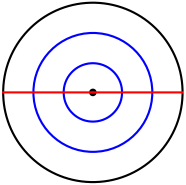


K-orbits

G/K Hermitian symm. space

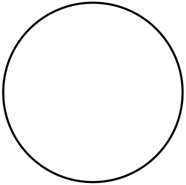
Ex.22
$$
\begin{aligned}
G &= SL(2,\mathbb{R})\\
K &= SO(2)\\
H &= \left\{\begin{pmatrix} a & 0\\ 0 & a^{-1}\end{pmatrix} : a > 0\right\}\\
N &= \left\{\begin{pmatrix} 1 & x\\ 0 & 1\end{pmatrix} : x \in \mathbb{R}\right\}\\
G/K &\simeq \{z \in \mathbb{C} : |z| < 1\}
\end{aligned}
$$

$K \curvearrowright G/K$ visible



K-orbits

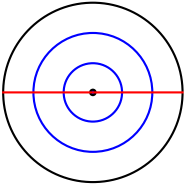
$H \curvearrowright G/K$



G/K Hermitian symm. space

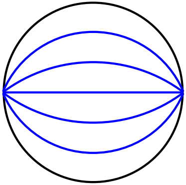
Ex.22
$$\begin{aligned} G &= SL(2,\mathbb{R}) \\ K &= SO(2) \\ H &= \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} : a > 0 \right\} \\ N &= \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in \mathbb{R} \right\} \\ G/K &\simeq \{ z \in \mathbb{C} : |z| < 1 \} \end{aligned}$$

$K \curvearrowright G/K$ visible



K-orbits

$H \curvearrowright G/K$



H-orbits

G/K Hermitian symm. space

Ex.22 $G = SL(2,\mathbb{R})$

$K = SO(2)$

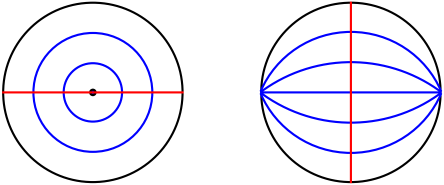
$H = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} : a > 0 \right\}$

$N = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in \mathbb{R} \right\}$

$G/K \simeq \{ z \in \mathbb{C} : |z| < 1 \}$

$K \curvearrowright G/K$ visible

$H \curvearrowright G/K$ visible



K-orbits

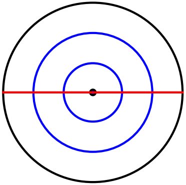
H-orbits

G/K Hermitian symm. space

Ex.22

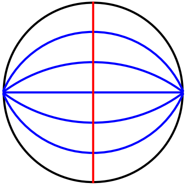
$$\begin{aligned} G &= SL(2,\mathbb{R}) \\ K &= SO(2) \\ H &= \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} : a > 0 \right\} \\ N &= \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in \mathbb{R} \right\} \\ G/K &\simeq \{ z \in \mathbb{C} : |z| < 1 \} \end{aligned}$$

$K \curvearrowright G/K$ visible



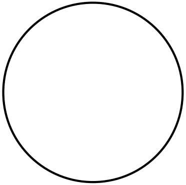
K-orbits

$H \curvearrowright G/K$ visible



H-orbits

$N \curvearrowright G/K$

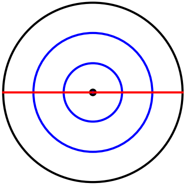


G/K Hermitian symm. space

Ex.22

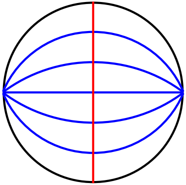
$$G = SL(2,\mathbb{R})$$
$$K = SO(2)$$
$$H = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} : a > 0 \right\}$$
$$N = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in \mathbb{R} \right\}$$
$$G/K \simeq \{z \in \mathbb{C} : |z| < 1\}$$

$K \curvearrowright G/K$ visible



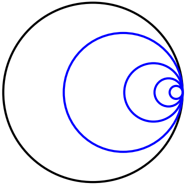
K-orbits

$H \curvearrowright G/K$ visible



H-orbits

$N \curvearrowright G/K$



N-orbits

G/K Hermitian symm. space

Ex.22 $G = SL(2,\mathbb{R})$

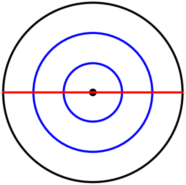
$K = SO(2)$

$H = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} : a > 0 \right\}$

$N = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in \mathbb{R} \right\}$

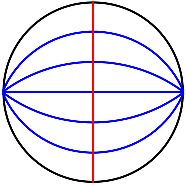
$G/K \simeq \{z \in \mathbb{C} : |z| < 1\}$

$K \curvearrowright G/K$ visible



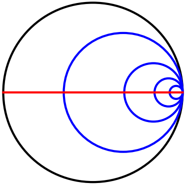
K-orbits

$H \curvearrowright G/K$ visible



H-orbits

$N \curvearrowright G/K$ visible



N-orbits

Visible actions on symmetric spaces

Theorem (Transf. Groups (2007))

Assume $\begin{cases} G/K & \text{Hermitian symm. sp.} \\ (G,H) & \text{symmetric pair} \end{cases}$

$\Longrightarrow H \curvearrowright G/K$ is (strongly) visible

Visible actions on symmetric spaces

Theorem (Transf. Groups (2007))

Assume $\begin{cases} G/K & \text{Hermitian symm. sp.} \\ (G,H) & \text{symmetric pair} \end{cases}$

$\Longrightarrow H \curvearrowright G/K$ is (strongly) visible

$\Downarrow$ Theorem 2

Ex.19 π_λ, π_μ highest wt. modules of scalar type

$\Longrightarrow \pi_\lambda \otimes \pi_\mu$ is MF

Visible actions on symmetric spaces

Theorem (Transf. Groups (2007))

Assume $\begin{cases} G/K & \text{Hermitian symm. sp.} \\ (G,H) & \text{symmetric pair} \end{cases}$

$\Longrightarrow H \curvearrowright G/K$ is (strongly) visible

$\Downarrow$ Theorem 2

Ex.19 π_λ, π_μ highest wt. modules of scalar type

$\Longrightarrow \pi_\lambda \otimes \pi_\mu$ is MF

Ex.20 $\begin{cases} \pi_\lambda & \text{highest wt. module of scalar type} \\ (G,H) & \text{symmetric pair} \end{cases}$

$\Longrightarrow \pi_\lambda|_H$ is MF

Finite dimensional case

Also, for finite dimensional case

$\Downarrow$ Theorem 2

Eg.23 (Okada, '98, rectangular shaped rep)

$\mathfrak{g}_{\mathbb{C}} = \mathfrak{gl}(n, \mathbb{C})$

$$\lambda = (\underbrace{a, \cdots, a}_{p}, \underbrace{b, \cdots, b}_{n-p}) \in \mathbb{Z}^n, \ a \geq b$$

$\pi_\lambda |_{\mathfrak{h}_{\mathbb{C}}}$ is MF if

Finite dimensional case

Also, for finite dimensional case

$\Downarrow$ Theorem 2

Eg.23 (Okada, '98, rectangular shaped rep)

$\mathfrak{g}_{\mathbb{C}} = \mathfrak{gl}(n, \mathbb{C})$

$$\lambda = (\underbrace{a, \cdots, a}_{p}, \underbrace{b, \cdots, b}_{n-p}) \in \mathbb{Z}^n, \ a \geq b$$

$\pi_\lambda|_{\mathfrak{h}_{\mathbb{C}}}$ is MF if

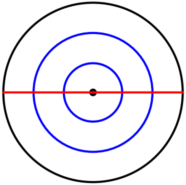
$$\mathfrak{h}_{\mathbb{C}} = \begin{cases} \mathfrak{gl}(k, \mathbb{C}) + \mathfrak{gl}(n-k, \mathbb{C}) & (1 \leq k \leq n) \\ \mathfrak{o}(n, \mathbb{C}) & \\ \mathfrak{sp}(\frac{n}{2}, \mathbb{C}) & (n : \text{even}) \end{cases}$$

G/K Hermitian symm. space

<u>Ex.22</u>

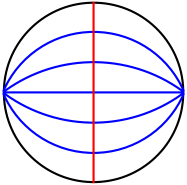
$$\begin{aligned} G &= SL(2,\mathbb{R}) \\ K &= SO(2) \\ H &= \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} : a > 0 \right\} \\ N &= \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in \mathbb{R} \right\} \\ G/K &\simeq \{ z \in \mathbb{C} : |z| < 1 \} \end{aligned}$$

$K \curvearrowright G/K$ visible



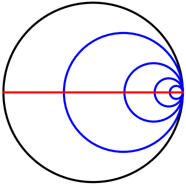
K-orbits

$H \curvearrowright G/K$ visible



H-orbits

$N \curvearrowright G/K$ visible



N-orbits

Non-reductive example

Theorem $N \subset G \supset K$

Assume $\begin{cases} G/K & \text{Hermitian symm. of non-cpt. type} \\ N & \text{max. unipotent subgp.} \end{cases}$

$\Longrightarrow N \curvearrowright G/K$ (strongly) visible

$\Downarrow$ Theorem 2

Ex.24 π_λ: highest wt. module of scalar type

$\Longrightarrow \pi_\lambda|_N$ is MF

Geometry on base spaces and rep theory

Suppose $\mathcal{L} \to D$ is an H-equivariant holomorphic line bundle.

Theorem 1 If $H \curvearrowright D$ transitively, then any unitary subrep of $\mathcal{O}(D, \mathcal{L})$ is irreducible.

Theorem 2 If $H \curvearrowright D$ strongly visibly, then any unitary subrep of $\mathcal{O}(D, \mathcal{L})$ is MF.

From line bundles to vector bundles

Observation

Properties of one-dimensional representations

- they are always irreducible
- they are always multiplicity-free

From line bundles to vector bundles

Observation

Properties of one-dimensional representations

- they are always irreducible
- they are always multiplicity-free

We interpret these as the properties for the fiber in the line bundle case.

Propagation Theorem

$$\boxed{\begin{array}{c}\mathcal{V}\\ \downarrow\\ D\end{array}}$$

H-equivariant holomorphic vector bundle

Propagation Theorem

$$\boxed{\begin{array}{c}\mathcal{V}\\ \downarrow\\ D\end{array}} \rightsquigarrow \boxed{H \curvearrowright \mathcal{O}(D,\mathcal{V})}$$

H-equivariant holomorphic vector bundle

Propagation Theorem

$$\begin{tikzcd}
\mathcal{V}_x \arrow[r, phantom, "\subset"] \arrow[d] & \mathcal{V} \arrow[d] \\
\{x\} \arrow[r, phantom, "\subset"] & D
\end{tikzcd}
\rightsquigarrow \boxed{H \curvearrowright \mathcal{O}(D, \mathcal{V})}$$

H-equivariant holomorphic vector bundle

Propagation Theorem

$$\begin{tikzcd} \boxed{H_x \curvearrowright \mathcal{V}_x} \arrow[dd] & \subset & \boxed{\mathcal{V}} \arrow[dd] & & \\ & & & \rightsquigarrow & \boxed{H \curvearrowright \mathcal{O}(D, \mathcal{V})} \\ \{x\} & \subset & \boxed{D} & & \end{tikzcd}$$

H-equivariant holomorphic vector bundle

$$H_x = \{h \in H : h \cdot x = x\}$$

Propagation Theorem

$$\begin{tikzcd}
\boxed{H_x \curvearrowright \mathcal{V}_x} \arrow[d] \arrow[r, phantom, "\subset"] & \mathcal{V} \arrow[d] \arrow[rd, phantom, "\rightsquigarrow"] & \\
\{x\} \arrow[r, phantom, "\subset"] & D & \boxed{H \curvearrowright \mathcal{O}(D, \mathcal{V})}
\end{tikzcd}$$

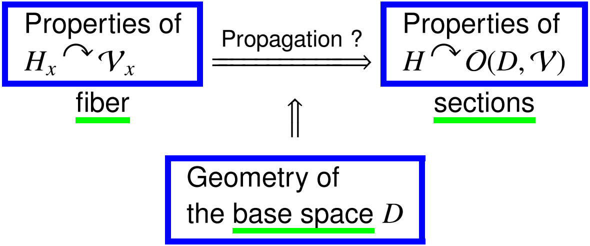
H-equivariant holomorphic vector bundle

$$\underset{\text{“easy”}}{\boxed{\text{Properties of } H_x \curvearrowright \mathcal{V}_x}} \xRightarrow{\text{Propagation ?}} \underset{\text{“difficult”}}{\boxed{\text{Properties of } H \curvearrowright \mathcal{O}(D, \mathcal{V})}}$$

Propagation Theorem

$$\begin{tikzcd}
\boxed{H_x \curvearrowright \mathcal{V}_x} \arrow[d] \arrow[r, phantom, "\subset"] & \mathcal{V} \arrow[d] & \\
\{x\} \arrow[r, phantom, "\subset"] & D \arrow[r, phantom, "\rightsquigarrow"] & \boxed{H \curvearrowright \mathcal{O}(D, \mathcal{V})}
\end{tikzcd}$$

H-equivariant holomorphic vector bundle



Properties of $H_x \curvearrowright \mathcal{V}_x$

fiber

Propagation ?

Properties of $H \curvearrowright \mathcal{O}(D, \mathcal{V})$

sections

Geometry of the base space D

Propagation Theorem

$$\begin{tikzcd} \boxed{H_x \curvearrowright \mathcal{V}_x} \arrow[d] \arrow[r, phantom, "\subset"] & \mathcal{V} \arrow[d] \arrow[r, rightsquigarrow] & \boxed{H \curvearrowright \mathcal{O}(D, \mathcal{V})} \\ \{x\} \arrow[r, phantom, "\subset"] & D & \end{tikzcd}$$

H-equivariant holomorphic vector bundle

<u>Theorem 1′</u>

$$\begin{tikzcd} \boxed{\begin{array}{l}\text{Irreducibility} \\ H_x \curvearrowright \mathcal{V}_x\end{array}} \arrow[rr, Rightarrow, "\text{Propagation}"] & & \boxed{\begin{array}{l}\text{Irreducibility} \\ H \curvearrowright \mathcal{O}(D, \mathcal{V})\end{array}} \\ \text{fiber} & \Uparrow & \text{sections} \\ & \boxed{\begin{array}{l} H \text{ acts transitively on} \\ \text{the base space } D \end{array}} & \end{tikzcd}$$

Propagation Theorem

$$\begin{tikzcd}
\boxed{H_x \curvearrowright \mathcal{V}_x} \arrow[d] & \subset & \mathcal{V} \arrow[d] & & \\
\{x\} & \subset & D \arrow[r, phantom, "\rightsquigarrow"] & \boxed{H \curvearrowright \mathcal{O}(D, \mathcal{V})} &
\end{tikzcd}$$

H-equivariant holomorphic vector bundle

Theorem 2′ (Progress in Mathematics, 2013)

$$\begin{tikzcd}
\boxed{\begin{array}{l}\text{MF}\\ H_x \curvearrowright \mathcal{V}_x\end{array}} \arrow[rr, Rightarrow, "\text{Propagation}"] & & \boxed{\begin{array}{l}\text{MF}\\ H \curvearrowright \mathcal{O}(D, \mathcal{V})\end{array}} \\
\text{fiber} & {} & \text{sections} \\
& \boxed{\begin{array}{l}H \text{ acts strong visibly on}\\ \text{the base space } D\end{array}} \arrow[u, Rightarrow] &
\end{tikzcd}$$

Automorphism of group action

$$H \curvearrowright D$$

Automorphism of group action

$$\underset{\text{Lie group}}{H} \curvearrowright \underset{\text{manifold}}{D}$$

<u>Def</u> $\sigma \in \mathrm{Aut}(H; D)$

$$\Longleftrightarrow \begin{cases} \sigma \curvearrowright H & \text{automorphism of Lie group} \\ \sigma \curvearrowright D & \text{diffeomorphism} \\ \sigma(g \cdot x) = \sigma(g) \cdot \sigma(x) & ({}^{\forall} g \in H,\ {}^{\forall} x \in D) \end{cases}$$

Automorphism of group action

$$\underset{\text{Lie group}}{H} \curvearrowright \underset{\text{manifold}}{D}$$

<u>Def</u> $\sigma \in \mathrm{Aut}(H; D)$

$$\iff \begin{cases} \sigma \curvearrowright H & \text{automorphism of Lie group} \\ \sigma \curvearrowright D & \text{diffeomorphism} \\ \sigma(g \cdot x) = \sigma(g) \cdot \sigma(x) & ({}^\forall g \in H,\ {}^\forall x \in D) \end{cases}$$

$\Longrightarrow$ σ preserves every H-orbit

Automorphism of group action

$$\underset{\text{Lie group}}{H} \curvearrowright \underset{\text{manifold}}{D}$$

<u>Def</u> $\sigma \in \mathrm{Aut}(H; D)$

$$\iff \begin{cases} \sigma \curvearrowright H & \text{automorphism of Lie group} \\ \sigma \curvearrowright D & \text{diffeomorphism} \\ \sigma(g \cdot x) = \sigma(g) \cdot \sigma(x) & ({}^{\forall} g \in H,\ {}^{\forall} x \in D) \end{cases}$$

$\Longrightarrow \sigma$ ~~preserves every H orbit~~

permutes H-orbits

Automorphism of group action

$$\underset{\text{Lie group}}{H} \curvearrowright \underset{\text{manifold}}{D}$$

<u>Def</u> $\sigma \in \mathrm{Aut}(H; D)$

$$\iff \begin{cases} \sigma \curvearrowright H & \text{automorphism of Lie group} \\ \sigma \curvearrowright D & \text{diffeomorphism} \\ \sigma(g \cdot x) = \sigma(g) \cdot \sigma(x) & ({}^{\forall} g \in H,\ {}^{\forall} x \in D) \end{cases}$$

$\Longrightarrow \sigma$ ~~preserves every H orbit~~

permutes H-orbits

Write simply $\sigma \curvearrowright D$ instead of $\sigma \in \mathrm{Aut}(H; D)$

Assumptions of MF theorem

$\mathcal{V} \to D$: H-equivariant

Assumption 1 $\quad {}^{\exists}\sigma \curvearrowright D$ anti-holomorphic s.t.
σ preserves every H-orbit.

Assumptions of MF theorem

$\mathcal{V} \to D$: H-equivariant

<u>Assumption 1</u> $\;{}^{\exists}\sigma \curvearrowright D$ anti-holomorphic s.t.
σ preserves every H-orbit.

$$\begin{tikzcd}
\mathcal{V} \arrow[dd] & \supset & \mathcal{V}_x \arrow[dd] & \simeq & \mathcal{V}_x^{(1)} \oplus \cdots \oplus \mathcal{V}_x^{(m)} \\
 & & & & \underline{\text{MF}} \text{ as an } H_x \text{ module} \\
D & \ni & x & &
\end{tikzcd}$$

Assumptions of MF theorem

$\mathcal{V} \to D$: H-equivariant

Assumption 1 $\quad {}^\exists\sigma \curvearrowright D$ anti-holomorphic s.t. σ preserves every H-orbit.

$$\begin{tikzcd}
\mathcal{V} \arrow[r, phantom, "\supset"] \arrow[d] & \mathcal{V}_x \arrow[r, phantom, "\simeq"] \arrow[d] & \mathcal{V}_x^{(1)} \oplus \cdots \oplus \mathcal{V}_x^{(m)} \\
D \arrow[r, phantom, "\ni"] & x &
\end{tikzcd}$$

MF as an H_x module

Assumption 2 $\quad \sigma$ lifts to an anti-holomorphic bundle map of $\mathcal{V} \to D$ s.t. $\sigma(\mathcal{V}_x^{(i)}) = \mathcal{V}_x^{(i)}$ $({}^\forall i)$ if $\sigma(x) = x$

Assumptions of MF theorem

$\mathcal{V} \to D$: H-equivariant

Assumption 1 $\quad {}^{\exists}\sigma \curvearrowright D$ anti-holomorphic s.t.
σ preserves every H-orbit.

$$\begin{tikzcd}
\mathcal{V} \arrow[d] \arrow[r, phantom, "\supset"] & \mathcal{V}_x \arrow[d] \arrow[r, phantom, "\simeq"] & \mathcal{V}_x^{(1)} \oplus \cdots \oplus \mathcal{V}_x^{(m)} \\
D \arrow[r, phantom, "\ni"] & x & \text{\underline{MF} as an } H_x \text{ module}
\end{tikzcd}$$

Assumption 2 $\quad \sigma$ lifts to an anti-holomorphic
bundle map of $\mathcal{V} \to D$ s.t.
$\sigma(\mathcal{V}_x^{(i)}) = \mathcal{V}_x^{(i)}$ $({}^{\forall}i)$ if $\sigma(x) = x$

Note: σ permutes $\mathcal{V}_x^{(1)}, \ldots, \mathcal{V}_x^{(m)}$ if $\sigma(x) = x$.

Assumptions of MF theorem

$\mathcal{V} \to D$: H-equivariant

Assumption 1 $\quad {}^{\exists}\sigma \curvearrowright D$ anti-holomorphic s.t.
σ preserves every H-orbit.

$$\begin{tikzcd}
\mathcal{V} \arrow[dd] & \supset & \mathcal{V}_x \arrow[dd] & \simeq & \mathcal{V}_x^{(1)} \oplus \cdots \oplus \mathcal{V}_x^{(m)} \\
 & & & & \text{MF as an } H_x \text{ module} \\
D & \ni & x & &
\end{tikzcd}$$

Assumption 2 $\quad \sigma$ lifts to an anti-holomorphic
bundle map of $\mathcal{V} \to D$ s.t.
$\sigma(\mathcal{V}_x^{(i)}) = \mathcal{V}_x^{(i)}$ $({}^{\forall}i)$ if $\sigma(x) = x$

Note: Assumption 2 is automatic for line bundles

Propagation of MF property

Progr. Math (2013)

H: Lie group

H-equiv. holo. vector b'dle:

$\mathcal{V} \to D$

$\Downarrow$

$H \curvearrowright \mathcal{O}(D, \mathcal{V})$ = {holo. sections}

Theorem 2′ (Propagation theorem)

$H_x \curvearrowright \mathcal{V}_x$ MF $(\forall x \in D)$

$\Longrightarrow H \curvearrowright \mathcal{O}(D, \mathcal{V})$ MF

if assumptions 1 & 2 hold.

Observations of MF theorem

Points: H has infinitely many orbits on D

- propagation of MF property

Observations of MF theorem

Points: H has infinitely many orbits on D

- propagation of MF property

$$\begin{tikzcd} & \mathcal{V} \arrow[d] \\ H \arrow[ur, phantom, "\curvearrowright"] \arrow[dr, phantom, "\curvearrowright"] & \\ & D \end{tikzcd}$$

fiber MF

Observations of MF theorem

Points: H has infinitely many orbits on D

- propagation of MF property

$$\begin{tikzcd} & \curvearrowright & \mathcal{V} \arrow[dd] & \\ H & & & \Longrightarrow \\ & \curvearrowright & D & \end{tikzcd}$$

fiber MF

Observations of MF theorem

Points: H has infinitely many orbits on D

- propagation of MF property

$$\begin{tikzcd} & \mathcal{V} \arrow[d] \\ H \arrow[ur, phantom, "\curvearrowright"] \arrow[dr, phantom, "\curvearrowright"] & \\ & D \end{tikzcd} \Longrightarrow \quad H \curvearrowright \mathcal{O}(D, \mathcal{V})$$

fiber MF — sections MF

Observations of MF theorem

Points: H has infinitely many orbits on D

- propagation of MF property

$$\begin{tikzcd} & \mathcal{V} \arrow[dd] \\ H \arrow[ur, phantom, "\curvearrowright"] \arrow[dr, phantom, "\curvearrowright"] & \\ & D \end{tikzcd} \quad \Longrightarrow \quad H \curvearrowright \mathcal{O}(D, \mathcal{V})$$

fiber MF $\qquad\qquad$ sections MF

$\Uparrow$

geometry of base space

. . . '(strongly) visible action'

Examples of MF theorem

Ex.20 $H = U(m) \times U(n)$

$D = M(m, n; \mathbb{C}) \simeq \mathbb{C}^{mn}$

Examples of MF theorem

<u>Ex.20</u> $H = U(m) \times U(n)$

$D = M(m, n; \mathbb{C}) \simeq \mathbb{C}^{mn}$

$\sigma(z) := \overline{z} \quad \Longrightarrow$ Assumption 1 O.K.

Examples of MF theorem

Ex.20 $H = U(m) \times U(n)$

$D = M(m, n; \mathbb{C}) \simeq \mathbb{C}^{mn}$

$\sigma(z) := \overline{z}$ $\Longrightarrow$ Assumption 1 O.K.

$\mathcal{V} = D \times \mathbb{C}$ $\Longrightarrow$ Assumption 2 O.K.

Examples of MF theorem

Ex.20 $H = U(m) \times U(n)$

$D = M(m, n; \mathbb{C}) \simeq \mathbb{C}^{mn}$

$\sigma(z) := \overline{z}$ $\Longrightarrow$ Assumption 1 O.K.

$\mathcal{V} = D \times \mathbb{C}$ $\Longrightarrow$ Assumption 2 O.K.

$\Downarrow$ Propagation theorem

$H \curvearrowright$ **Pol**(D) MF

$GL_m \times GL_n \curvearrowright$ **Pol**$(M(m, n; \mathbb{C}))$

Totally real submanifold

(D, J) complex manifold

$$J_x : T_xD \to T_xD, \quad J_x^2 = -\mathrm{id.}\ (x \in D)$$

Totally real submanifold

(D, J) complex manifold

$$J_x : T_xD \to T_xD, \quad J_x^2 = -\mathrm{id}. \ (x \in D)$$

Definition A real submanifold S of D is totally real if

$$T_xS \cap J_x(T_xS) = \{0\} \quad \text{for any } x \in S,$$

Totally real submanifold

(D, J) complex manifold

$$J_x : T_xD \to T_xD, \quad J_x^2 = -\mathrm{id}.\ (x \in D)$$

Definition A real submanifold S of D is totally real if

$$T_xS \cap J_x(T_xS) = \{0\} \quad \text{for any } x \in S,$$

or equivalently, if

T_xS does not contain any complex subspace.

Totally real submanifold

(D, J) complex manifold

$$J_x : T_xD \to T_xD, \quad J_x^2 = -\mathrm{id}.\ (x \in D)$$

Definition A real submanifold S of D

is totally real if

$T_xS \cap J_x(T_xS) = \{0\}$ for any $x \in S$,

or equivalently, if

T_xS does not contain any complex subspace.

Example Let $k \leq n$

$\mathbb{C}^n \supset \mathbb{R}^k$ totally real

$\mathbb{C}^n \supset \mathbb{C}^k$ not totally real

Totally real submanifold

(D, J) complex manifold

$$J_x : T_xD \to T_xD, \quad J_x^2 = -\mathrm{id}.\ (x \in D)$$

Definition A real submanifold S of D is totally real if

$$T_xS \cap J_x(T_xS) = \{0\} \quad \text{for any } x \in S,$$

or equivalently, if

T_xS does not contain any complex subspace.

Example Let $k \leq n$

$\mathbb{C}^n \supset \mathbb{R}^k$ totally real

$\mathbb{C}^n \supset \mathbb{C}^k$ not totally real

Rem We do not request S to be of maximal dimension (e.g. $S = \mathbb{R}^n$ in $\mathbb{C}^n$.)

§3 Visible actions

(D, J) complex mfd, connected

Def. A real submanifold S of D is totally real if $T_x S$ does not contain any complex subspace.

§3 Visible actions

$H \overset{\text{holomorphic}}{\curvearrowright} (D, J)$ complex mfd, connected

Def.(2003) A holomorphic action of H is visible w.r.t. a slice S if
$^{\exists}D' \underset{\text{open}}{\subset} D$,

Def. A real submanifold S of D is totally real if T_xS does not contain any complex subspace.

§3 Visible actions

$H \overset{\text{holomorphic}}{\curvearrowright} (D, J)$ complex mfd, connected

Def.(2003) A holomorphic action of H is visible w.r.t. a slice S if

$^{\exists}D' \underset{\text{open}}{\subset} D$,

$^{\exists}S \underset{\text{totally real}}{\subset} D'$ s.t.

$$\begin{cases} S \text{ meets every } H\text{-orbit} \\ J_x(T_x S) \subset T_x(H \cdot x) \ (x \in S) \end{cases}$$

Def. A real submanifold S of D is totally real if $T_x S$ does not contain any complex subspace.

Example of visible actions

$$H = \mathbb{T} := \{a \in \mathbb{C} : |a| = 1\} \quad (\simeq S^1)$$

Example of visible actions

$$H = \mathbb{T} := \{a \in \mathbb{C} : |a| = 1\} \quad (\simeq S^1)$$

Ex.21

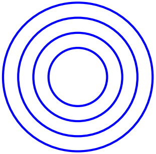
$$\begin{array}{lll} \mathbb{T} \curvearrowright & & \mathbb{C} \supset \mathbb{R} \\ \rotatebox{90}{$\in$} & & \\ a & & z \mapsto az \end{array}$$

Example of visible actions

$$H = \mathbb{T} := \{a \in \mathbb{C} : |a| = 1\} \quad (\simeq S^1)$$

Ex.21

$$\begin{array}{ccc} \mathbb{T} \curvearrowright & & \mathbb{C} \supset \mathbb{R} \\ \rotatebox{90}{$\in$} & & \\ a & & z \mapsto az \end{array}$$

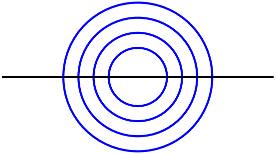


Example of visible actions

$$H = \mathbb{T} := \{a \in \mathbb{C} : |a| = 1\} \quad (\simeq S^1)$$

Ex.21

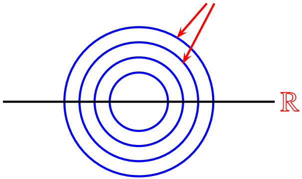
$$\begin{array}{ccc} \mathbb{T} \curvearrowright & & \mathbb{C} \supset \mathbb{R} \\ \rotatebox{90}{$\in$} & & \\ a & & z \mapsto az \end{array}$$



Example of visible actions

$$H = \mathbb{T} := \{a \in \mathbb{C} : |a| = 1\} \quad (\simeq S^1)$$

Ex.21 $$\begin{array}{ccc} \mathbb{T} \curvearrowright & & \mathbb{C} \supset \mathbb{R} \\ \Cup & & \\ a & & z \mapsto az \end{array}$$



$\mathbb{R}$ meets every $\mathbb{T}$-orbit

$$J_x(T_x S) \subset T_x(H \cdot x), \quad {}^{\forall} x \in \mathbb{R} \setminus \{0\} =: S$$

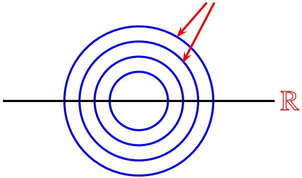
Example of visible actions

$$H = \mathbb{T} := \{a \in \mathbb{C} : |a| = 1\} \quad (\simeq S^1)$$

<u>Ex.21</u>

$$\begin{array}{ccc} \mathbb{T} \curvearrowright & & \mathbb{C} \supset \mathbb{R} \\ \rotatebox[origin=c]{90}{$\in$} & & \\ a & & z \mapsto az \end{array}$$

$\mathbb{R}$ meets every $\mathbb{T}$-orbit



$\Longrightarrow$ $\mathbb{T}$-action on $\mathbb{C}$ is visible

Strongly visible actions

$H \overset{\text{holomorphic}}{\curvearrowright} D$ complex mfd, connected

Strongly visible actions

$H \overset{\text{holomorphic}}{\curvearrowright} D$ complex mfd, connected

Def. ♣ ref. A holomorphic action is strongly visible
if

$^{\exists}\sigma$ $\curvearrowleft$ H as Lie group auto
$\curvearrowright$ D as anti-holomorphic diffeo
in a compatible way

s.t. $H \cdot D^{\sigma}$ contains a non-empty open set of D.
(generic H-orbits meets the fixed point set D^{σ})

Strongly visible actions

$H \overset{\text{holomorphic}}{\curvearrowright} D$ complex mfd, connected

<u>Def.</u> ♣ ref. A holomorphic action is strongly visible

if

$^{\exists}\sigma$ $\curvearrowleft$ H as Lie group auto

$\curvearrowright$ D as anti-holomorphic diffeo

in a compatible way

i.e. $\sigma(g \cdot x) = \sigma(g) \cdot \sigma(x)$

s.t. $H \cdot D^{\sigma}$ contains a non-empty open set of D.

(generic H-orbits meets the fixed point set D^{σ})

Strongly visible actions

$H \overset{\text{holomorphic}}{\curvearrowright} D$ complex mfd, connected

Def. ♣ ref. A holomorphic action is strongly visible

if

$^{\exists}\sigma$ $\curvearrowleft H$ as Lie group auto

$\curvearrowright D$ as anti-holomorphic diffeo

in a compatible way

i.e. $\sigma(g \cdot x) = \sigma(g) \cdot \sigma(x)$

s.t. $H \cdot D^{\sigma}$ contains a non-empty open set of D.

(generic H-orbits meets the fixed point set D^{σ})

$$D^{\sigma} := \{x \in D : \sigma(x) = x\}$$

Strongly visible actions

$H \overset{\text{holomorphic}}{\curvearrowright} D$ complex mfd, connected

<u>Def.</u> ♣ ref. A holomorphic action is <u>strongly visible</u>

if

$^{\exists}\sigma$ $\curvearrowleft$ H as Lie group auto

$\curvearrowright$ D as <u>anti-holomorphic</u> diffeo

in a compatible way

i.e. $\sigma(g \cdot x) = \sigma(g) \cdot \sigma(x)$

s.t. $H \cdot D^{\sigma}$ contains a non-empty open set of D.

(generic H-orbits meets the fixed point set D^{σ})

$$D^{\sigma} := \{x \in D : \sigma(x) = x\}$$

Remark. Not necessarily $\sigma^2 = \mathrm{id}$

Example of strongly visible actions

$$H = \mathbb{T} = \{a \in \mathbb{C} : |a| = 1\} \quad (\simeq S^1)$$

Example of strongly visible actions

$$H = \mathbb{T} = \{a \in \mathbb{C} : |a| = 1\} \quad (\simeq S^1)$$

Ex.21

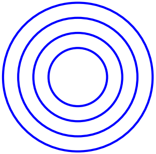
$$\begin{array}{lll} \mathbb{T} \curvearrowright & & \mathbb{C} \supset \mathbb{R} \\ \rotatebox[origin=c]{90}{$\in$} & & \\ a & & z \mapsto az \end{array}$$

Example of strongly visible actions

$$H = \mathbb{T} = \{a \in \mathbb{C} : |a| = 1\} \quad (\simeq S^1)$$

Ex.21

$$\begin{array}{ccc} \mathbb{T} \curvearrowright & & \mathbb{C} \supset \mathbb{R} \\ \rotatebox{90}{$\in$} & & \\ a & & z \mapsto az \end{array}$$

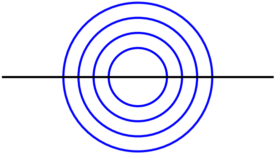


Example of strongly visible actions

$$H = \mathbb{T} = \{a \in \mathbb{C} : |a| = 1\} \quad (\simeq S^1)$$

<u>Ex.21</u> $\mathbb{T} \curvearrowright \mathbb{C} \supset \mathbb{R}$

$$\begin{array}{ccc} \mathbb{T} & & \mathbb{C} \\ \Cup & & \\ a & & z \mapsto az \end{array}$$



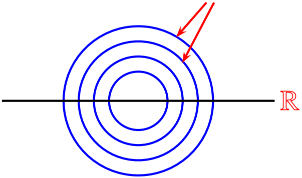
Example of strongly visible actions

$$H = \mathbb{T} = \{a \in \mathbb{C} : |a| = 1\} \quad (\simeq S^1)$$

Ex.21

$$\begin{array}{ccc} \mathbb{T} & \curvearrowright & \mathbb{C} \supset \mathbb{R} \\ \rotatebox[origin=c]{90}{$\in$} & & \\ a & & z \mapsto az \end{array}$$

ℝ meets every 𝕋-orbit



$\sigma(a) := \overline{a}$, $\sigma(z) := \overline{z}$ anti-holomorphic.
Then $\sigma(a \cdot z) = \sigma(a) \cdot \sigma(z)$ (compatibility), and $\sigma|_{\mathbb{R}} = \mathrm{id}$.

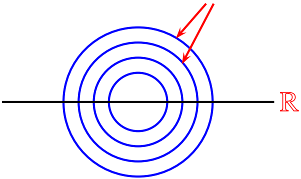
Example of strongly visible actions

$$H = \mathbb{T} = \{a \in \mathbb{C} : |a| = 1\} \quad (\simeq S^1)$$

Ex.21

$$\begin{array}{ccc} \mathbb{T} \curvearrowright & & \mathbb{C} \supset \mathbb{R} \\ \rotatebox{90}{$\in$} & & \\ a & & z \mapsto az \end{array}$$

$\mathbb{R}$ meets every $\mathbb{T}$-orbit



$\Longrightarrow$ $\mathbb{T}$-action on $\mathbb{C}$ is strongly visible

Strongly visible actions with slice S

holomorphic

$H \curvearrowright D$ complex mfd, connected

Strongly visible actions with slice S

holomorphic

$H \curvearrowright D$ complex mfd, connected

Def. ♣ ref. A holomorphic action is strongly visible w.r.t. a slice S if

$^\exists\sigma$ $\curvearrowleft H$

$\curvearrowright D$ anti-holomorphic

in a compatible way

s.t.

$\sigma|_S = \mathrm{id}$

$H \cdot S$ contains a non-empty open set of D.

Remark. S is automatically totally real.

Point Try to find a smallest possible $S \subset D^\sigma$.

Strongly visible actions

<u>Proposition</u> Strongly visible $\Longrightarrow$ Visible

Strongly visible actions

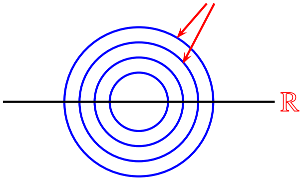
Proposition Strongly visible $\Longrightarrow$ Visible

To be more precise,

strongly visible w.r.t. a slice S

$\Longrightarrow$ visible w.r.t. S' for some $S' \underset{\text{open dense}}{\subset} S$.

$\mathbb{R}$ meets every $\mathbb{T}$-orbit



G/K Hermitian symm. space

Ex.22 $G = SL(2,\mathbb{R})$

$K = SO(2)$

$H = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} : a > 0 \right\}$

$N = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in \mathbb{R} \right\}$

$G/K \simeq \{z \in \mathbb{C} : |z| < 1\}$

G/K Hermitian symm. space

<u>Ex.22</u> $G = SL(2,\mathbb{R})$

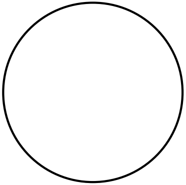
$K = SO(2)$

$H = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} : a > 0 \right\}$

$N = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in \mathbb{R} \right\}$

$G/K \simeq \{z \in \mathbb{C} : |z| < 1\}$

$K \curvearrowright G/K$



G/K Hermitian symm. space

Ex.22 $G = SL(2,\mathbb{R})$

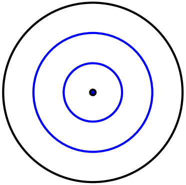
$K = SO(2)$

$H = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} : a > 0 \right\}$

$N = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in \mathbb{R} \right\}$

$G/K \simeq \{z \in \mathbb{C} : |z| < 1\}$

$K \curvearrowright G/K$



K-orbits

G/K Hermitian symm. space

Ex.22 $G = SL(2,\mathbb{R})$

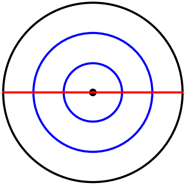
$K = SO(2)$

$H = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} : a > 0 \right\}$

$N = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in \mathbb{R} \right\}$

$G/K \simeq \{z \in \mathbb{C} : |z| < 1\}$

$K \curvearrowright G/K$ visible



K-orbits

G/K Hermitian symm. space

Ex.22 $G = SL(2,\mathbb{R})$

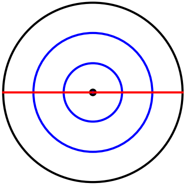
$K = SO(2)$

$H = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} : a > 0 \right\}$

$N = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in \mathbb{R} \right\}$

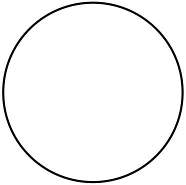
$G/K \simeq \{z \in \mathbb{C} : |z| < 1\}$

$K \curvearrowright G/K$ visible



K-orbits

$H \curvearrowright G/K$

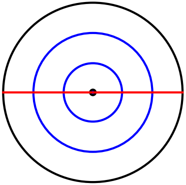


G/K Hermitian symm. space

<ins>Ex.22</ins>

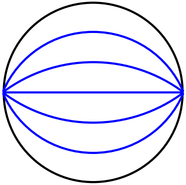
$$\begin{aligned} G &= SL(2,\mathbb{R}) \\ K &= SO(2) \\ H &= \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} : a > 0 \right\} \\ N &= \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in \mathbb{R} \right\} \\ G/K &\simeq \{ z \in \mathbb{C} : |z| < 1 \} \end{aligned}$$

$K \curvearrowright G/K$ visible



K-orbits

$H \curvearrowright G/K$



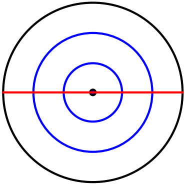
H-orbits

G/K Hermitian symm. space

Ex.22

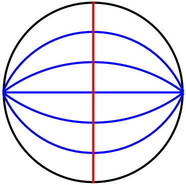
$$
\begin{aligned}
G &= SL(2,\mathbb{R})\\
K &= SO(2)\\
H &= \left\{\begin{pmatrix} a & 0\\ 0 & a^{-1}\end{pmatrix} : a > 0\right\}\\
N &= \left\{\begin{pmatrix} 1 & x\\ 0 & 1\end{pmatrix} : x \in \mathbb{R}\right\}\\
G/K &\simeq \{z \in \mathbb{C} : |z| < 1\}
\end{aligned}
$$

$K \curvearrowright G/K$ visible



K-orbits

$H \curvearrowright G/K$ visible

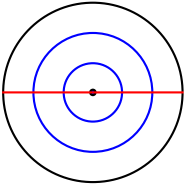


H-orbits

G/K Hermitian symm. space

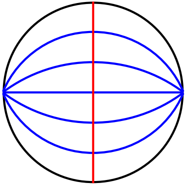
Ex.22
$$\begin{aligned} G &= SL(2,\mathbb{R}) \\ K &= SO(2) \\ H &= \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} : a > 0 \right\} \\ N &= \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in \mathbb{R} \right\} \\ G/K &\simeq \{ z \in \mathbb{C} : |z| < 1 \} \end{aligned}$$

$K \curvearrowright G/K$ visible



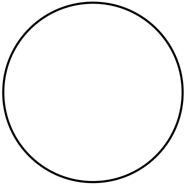
K-orbits

$H \curvearrowright G/K$ visible



H-orbits

$N \curvearrowright G/K$

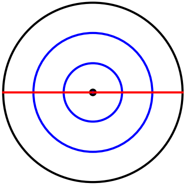


G/K Hermitian symm. space

<u>Ex.22</u>

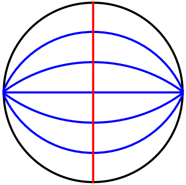
$$G = SL(2, \mathbb{R})$$
$$K = SO(2)$$
$$H = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} : a > 0 \right\}$$
$$N = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in \mathbb{R} \right\}$$
$$G/K \simeq \{z \in \mathbb{C} : |z| < 1\}$$

$K \curvearrowright G/K$ visible



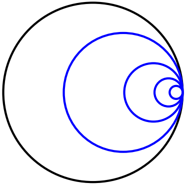
K-orbits

$H \curvearrowright G/K$ visible



H-orbits

$N \curvearrowright G/K$



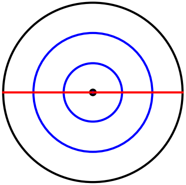
N-orbits

G/K Hermitian symm. space

Ex.22

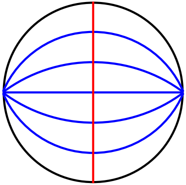
$$
\begin{aligned}
G &= SL(2,\mathbb{R})\\
K &= SO(2)\\
H &= \left\{\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} : a > 0\right\}\\
N &= \left\{\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in \mathbb{R}\right\}\\
G/K &\simeq \{z \in \mathbb{C} : |z| < 1\}
\end{aligned}
$$

$K \curvearrowright G/K$ visible



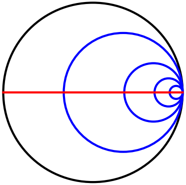
K-orbits

$H \curvearrowright G/K$ visible



H-orbits

$N \curvearrowright G/K$ visible



N-orbits

Visible actions on symmetric spaces

Theorem (Transf. Groups (2007))

Assume $\begin{cases} G/K & \text{Hermitian symm. sp.} \\ (G,H) & \text{symmetric pair} \end{cases}$

$\Longrightarrow H \curvearrowright G/K$ is (strongly) visible

§4 Compex / Riemannian / symplectic geometry

$H \overset{\text{holomorphic}}{\curvearrowright} (D, J)$ complex mfd, connected

<u>Def.</u> Action is visible if

$^\exists S \underset{\text{totally real}}{\subset} {}^\exists D' \underset{\text{open}}{\subset} D$ s.t.

$$\begin{cases} S \text{ meets every } H\text{-orbit in } D' \\ J_x(T_x S) \subset T_x(H \cdot x) \ (x \in S) \end{cases}$$

§4 Compex / Riemannian / symplectic geometry

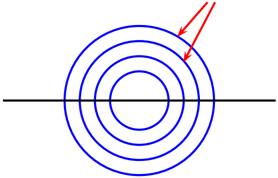
$H \overset{\text{holomorphic}}{\curvearrowright} (D, J)$ complex mfd, connected

<u>Def.</u> Action is <u>visible</u> if

$^\exists S \underset{\text{totally real}}{\subset} {}^\exists D' \underset{\text{open}}{\subset} D$ s.t.

$$\begin{cases} S \text{ meets every } H\text{-orbit in } D' \\ J_x(T_x S) \subset T_x(H \cdot x) \ (x \in S) \end{cases}$$

S meets <u>every $\mathbb{T}$-orbit</u>



$S = \mathbb{R}$

Complex / Riemannian / symplectic

$H \overset{\text{isometric}}{\curvearrowright} (D, g)$ Riemannian mfd

Def. Action is polar if $^{\exists} S \underset{\text{closed submfd}}{\subset} D$ s.t.

$$\begin{cases} S \text{ meets every } H\text{-orbit} \\ T_x S \perp T_x(H \cdot x) \quad (x \in S) \end{cases}$$

Complex / Riemannian / symplectic

$H \overset{\text{isometric}}{\curvearrowright} (D, g)$ Riemannian mfd

Def. Action is polar if $^{\exists}S \underset{\text{closed submfd}}{\subset} D$ s.t.

$$\begin{cases} S \text{ meets every } H\text{-orbit} \\ T_xS \perp T_x(H \cdot x) \quad (x \in S) \end{cases}$$

$H \overset{\text{symplectic}}{\curvearrowright} (D, \omega)$ symplectic mfd

Def. (Guillemin–Sternberg, Huckleberry–Wurzbacher)
Action is coisotropic (or multiplicity-free)
if $T_x(H \cdot x)^{\perp\omega} \subset T_x(H \cdot x)$ for principal orbits $H \cdot x$ in D

Three geometries

Complex geometry

Symplectic geometry

Riemannian geometry

Three geometries

Complex geometry

Visible action

K– (2004)

Symplectic geometry

Coisotropic action

Guillemin–Sternberg ('84)
Huckleberry–Wurzbacher ('90)

Riemannian geometry

Polar action

Bott–Samelson ('58), Conlon, Hermann, Palais, Terng, Dadok,
Eschenburg, Heintze, Podestà–Thorbergsson ('03), Kollross ('07), . . .

Three geometries

Complex geometry

Visible action

K– (2004)

Symplectic geometry

Coisotropic action

Guillemin–Sternberg ('84)
Huckleberry–Wurzbacher ('90)

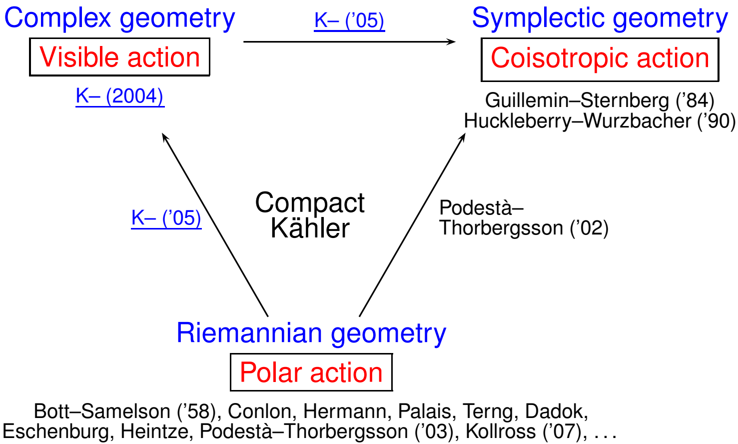
Compact
Kähler

Riemannian geometry

Polar action

Bott–Samelson ('58), Conlon, Hermann, Palais, Terng, Dadok,
Eschenburg, Heintze, Podestà–Thorbergsson ('03), Kollross ('07), . . .

Three geometries



Complex geometry

Visible action

K– (2004)

K– ('05)

Symplectic geometry

Coisotropic action

Guillemin–Sternberg ('84)
Hucklebery–Wurzbacher ('90)

K– ('05)

Compact
Kähler

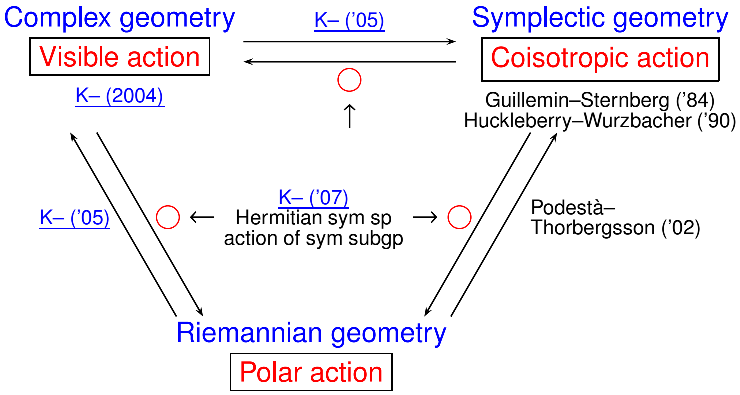
Podestà–
Thorbergsson ('02)

Riemannian geometry

Polar action

Bott–Samelson ('58), Conlon, Hermann, Palais, Terng, Dadok,
Eschenburg, Heintze, Podestà–Thorbergsson ('03), Kollross ('07), . . .

Three geometries



Complex geometry

Visible action

K– (2004)

K– ('05)

Symplectic geometry

Coisotropic action

Guillemin–Sternberg ('84)
Huckleberry–Wurzbacher ('90)

○
↑

K– ('05)

○ ← K– ('07)
Hermitian sym sp
action of sym subgp → ○

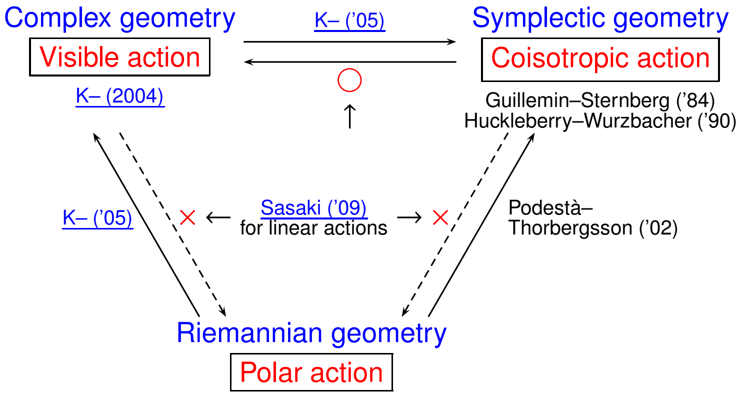
Podestà–
Thorbergsson ('02)

Riemannian geometry

Polar action

Bott–Samelson ('58), Conlon, Hermann, Palais, Terng, Dadok,
Eschenburg, Heintze, Podestà–Thorbergsson ('03), Kollross ('07), . . .

Three geometries



Complex geometry

Visible action

K– (2004)

K– ('05)

○

↑

Symplectic geometry

Coisotropic action

Guillemin–Sternberg ('84)
Huckleberry–Wurzbacher ('90)

K– ('05)

× ← Sasaki ('09) → ×
for linear actions

Podestà–
Thorbergsson ('02)

Riemannian geometry

Polar action

Bott–Samelson ('58), Conlon, Hermann, Palais, Terng, Dadok,
Eschenburg, Heintze, Podestà–Thorbergsson ('03), Kollross ('07), . . .

§5 Making examples of visible actions

Ex.20 $H = U(m) \times U(n)$

$D = M(m, n; \mathbb{C})$

$\Rightarrow$ Every H-orbit is preserved by $z \mapsto \overline{z}$

§5 Making examples of visible actions

Ex.20 $H = U(m) \times U(n)$

$D = M(m, n; \mathbb{C})$

$\Rightarrow$ Every H-orbit is preserved by $z \mapsto \overline{z}$

Proof Let $m \leq n$. Set

$$S := \left\{ \begin{pmatrix} a_1 & & 0 & \\ & \ddots & & 0 \\ 0 & & a_m & \end{pmatrix} : a_1, \ldots, a_m \in \mathbb{R} \right\}$$

§5 Making examples of visible actions

Ex.20 $H = U(m) \times U(n)$

$D = M(m, n; \mathbb{C})$

$\Rightarrow$ Every H-orbit is preserved by $z \mapsto \overline{z}$

Proof Let $m \leq n$. Set

$$S := \left\{ \begin{pmatrix} a_1 & & 0 & \\ & \ddots & & 0 \\ 0 & & a_m & \end{pmatrix} : a_1, \ldots, a_m \in \mathbb{R} \right\}$$

$\Rightarrow$ Every H-orbit meets S, i.e. $H \cdot S = D$

§5 Making examples of visible actions

Ex.20 $H = U(m) \times U(n)$

$D = M(m, n; \mathbb{C})$

$\Rightarrow$ Every H-orbit is preserved by $z \mapsto \overline{z}$

Proof Let $m \leq n$. Set

$$S := \left\{ \begin{pmatrix} a_1 & & 0 & \\ & \ddots & & 0 \\ 0 & & a_m & \end{pmatrix} : a_1, \ldots, a_m \in \mathbb{R} \right\} \quad \subset M(m, n; \mathbb{R})$$

$\Rightarrow$ Every H-orbit meets S, i.e. $H \cdot S = D$

$\Rightarrow$ Any H-orbit is of the form $H \cdot x$ (${}^{\exists} x \in S$)

$$\overline{H \cdot x} \underset{\text{compatibility}}{=} \overline{H} \cdot \overline{x} \underset{x \in M(m,n;\mathbb{R})}{=} H \cdot x$$

□

§5 Making examples of visible actions

Ex. $H = U(m) \times U(n)$

$D = M(m, n; \mathbb{C})$

$\Rightarrow$ Every H-orbit is preserved by $z \mapsto \bar{z}$

In general,

Strongly visible

(i.e. ${}^{\exists}\sigma$ anti-holo s.t. $(H \cdot D^{\sigma})^{\circ} \neq \varnothing$)

$\Rightarrow$ Assumption 1 of Theorem

(i.e. ${}^{\exists}\sigma$ anti-holo s.t. σ preserves generic H-orbits)

Analysis on ∞-many orbits

$\mathcal{V} \to X$: H-equiv. holo vector bundle.

Analysis on ∞-many orbits

$\mathcal{V} \to X$: H-equiv. holo vector bundle.

$\mathcal{V}_x \to \{x\}$: Fiber

Analysis on ∞-many orbits

$\mathcal{V} \to X$: H-equiv. holo vector bundle.
$\mathcal{V}_x \to \{x\}$: Fiber

Theorem (Propagation thm of MF property)

Sections

$\Longrightarrow$ $H \curvearrowright \mathcal{O}(X, \mathcal{V})$ multiplicity-free

∞ many orbits

Analysis on ∞-many orbits

$\mathcal{V} \to X$: H-equiv. holo vector bundle.
$\mathcal{V}_x \to \{x\}$: Fiber

Theorem (Propagation thm of MF property)

Fiber

$H_x \curvearrowright \mathcal{V}_x$
multiplicity-free

$\Longrightarrow$

Sections

$H \curvearrowright \mathcal{O}(X, \mathcal{V})$
multiplicity-free

∞ many orbits

Analysis on ∞-many orbits

$\mathcal{V} \to X$: H-equiv. holo vector bundle.
$\mathcal{V}_x \to \{x\}$: Fiber

<u>Theorem</u> (<u>Propagation thm of MF property</u>)

Fiber

$H_x \curvearrowright \mathcal{V}_x$
multiplicity-free

$\Longrightarrow$

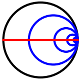
Sections

$H \curvearrowright \mathcal{O}(X, \mathcal{V})$
multiplicity-free

$\Uparrow$

$H \curvearrowright X$
(strongly) visible

Base sp.



∞ many orbits

Classification theory of visible actions

Methods to find visible action

Want to find visible actions systematically

Classification theory of visible actions

Methods to find visible action

Want to find visible actions systematically

- Structure theory
  - geometry of three involutions
  - generalized Cartan decomposition

Classification theory of visible actions

Methods to find visible action

Want to find visible actions systematically

- Structure theory
    - geometry of three involutions
    - generalized Cartan decomposition
- Make new from old
    - make ‘large’ from ‘small’
    - make ‘three’ from ‘one’ (triunity)

Classification theory of visible actions

Methods to find visible action

Want to find visible actions systematically

- Structure theory
    - geometry of three involutions
    - generalized Cartan decomposition
- Make new from old
    - make ‘large’ from ‘small’
    - make ‘three’ from ‘one’ (triunity)

Geometry of three involutions

$G_{\mathbb{C}}$ complex semi-simple Lie gp
$\sigma \in \mathrm{Aut}(G_{\mathbb{C}})$ involution $\Leftrightarrow \sigma^2 = \mathrm{id}$

Geometry of three involutions

$G_{\mathbb{C}}$ complex semi-simple Lie gp

$\sigma \in \mathrm{Aut}(G_{\mathbb{C}})$ involution $\Leftrightarrow \sigma^2 = \mathrm{id}$

One involution $\longleftrightarrow$ $G_{\mathbb{R}} \underset{\text{real simple Lie group}}{\subset} G_{\mathbb{C}}$

Geometry of three involutions

$G_{\mathbb{C}}$ complex semi-simple Lie gp
$\sigma \in \mathrm{Aut}(G_{\mathbb{C}})$ involution $\Leftrightarrow \sigma^2 = \mathrm{id}$

One involution $\longleftrightarrow$ $G_{\mathbb{R}} \underset{\text{real simple Lie group}}{\subset} G_{\mathbb{C}}$
$\longleftrightarrow$ Riemannian symmetric space
$\cdots$ classified: Killing–Cartan (1914)

Geometry of three involutions

$G_{\mathbb{C}}$ complex semi-simple Lie gp
$\sigma \in \mathrm{Aut}(G_{\mathbb{C}})$ involution $\Leftrightarrow \sigma^2 = \mathrm{id}$

One involution $\longleftrightarrow$ $G_{\mathbb{R}} \underset{\text{real simple Lie group}}{\subset} G_{\mathbb{C}}$
$\longleftrightarrow$ Riemannian symmetric space
$\cdots$ classified: Killing–Cartan (1914)
Two involutions $\longleftrightarrow$ semisimple symmetric space
$\cdots$ classified: M. Berger (1957)

Geometry of three involutions

$G_{\mathbb{C}}$ complex semi-simple Lie gp
$\sigma \in \mathrm{Aut}(G_{\mathbb{C}})$ involution $\Leftrightarrow \sigma^2 = \mathrm{id}$

One involution	$\longleftrightarrow$	$G_{\mathbb{R}} \underset{\text{real simple Lie group}}{\subset} G_{\mathbb{C}}$
	$\longleftrightarrow$	Riemannian symmetric space
	$\cdots$	classified: Killing–Cartan (1914)
Two involutions	$\longleftrightarrow$	semisimple symmetric space
	$\cdots$	classified: M. Berger (1957)
Three involutions (special case)	$\longleftrightarrow$	visible action (2004–) (special case)

Visible actions on symmetric spaces

Theorem (geometry of three involutions '07)

Assume $\begin{cases} G/K & \text{Hermitian symm. sp.} \\ (G,H) & \text{any symmetric pair} \end{cases}$

$\Longrightarrow H \curvearrowright G/K$ is (strongly) visible

Visible actions on symmetric spaces

Theorem (geometry of three involutions '07)

Assume $\begin{cases} G/K & \text{Hermitian symm. sp.} \\ (G,H) & \text{any symmetric pair} \end{cases}$

$\Longrightarrow H \curvearrowright G/K$ is (strongly) visible

$\Downarrow$ Propagation theorem

Thm $V_\lambda = U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} \mathbb{C}_\lambda$ (λ generic) is an algebraic MF direct sum of irreducible $\mathfrak{g}'$-modules if

- nilradical of $\mathfrak{p}_\mathbb{R}$ is abelian
- $G'_\mathbb{R} \cdot o$ is closed in $G_\mathbb{R}/P_\mathbb{R}$
- $(G_\mathbb{R}, G'_\mathbb{R})$ is symmetric pair.

Visible actions on symmetric spaces

Theorem (geometry of three involutions '07)

Assume $\begin{cases} G/K & \text{Hermitian symm. sp.} \\ (G,H) & \text{any symmetric pair} \end{cases}$

$\Longrightarrow H \curvearrowright G/K$ is (strongly) visible

$\Downarrow$ Propagation theorem

Thm $V_\lambda = U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} \mathbb{C}_\lambda$ (λ generic) is an algebraic MF direct sum of irreducible $\mathfrak{g}'$-modules if

- nilradical of $\mathfrak{p}_\mathbb{R}$ is abelian
- $G'_\mathbb{R} \cdot o$ is closed in $G_\mathbb{R}/P_\mathbb{R}$
- $(G_\mathbb{R}, G'_\mathbb{R})$ is symmetric pair.

$G'_\mathbb{R}$	$\subset$	$G_\mathbb{R}$	$\supset$	$P_\mathbb{R}$
subgp		real reductive		real parabolic

Classification theory of visible actions

Methods to find visible action

Want to find visible actions systematically

- Structure theory
  - geometry of three involutions (symmetric case)
  - generalized Cartan decomposition (non-symmetric case)
- Make new from old
  - make ‘large’ from ‘small’
  - make ‘three’ from ‘one’ (triunity)

Classification theory of visible actions

Methods to find visible action

Want to find visible actions systematically

- Structure theory
  - geometry of three involutions (symmetric case)
  - generalized Cartan decomposition (non-symmetric case)
- Make new from old
  - make ‘large’ from ‘small’
  - make ‘three’ from ‘one’ (triunity)

(Generalized) Cartan involutions

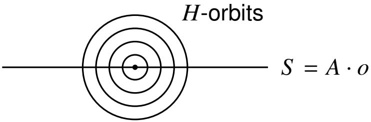
Observation

$$D = G/K$$

Suppose we have a decomposition

$$G = H\ A\ K$$

Set $S := A \cdot o \subset D$



$\Longrightarrow$ S is a candidate of ‘slice’ for (strongly) visible action

Classification theory of visible actions

Grassmannian $U(n)/(U(p) \times U(q)) \simeq Gr_p(\mathbb{C}^n) \quad (n = p + q)$

Ex.(symmetric case) $n_1 + n_2 = p + q = n$
$\Longrightarrow$ $U(n_1) \times U(n_2)$ acts on $Gr_p(\mathbb{C}^n)$
in a strongly visible fashion

Classification theory of visible actions

Grassmannian $U(n)/(U(p) \times U(q)) \simeq Gr_p(\mathbb{C}^n) \quad (n = p + q)$

Ex.(symmetric case) $n_1 + n_2 = p + q = n$
$\Longrightarrow$ $U(n_1) \times U(n_2)$ acts on $Gr_p(\mathbb{C}^n)$
in a strongly visible fashion

Ex.34 (JMSJ 2007)
$n_1 + n_2 + n_3 = p + q = n$
$U(n_1) \times U(n_2) \times U(n_3)$ acts on $Gr_p(\mathbb{C}^n)$
in a strongly visible fashion

Classification theory of visible actions

Grassmannian $U(n)/(U(p) \times U(q)) \simeq Gr_p(\mathbb{C}^n) \quad (n = p + q)$

Ex.(symmetric case) $n_1 + n_2 = p + q = n$
$\Longrightarrow$ $U(n_1) \times U(n_2)$ acts on $Gr_p(\mathbb{C}^n)$
in a strongly visible fashion

Ex.34 (JMSJ 2007)
$n_1 + n_2 + n_3 = p + q = n$
$U(n_1) \times U(n_2) \times U(n_3)$ acts on $Gr_p(\mathbb{C}^n)$
in a strongly visible fashion
$\Longleftrightarrow \min(n_1 + 1, n_2 + 1, n_3 + 1, p, q) \leq 2$

Classification theory of visible actions

Grassmannian $U(n)/(U(p) \times U(q)) \simeq Gr_p(\mathbb{C}^n) \quad (n = p + q)$

Ex.(symmetric case) $\quad n_1 + n_2 = p + q = n$

$\Longrightarrow$ $U(n_1) \times U(n_2)$ acts on $Gr_p(\mathbb{C}^n)$

in a strongly visible fashion

Ex.34 (JMSJ 2007)

$n_1 + n_2 + n_3 = p + q = n$

$U(n_1) \times U(n_2) \times U(n_3)$ acts on $Gr_p(\mathbb{C}^n)$

in a strongly visible fashion

$\Longleftrightarrow \min(n_1 + 1, n_2 + 1, n_3 + 1, p, q) \leq 2$

For type B, C, D and exceptional groups (Y. Tanaka, Tohoku J. (2013), J. Math. Soc. Japan (2013), B. Austrian Math Soc. (2013), J. Algebra (2014))

$\Downarrow$ Propagation theorem

MF property of the following

- $GL_m \times GL_n \curvearrowright S(\mathbb{C}^{mn})$ Ex.16
- $GL_{m-1} \times GL_n \curvearrowright S(\mathbb{C}^{mn})$ Ex.18 (Kac)
- the Stembridge list of $\pi_\lambda \otimes \pi_\nu$ Ex.11
- $GL_n \downarrow (GL_p \times GL_q)$ Ex.12
- $GL_n \downarrow (GL_{n_1} \times GL_{n_2} \times GL_{n_3})$ Ex.13
- ∞-dimensional versions

……

Classification theory of visible actions

Methods to find visible action

Want to find visible actions systematically

- Structure theory
  - geometry of three involutions
  - generalized Cartan decomposition
- Make new from old
  - make ‘large’ from ‘small’
  - make ‘three’ from ‘one’ (triunity)

Classification theory of visible actions

Methods to find visible action

Want to find visible actions systematically

- Structure theory
  - geometry of three involutions
  - generalized Cartan decomposition
- Make new from old
  - make ‘large’ from ‘small’
  - make ‘three’ from ‘one’ (triunity)

Make ‘large’ from ‘small’

Idea: induced action preserving visibility

Make ‘large’ from ‘small’

Idea: induced action preserving visibility

$$H \subset G$$

Make ‘large’ from ‘small’

Idea: induced action preserving visibility

$$H \subset G$$

$H \curvearrowright Y$ visible w.r.t. S

$\rotatebox[origin=c]{-90}{$\rightsquigarrow$}$ $\Leftarrow$ certain assumption

$G \curvearrowright X := G \times_H Y$ visible w.r.t. $S \simeq [\{e\}, S]$

Make ‘large’ from ‘small’

Idea: induced action preserving visibility

$$H \subset G$$

$H \curvearrowright Y$ visible w.r.t. S

$\rightsquigarrow$ ⇐ certain assumption

$G \curvearrowright X := G \times_H Y$ visible w.r.t. $S \simeq [\{e\}, S\,]$

Ex. $H = U(p) \times U(q)$, $Y = M(p, q; \mathbb{C})\ (p \geq q)$

$G = U(p+q)$, $X = T^*(G/H) = T^*(Gr_p(\mathbb{C}^{p+q}))$

Make ‘large’ from ‘small’

Idea: induced action preserving visibility

$$H \subset G$$

$$\begin{tikzcd} H \curvearrowright Y \text{ visible w.r.t. } S \arrow[d, squiggly, "\Leftarrow \text{certain assumption}"] \\ G \curvearrowright X := G \times_H Y \text{ visible w.r.t. } S \simeq [\{e\}, S] \end{tikzcd}$$

Ex. $H = U(p) \times U(q)$, $Y = M(p, q; \mathbb{C})$ $(p \geq q)$

$G = U(p + q)$, $X = T^*(G/H) = T^*(Gr_p(\mathbb{C}^{p+q}))$

$\rightsquigarrow$ momentum map

nilpotent orbit for $GL(p + q, \mathbb{C})$
for partition $(2^q, 1^{p-q})$ is spherical (Panyushev)

Classification theory of visible actions

Methods to find visible action

Want to find visible actions systematically

- Structure theory
  - geometry of three involutions
  - generalized Cartan decomposition
- Make new from old
  - make ‘large’ from ‘small’
  - make ‘three’ from ‘one’ (triunity)

Classification theory of visible actions

Methods to find visible action

Want to find visible actions systematically

- Structure theory
    - geometry of three involutions
    - generalized Cartan decomposition
- Make new from old
    - make ‘large’ from ‘small’
    - make ‘three’ from ‘one’ (triunity)

Examples of (strongly) visible actions

Ex.22 $\mathbb{T} \curvearrowright \mathbb{C} \supset \mathbb{R}$ is visible.

Examples of (strongly) visible actions

<u>Ex.22</u> $\mathbb{T} \curvearrowright \mathbb{C} \supset \mathbb{R}$ is visible.

$$\Downarrow$$

<u>Ex.23</u> $\mathbb{T}^n \curvearrowright \mathbb{C}^n \supset \mathbb{R}^n$ is visible.

Examples of (strongly) visible actions

Ex.22 $\mathbb{T} \curvearrowright \mathbb{C} \supset \mathbb{R}$ is visible.

$$\Downarrow$$

Ex.23 $\mathbb{T}^n \curvearrowright \mathbb{C}^n \supset \mathbb{R}^n$ is visible.

$$\Downarrow$$

Ex.24 $\mathbb{T}^n \curvearrowright \mathbb{P}^{n-1}\mathbb{C} \supset \mathbb{P}^{n-1}\mathbb{R}$ is visible.

Examples of (strongly) visible actions

Ex.22 $\mathbb{T} \curvearrowright \mathbb{C} \supset \mathbb{R}$ is visible.

$\Downarrow$

Ex.23 $\mathbb{T}^n \curvearrowright \mathbb{C}^n \supset \mathbb{R}^n$ is visible.

$\Downarrow$

Ex.24 $\mathbb{T}^n \curvearrowright \mathbb{P}^{n-1}\mathbb{C} \supset \mathbb{P}^{n-1}\mathbb{R}$ is visible.

Ex.25 $U(1) \times U(n-1) \curvearrowright \mathcal{B}_n$ (full flag variety) is visible.

Ex.26 $U(n) \curvearrowright \mathbb{P}^{n-1}\mathbb{C} \times \mathcal{B}_n$ is visible.

Triunity of visible actions

$$\left(\begin{tikzcd}[column sep=small, row sep=small] H & & L \\ & G \arrow[ul, phantom, sloped, "\subset"] \arrow[ur, phantom, sloped, "\supset"] & \\ & {\color{blue}G^{\sigma}} \arrow[u, phantom, sloped, "\subset"] & \end{tikzcd}\right) := \left(\begin{tikzcd}[column sep=small, row sep=small] \mathbb{T}^n & & U(1) \times U(n-1) \\ & U(n) \arrow[ul, phantom, sloped, "\subset"] \arrow[ur, phantom, sloped, "\supset"] & \\ & O(n) \arrow[u, phantom, sloped, "\subset"] & \end{tikzcd}\right)$$

Triunity of visible actions

$$\left(\begin{array}{ccccc} H & & & & L \\ & \rotatebox{-45}{$\subset$} & & \rotatebox{45}{$\supset$} & \\ & & G & & \\ & & \cup & & \\ & & {\color{blue}G^{\sigma}} & & \end{array}\right) := \left(\begin{array}{ccccc} \mathbb{T}^n & & & & U(1) \times U(n-1) \\ & \rotatebox{-45}{$\subset$} & & \rotatebox{45}{$\supset$} & \\ & & U(n) & & \\ & & \cup & & \\ & & O(n) & & {\color{blue}\sigma(g) := \overline{g}} \end{array}\right)$$

Triunity of visible actions

$$\begin{pmatrix} H & & L \\ & \rotatebox{-45}{$\subset$} \quad \rotatebox{45}{$\supset$} & \\ & G & \\ & \cup & \\ & G^{\sigma} & \end{pmatrix} := \begin{pmatrix} \mathbb{T}^n & & U(1) \times U(n-1) \\ \rotatebox{-45}{$\subset$} & & \rotatebox{45}{$\supset$} \\ & U(n) & \\ & \cup & \\ & O(n) & \sigma(g) := \overline{g} \end{pmatrix}$$

Geometry

(visible actions)

$\boxed{\mathbb{P}^{n-1}\mathbb{R} \text{ meets every } \mathbb{T}^n\text{-orbit on } \mathbb{P}^{n-1}\mathbb{C}}$

Triunity of visible actions

$$\left(\begin{array}{ccc} H & & L \\ \subset & & \supset \\ & G & \\ & \cup & \\ & G^{\sigma} & \end{array}\right) := \left(\begin{array}{ccc} \mathbb{T}^n & & U(1)\times U(n-1) \\ \subset & & \supset \\ & U(n) & \\ & \cup & \\ & O(n) & \sigma(g) := \overline{g} \end{array}\right)$$

Geometry
(visible actions)

$$\boxed{\mathbb{P}^{n-1}\mathbb{R} \text{ meets every } \mathbb{T}^n\text{-orbit on } \mathbb{P}^{n-1}\mathbb{C}}$$

$$\updownarrow \quad \underset{G^{\sigma}/G^{\sigma}\cap L}{\|} \qquad \underset{H}{\|} \qquad \underset{G/L}{\|}$$

Group $\boxed{G = HG^{\sigma}L} \Rightarrow H \curvearrowright G/L$

Triunity of visible actions

$$\left(\begin{array}{ccc} H & & L \\ \rotatebox{-45}{$\subset$} & & \rotatebox{45}{$\supset$} \\ & G & \\ & \cup & \\ & G^{\sigma} & \end{array}\right) := \left(\begin{array}{cccc} \mathbb{T}^n & & U(1) \times U(n-1) & \\ \rotatebox{-45}{$\subset$} & & \rotatebox{45}{$\supset$} & \\ & U(n) & & \\ & \cup & & \\ & O(n) & & \sigma(g) := \overline{g} \end{array}\right)$$

Geometry

(visible actions)

$$\boxed{\underset{G^{\sigma}/G^{\sigma} \cap L}{\underset{\parallel}{\mathbb{P}^{n-1}\mathbb{R}}} \text{ meets every } \underset{H}{\underset{\parallel}{\mathbb{T}^n}}\text{-orbit on } \underset{G/L}{\underset{\parallel}{\mathbb{P}^{n-1}\mathbb{C}}}}$$

$\updownarrow$

Group $\boxed{G = HG^{\sigma}L} \Rightarrow H \curvearrowright G/L$

$\updownarrow$

Group $\boxed{G = LG^{\sigma}H} \Rightarrow L \curvearrowright G/H$

Triunity of visible actions

$$\left(\begin{tikzcd} H \arrow[dr, phantom, "\subset" sloped] & & L \arrow[dl, phantom, "\supset" sloped] \\ & G \arrow[d, phantom, "\cup"] & \\ & G^{\sigma} & \end{tikzcd}\right) := \left(\begin{tikzcd} \mathbb{T}^n \arrow[dr, phantom, "\subset" sloped] & & U(1) \times U(n-1) \arrow[dl, phantom, "\supset" sloped] \\ & U(n) \arrow[d, phantom, "\cup"] & \\ & O(n) & \sigma(g) := \overline{g} \end{tikzcd}\right)$$

Geometry

(visible actions)

$$\boxed{\underset{\substack{\parallel \\ G^{\sigma}/G^{\sigma} \cap L}}{\mathbb{P}^{n-1}\mathbb{R}} \text{ meets every } \underset{\substack{\parallel \\ H}}{\mathbb{T}^n}\text{-orbit on } \underset{\substack{\parallel \\ G/L}}{\mathbb{P}^{n-1}\mathbb{C}}}$$

$\updownarrow$

Group $\boxed{G = HG^{\sigma}L} \Rightarrow H \curvearrowright G/L$

$\updownarrow$

Group $\boxed{G = LG^{\sigma}H} \Rightarrow L \curvearrowright G/H$

$\updownarrow$

Group $\boxed{(G \times G) = \text{diag}(G)(G^{\sigma} \times G^{\sigma})(H \times L)} \Rightarrow$ diag.

Examples of visible actions

<u>Ex.22</u> $\mathbb{T} \curvearrowright \mathbb{C} \supset \mathbb{R}$ is visible.

$\Downarrow$

<u>Ex.23</u> $\mathbb{T}^n \curvearrowright \mathbb{C}^n \supset \mathbb{R}^n$ is visible.

$\Downarrow$

<u>Ex.24</u> $\mathbb{T}^n \curvearrowright \mathbb{P}^{n-1}\mathbb{C} \supset \mathbb{P}^{n-1}\mathbb{R}$ is visible.

<u>Ex.25</u> $U(1) \times U(n-1) \curvearrowright \mathcal{B}_n$ (full flag variety) is visible.

<u>Ex.26</u> $U(n) \curvearrowright \mathbb{P}^{n-1}\mathbb{C} \times \mathcal{B}_n$ is visible.

$\Downarrow$ Propagation theorem

Three kinds of MF results:

- (Taylor series) $\mathbb{T}^n \curvearrowright \mathcal{O}(\mathbb{C}^n)$ Ex.2
- ($GL_n \downarrow GL_{n-1}$) Restriction $\pi|_{GL_{n-1}}$ Ex.14
- (Pieri) $\pi \otimes S^k(\mathbb{C}^n)$ Ex.9

$\otimes$-product rep.

$$\lambda = (\underbrace{a, \cdots, a}_{p}, \underbrace{b, \cdots, b}_{q}) \in \mathbb{Z}^n, \ a \geq b$$

Ex.11 (Stembridge 2001, K– 2002)

$\pi_\lambda \otimes \pi_\nu$ is MF as a $GL_n(\mathbb{C})$-module if

1) $\min(a-b, p, q) = 1$ (and ν is any),

or

2) $\min(a-b, p, q) = 2$ and

★ $\boxed{\nu \text{ is of the form } \nu = (\underbrace{x \cdots x}_{n_1} \ \underbrace{y \cdots y}_{n_2} \ \underbrace{z \cdots z}_{n_3}) \ (x \geq y \geq z)}$,

or

3) $\min(a-b, p, q) \geq 3$, ★ &

$\min(x-y, \ y-z, n_1, n_2, n_3) = 1.$

Restriction $(GL_n \downarrow G_{n_1} \times GL_{n_2} \times GL_{n_3})$

Ex.12 $(GL_n \downarrow (GL_p \times GL_q)) \quad n = p + q$

$$\pi^{GL_n}_{(\underbrace{x,\dots,x}_{n_1},\underbrace{y,\dots,y}_{n_2},\underbrace{z,\dots,z}_{n_3})}|_{GL_p \times GL_q}$$ is MF

if $\min(p, q) \leq 2$ or

if $\min(n_1, n_2, n_3, x - y, y - z) \leq 1$

(Kostant $n_3 = 0$; Krattenthaler 1998)

Ex.13 $(GL_n \downarrow (GL_{n_1} \times GL_{n_2} \times GL_{n_3}))$ ([5])

$n = n_1 + n_2 + n_3$

$$\pi^{GL_n}_{(\underbrace{a,\dots,a}_{p},\underbrace{b,\dots,b}_{q})}|_{GL_{n_1} \times GL_{n_2} \times GL_{n_3}}$$ is MF

if $\min(n_1, n_2, n_3) \leq 1$ or

if $\min(p, q, a - b) \leq 2$

Plan of Lecture 5

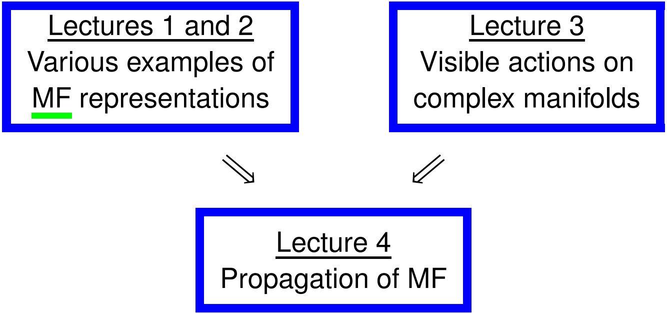
<u>Lectures 1 and 2</u>
Various examples of
MF representations

Plan of Lecture 5

<u>Lectures 1 and 2</u>
Various examples of
MF representations

<u>Lecture 3</u>
Visible actions on
complex manifolds

Plan of Lecture 5



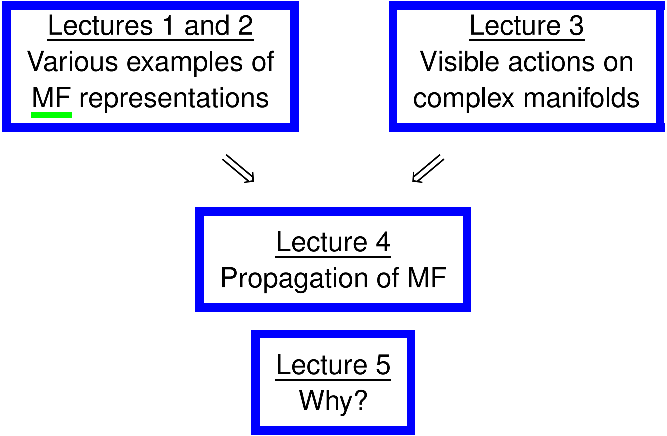
Lectures 1 and 2
Various examples of
MF representations

Lecture 3
Visible actions on
complex manifolds

⇘ ⇙

Lecture 4
Propagation of MF

Plan of Lecture 5



Lectures 1 and 2
Various examples of
MF representations

Lecture 3
Visible actions on
complex manifolds

⇘ ⇙

Lecture 4
Propagation of MF

Lecture 5
Why?

Strategy of proof

$\mathcal{V} \to D$ G-equivariant holomorphic vector bundle

$\rightsquigarrow G \curvearrowright \mathcal{O}(D, \mathcal{V}) \supset \underset{\text{Hilbert space}}{\mathcal{H}}$

Strategy of proof

$\mathcal{V} \to D$ G-equivariant holomorphic vector bundle

$\rightsquigarrow G \curvearrowright \mathcal{O}(D, \mathcal{V}) \supset \underset{\text{Hilbert space}}{\mathcal{H}}$

Goal Prove that $\mathrm{End}_G(\mathcal{H})$ is commutative
under the assumptions

(base space)	$G \curvearrowright D$	strongly visible
(fiber)	$G_x \curvearrowright \mathcal{V}_x$	multiplicity-free

Strategy of proof

$\mathcal{V} \to D$ G-equivariant holomorphic vector bundle

$\rightsquigarrow G \curvearrowright \mathcal{O}(D, \mathcal{V}) \quad \supset \quad \mathcal{H}$

Hilbert space

Goal Prove that $\mathrm{End}_G(\mathcal{H})$ is commutative
under the assumptions
(base space) $G \curvearrowright D$ strongly visible
(fiber) $G_x \curvearrowright \mathcal{V}_x$ multiplicity-free

Idea Construct a bijection $T^\# : \mathrm{End}_G(\mathcal{H}) \to \mathrm{End}_G(\mathcal{H})$
such that

$$T^\#(ab) = T^\#(a)T^\#(b)$$

and simultaneously

$$T^\#(ab) = T^\#(b)T^\#(a)$$

for all $a, b \in \mathrm{End}_G(\mathcal{H})$

Construction of $T^{\#}$

$\sigma : D \to D$ anti-holomorphic diffeo

$\rightsquigarrow$ $f \circ \sigma$ is anti-holomorphic if f is holomorphic

$\rightsquigarrow$ $\overline{f \circ \sigma}$ is holomorphic

Construction of $T^{\#}$

$\sigma : D \to D$ anti-holomorphic diffeo

$\rightsquigarrow$ $f \circ \sigma$ is anti-holomorphic if f is holomorphic

$\rightsquigarrow$ $\overline{f \circ \sigma}$ is holomorphic

$$T : \mathcal{O}(D, \mathcal{V}) \to \mathcal{O}(D, \mathcal{V}), \quad f \mapsto \overline{f \circ \sigma}$$

Construction of $T^\#$

$\sigma : D \to D$ anti-holomorphic diffeo

$\rightsquigarrow$ $f \circ \sigma$ is anti-holomorphic if f is holomorphic

$\rightsquigarrow$ $\overline{f \circ \sigma}$ is holomorphic

$$T : \begin{array}{c} \mathcal{O}(D,\mathcal{V}) \\ \cup \\ \mathcal{H} \end{array} \to \mathcal{O}(D,\mathcal{V}), \quad f \mapsto \overline{f \circ \sigma}$$

Hilbert space

Construction of $T^\#$

$\sigma : D \to D$ anti-holomorphic diffeo

$\rightsquigarrow$ $f \circ \sigma$ is anti-holomorphic if f is holomorphic

$\rightsquigarrow$ $\overline{f \circ \sigma}$ is holomorphic

$$\begin{array}{c} T : \mathcal{O}(D, \mathcal{V}) \to \mathcal{O}(D, \mathcal{V}), \quad f \mapsto \overline{f \circ \sigma} \\ \cup \\ \mathcal{H} \end{array}$$

Hilbert space

<u>Step 1</u> Define another Hilbert space $\widetilde{\mathcal{H}} := T(\mathcal{H})$.

$\Longrightarrow$ If G acts unitarily on $\mathcal{H}$, then so does G on $\widetilde{\mathcal{H}}$.

Construction of $T^{\#}$

$\sigma : D \to D$ anti-holomorphic diffeo

$\rightsquigarrow$ $f \circ \sigma$ is anti-holomorphic if f is holomorphic

$\rightsquigarrow$ $\overline{f \circ \sigma}$ is holomorphic

$$T : \underset{\underset{\underset{\text{Hilbert space}}{\mathcal{H}}}{\cup}}{\mathcal{O}(D, \mathcal{V})} \to \mathcal{O}(D, \mathcal{V}), \quad f \mapsto \overline{f \circ \sigma}$$

Step 1 Define another Hilbert space $\widetilde{\mathcal{H}} := T(\mathcal{H})$.

$\Longrightarrow$ If G acts unitarily on $\mathcal{H}$, then so does G on $\widetilde{\mathcal{H}}$.

Step 2 $\widetilde{\mathcal{H}} = \mathcal{H}$

Construction of $T^{\#}$

$\sigma : D \to D$ anti-holomorphic diffeo

$\rightsquigarrow$ $f \circ \sigma$ is anti-holomorphic if f is holomorphic

$\rightsquigarrow$ $\overline{f \circ \sigma}$ is holomorphic

$$\begin{array}{l} T : \mathcal{O}(D, \mathcal{V}) \to \mathcal{O}(D, \mathcal{V}), \quad f \mapsto \overline{f \circ \sigma} \\ \qquad\quad \cup \\ \qquad\quad \mathcal{H} \end{array}$$

Hilbert space

Step 1 Define another Hilbert space $\widetilde{\mathcal{H}} := T(\mathcal{H})$.

$\Longrightarrow$ If G acts unitarily on $\mathcal{H}$, then so does G on $\widetilde{\mathcal{H}}$.

Step 2 $\widetilde{\mathcal{H}} = \mathcal{H}$

Step 3 Define $T^{\#} : \mathrm{End}_{\mathbb{C}}(\mathcal{H}) \to \mathrm{End}_{\mathbb{C}}(\mathcal{H}),\ A \mapsto T \circ A \circ T^{-1}$.

$\Longrightarrow$ $T^{\#}(A) = A^*$ if $A \in \mathrm{End}_G(\mathcal{H})$.

Reproducing kernel

Prototype (Scalar valued) holomorphic functions

D : complex manifold

Reproducing kernel

Prototype (Scalar valued) holomorphic functions

D : complex manifold

$$\underset{\text{Hilbert space}}{\mathcal{H}} \subset \underset{\{\text{holomorphic functions on } D\}}{\mathcal{O}(D)}$$

Reproducing kernel

Prototype (Scalar valued) holomorphic functions

D : complex manifold

$$\underset{\text{Hilbert space}}{\mathcal{H}} \subset \underset{\{\text{holomorphic functions on } D\}}{\mathcal{O}(D)}$$

Definition (reproducing kernel)

Let $\{\varphi_l\}$ be an orthonormal basis of $\mathcal{H}$.

$$K_{\mathcal{H}}(z,w) := \sum_l \varphi_l(z)\overline{\varphi_l(w)}$$

is independent of the choice of the basis.

Examples of reproducing kernels

$$\{\varphi_j\} \quad \subset \quad \mathcal{H} \quad \subset \quad \mathcal{O}(D)$$

orthonormal basis   Hilbert space

$$K_{\mathcal{H}}(z,w) = \sum_{l} \varphi_l(z)\overline{\varphi_l(w)}$$

Examples of reproducing kernels

$$\{\varphi_j\} \quad \subset \quad \mathcal{H} \quad \subset \quad \mathcal{O}(D)$$

orthonormal basis   Hilbert space

$$K_{\mathcal{H}}(z,w) = \sum_{l} \varphi_l(z)\overline{\varphi_l(w)}$$

Example 1 (weighted Bergman space)

$$D := \{z \in \mathbb{C} : |z| < 1\}$$

Fix $\lambda > 1$.

$$\mathcal{H}_\lambda := \{f \in \mathcal{O}(D) : \|f\|_\lambda < \infty\}$$

$$\|f\|_\lambda := \left(\int_D |f(x+iy)|^2(1-x^2-y^2)^{\lambda-2}dxdy\right)^{\frac{1}{2}}$$

Examples of reproducing kernels

$$\{\varphi_j\} \quad \subset \quad \mathcal{H} \quad \subset \quad \mathcal{O}(D)$$

orthonormal basis Hilbert space

$$K_{\mathcal{H}}(z,w) = \sum_l \varphi_l(z)\overline{\varphi_l(w)}$$

Example 1 (weighted Bergman space)

$$D := \{z \in \mathbb{C} : |z| < 1\}$$

Fix $\lambda > 1$.

$$\mathcal{H}_\lambda := \{f \in \mathcal{O}(D) : \|f\|_\lambda < \infty\}$$

$$\|f\|_\lambda := \left(\int_D |f(x+iy)|^2(1-x^2-y^2)^{\lambda-2}dxdy\right)^{\frac{1}{2}}$$

$$(z^l, z^m)_\lambda = \frac{\pi\Gamma(\lambda-1)}{\Gamma(\lambda+l)}\delta_{lm}$$

Examples of reproducing kernels

$$\{\varphi_j\} \quad\subset\quad \mathcal{H} \quad\subset\quad \mathcal{O}(D)$$

orthonormal basis Hilbert space

$$K_{\mathcal{H}}(z,w) = \sum_l \varphi_l(z)\overline{\varphi_l(w)}$$

<u>Example 1</u> (weighted Bergman space)

$$D := \{z \in \mathbb{C} : |z| < 1\}$$

Fix $\lambda > 1$.

$$\mathcal{H}_\lambda := \{f \in \mathcal{O}(D) : \|f\|_\lambda < \infty\}$$

$$\|f\|_\lambda := \left(\int_D |f(x+iy)|^2(1-x^2-y^2)^{\lambda-2}dxdy\right)^{\frac{1}{2}}$$

$$(z^l, z^m)_\lambda = \frac{\pi\Gamma(\lambda-1)}{\Gamma(\lambda+l)}\delta_{lm}$$

$$K_{\mathcal{H}_\lambda}(z,w) = \sum_{l=0}^{\infty} \frac{\Gamma(\lambda+l)}{\pi\Gamma(\lambda-1)} z^l \overline{w}^l$$

$$= \frac{\lambda-1}{\pi}(1-z\overline{w})^{-\lambda}$$

Examples of reproducing kernels

$$\{\varphi_j\} \quad \subset \quad \mathcal{H} \quad \subset \quad \mathcal{O}(D)$$

orthonormal basis   Hilbert space

$$K_{\mathcal{H}}(z,w) = \sum_l \varphi_l(z)\overline{\varphi_l(w)}$$

Example 2 (Fock space)

$D = \mathbb{C}$

$\mathcal{F} := \{f \in \mathcal{O}(\mathbb{C}) : \|f\|_{\mathcal{F}} < \infty\}$

$$\|f\|_{\mathcal{F}} := \left(\int_{\mathbb{C}} |f(x+iy)|^2 e^{-x^2-y^2} dxdy\right)^{\frac{1}{2}}$$

Examples of reproducing kernels

$$\{\varphi_j\} \quad \subset \quad \mathcal{H} \quad \subset \quad \mathcal{O}(D)$$

orthonormal basis   Hilbert space

$$K_{\mathcal{H}}(z,w) = \sum_l \varphi_l(z)\overline{\varphi_l(w)}$$

<u>Example 2</u> (Fock space)

$$D = \mathbb{C}$$

$$\mathcal{F} := \{f \in \mathcal{O}(\mathbb{C}) : \|f\|_{\mathcal{F}} < \infty\}$$

$$\|f\|_{\mathcal{F}} := \left(\int_{\mathbb{C}} |f(x+iy)|^2 e^{-x^2-y^2} dxdy\right)^{\frac{1}{2}}$$

$$(z^l, z^m)_{\mathcal{F}} = \pi l!\, \delta_{lm}$$

Examples of reproducing kernels

$$\{\varphi_j\} \quad \subset \quad \mathcal{H} \quad \subset \quad \mathcal{O}(D)$$

orthonormal basis   Hilbert space

$$K_{\mathcal{H}}(z,w) = \sum_l \varphi_l(z)\overline{\varphi_l(w)}$$

Example 2 (Fock space)

$$D = \mathbb{C}$$

$$\mathcal{F} := \{f \in \mathcal{O}(\mathbb{C}) : \|f\|_{\mathcal{F}} < \infty\}$$

$$\|f\|_{\mathcal{F}} := \left(\int_{\mathbb{C}} |f(x+iy)|^2 e^{-x^2-y^2}\,dxdy\right)^{\frac{1}{2}}$$

$$(z^l, z^m)_{\mathcal{F}} = \pi l!\,\delta_{lm}$$

$$K_{\mathcal{F}}(z,w) = \sum_{l=0}^{\infty} \frac{z^l\overline{w}^l}{\pi l!}$$

$$= \frac{1}{\pi} e^{z\overline{w}}$$

Properties of reproducing kernel

$$\begin{array}{c} \mathcal{O}(D) \\ \cup \\ \mathcal{H} \end{array} \quad \rightsquigarrow K_{\mathcal{H}}(z, w) = \sum_j \varphi(x)\overline{\varphi(w)}$$

Hilbert space

Properties of reproducing kernel

$$\begin{array}{c}\mathcal{O}(D)\\ \cup\\ \mathcal{H}\end{array} \quad \rightsquigarrow K_{\mathcal{H}}(z,w) = \sum_{j} \varphi(x)\overline{\varphi(w)}$$

Hilbert space

- $K_{\mathcal{H}}(z,w)$ is holomorphic in z; anti-holomorphic in w

Properties of reproducing kernel

$$\begin{array}{c}\mathcal{O}(D)\\ \cup\\ \mathcal{H}\end{array} \quad \rightsquigarrow K_{\mathcal{H}}(z,w) = \sum_{j} \varphi(x)\overline{\varphi(w)}$$

Hilbert space

- $K_{\mathcal{H}}(z,w)$ is holomorphic in z; anti-holomorphic in w
- $K_{\mathcal{H}}(z,w)$ recovers the Hilbert space $\mathcal{H}$
  (i.e. subspace of $\mathcal{O}(D)$ & inner product on $\mathcal{H}$)

Properties of reproducing kernel

$$\begin{array}{c}\mathcal{O}(D)\\ \cup\\ \mathcal{H}\end{array} \quad \rightsquigarrow K_{\mathcal{H}}(z,w) = \sum_j \varphi(x)\overline{\varphi(w)}$$

Hilbert space

- $K_{\mathcal{H}}(z,w)$ is holomorphic in z; anti-holomorphic in w
- $K_{\mathcal{H}}(z,w)$ recovers the Hilbert space $\mathcal{H}$
  (i.e. subspace of $\mathcal{O}(D)$ & inner product on $\mathcal{H}$)

Corollary Suppose a group G acts on D as biholomorphic transformations. Then G acts on $\mathcal{H}$ as a unitary representation if and only if

$$K_{\mathcal{H}}(gz,gw) = K_{\mathcal{H}}(z,w) \quad {}^{\forall}g \in G, {}^{\forall}z, {}^{\forall}w \in D. \qquad (\star)$$

Properties of reproducing kernel

$$\begin{array}{c} \mathcal{O}(D) \\ \cup \\ \mathcal{H} \end{array} \qquad \rightsquigarrow K_{\mathcal{H}}(z,w) = \sum_{j} \varphi(x)\overline{\varphi(w)}$$

Hilbert space

- $K_{\mathcal{H}}(z,w)$ is holomorphic in z; anti-holomorphic in w
- $K_{\mathcal{H}}(z,w)$ recovers the Hilbert space $\mathcal{H}$
  (i.e. subspace of $\mathcal{O}(D)$ & inner product on $\mathcal{H}$)

<ins>Corollary</ins> Suppose a group G acts on D
as biholomorphic transformations.
Then G acts on $\mathcal{H}$ as a unitary representation
if and only if

$$K_{\mathcal{H}}(gz,gw) = K_{\mathcal{H}}(z,w) \quad {}^{\forall}g \in G, {}^{\forall}z, {}^{\forall}w \in D. \qquad (\star)$$

$$(\star) \Longleftrightarrow K_{\mathcal{H}}(gz,gz) = K_{\mathcal{K}}(z,z) \quad {}^{\forall}g \in G, {}^{\forall}z \in D$$

Scalar-valued reproducing kernel

$$\underset{\text{Hilbert space}}{\mathcal{H}} \subset \mathcal{O}(D)$$

Assume that for each $x \in D$,

$$\begin{array}{cccccl} \mathrm{ev}_x : & \mathcal{H} & \to & \mathbb{C} & & \text{is continuous.} \\ & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} & & \\ & f & \mapsto & f(x) & & \end{array}$$

Scalar-valued reproducing kernel

$$\underset{\text{Hilbert space}}{\mathcal{H}} \subset \mathcal{O}(D)$$

Assume that for each $x \in D$,

$$\begin{array}{cccccl} \mathrm{ev}_x : & \mathcal{H} & \to & \mathbb{C} & & \text{is continuous.} \\ & \Psi & & \Psi & & \\ & f & \mapsto & f(x) & & \end{array}$$

$\mathrm{ev}_x^* : \mathbb{C} \to \mathcal{H}$ adjoint

Scalar-valued reproducing kernel

$$\underset{\text{Hilbert space}}{\mathcal{H}} \subset \mathcal{O}(D)$$

Assume that for each $x \in D$,

$$\begin{array}{cccccl} \mathrm{ev}_x : & \mathcal{H} & \rightarrow & \mathbb{C} & & \text{is continuous.} \\ & \rotatebox[origin=c]{90}{$\in$} & & \rotatebox[origin=c]{90}{$\in$} & & \\ & f & \mapsto & f(x) & & \end{array}$$

$\mathrm{ev}_x^* : \mathbb{C} \rightarrow \mathcal{H}$ adjoint

$$\begin{aligned} K_{\mathcal{H}}(z, w) &= \mathrm{ev}_w \circ \mathrm{ev}_z^* \\ &= \sum_j \varphi_j(z)\overline{\varphi_j(w)} \end{aligned}$$

Operator-valued reproducing kernel

$\mathcal{V} \to D$: holomorphic vector bundle

$\mathcal{H} \underset{\text{Hilbert space}}{\hookrightarrow} \mathcal{O}(D, \mathcal{V})$

Assume that for each $x \in D$,

$$\begin{array}{cccccl} \mathrm{ev}_x : & \mathcal{H} & \to & \mathcal{V}_x & \text{is continuous} \\ & \Psi & & \Psi & \\ & f & \mapsto & f(x) & \end{array}$$

Operator-valued reproducing kernel

$\mathcal{V} \to D$: holomorphic vector bundle

$\mathcal{H} \underset{\text{Hilbert space}}{\hookrightarrow} \mathcal{O}(D, \mathcal{V})$

Assume that for each $x \in D$,

$$\begin{array}{cccccc} \mathrm{ev}_x : & \mathcal{H} & \to & \mathcal{V}_x & \text{is continuous} \\ & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} & \\ & f & \mapsto & f(x) & \end{array}$$

$$\mathrm{ev}_x^* : \quad \mathcal{V}_x^* \quad \to \quad \mathcal{H} \qquad \text{adjoint}$$

Operator-valued reproducing kernel

$\mathcal{V} \to D$: holomorphic vector bundle

$\mathcal{H} \underset{\text{Hilbert space}}{\hookrightarrow} \mathcal{O}(D, \mathcal{V})$

Assume that for each $x \in D$,

$$\begin{array}{ccccc} \mathrm{ev}_x : & \mathcal{H} & \to & \mathcal{V}_x & \text{is continuous} \\ & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} & \\ & f & \mapsto & f(x) & \end{array}$$

$$\mathrm{ev}_x^* : \quad \mathcal{V}_x^* \quad \to \quad \mathcal{H} \quad \text{adjoint}$$

$$K_{\mathcal{H}}(x, y) := \mathrm{ev}_y \circ \mathrm{ev}_x^* \in \mathrm{Hom}_{\mathbb{C}}(\mathcal{V}_x^*, \mathcal{V}_y)$$

operator-valued reproducing kernel

Operator-valued reproducing kernel

$$\mathcal{V} = \coprod_{x \in D} \mathcal{V}_x \longrightarrow D$$

$$\mathcal{H}om(\mathcal{V}^*, \mathcal{V}) = \coprod_{x,y} \mathrm{Hom}(\mathcal{V}_x^*, \mathcal{V}_y) \longrightarrow D \times D$$

Operator-valued reproducing kernel

$$\mathcal{V} = \coprod_{x \in D} \mathcal{V}_x \longrightarrow D$$

$$\mathcal{H}om(\mathcal{V}^*, \mathcal{V}) = \coprod_{x,y} \mathrm{Hom}(\mathcal{V}_x^*, \mathcal{V}_y) \longrightarrow D \times D$$

$$\boxed{\underset{\text{Hilbert space}}{\mathcal{H} \qquad \hookrightarrow} \qquad \mathcal{O}(D, \mathcal{V})}$$

Operator-valued reproducing kernel

$$\mathcal{V} = \coprod_{x \in D} \mathcal{V}_x \longrightarrow D$$

$$\mathcal{H}om(\mathcal{V}^*, \mathcal{V}) = \coprod_{x,y} \mathrm{Hom}(\mathcal{V}_x^*, \mathcal{V}_y) \longrightarrow D \times D$$

$$\mathcal{H} \underset{\text{Hilbert space}}{\hookrightarrow} \mathcal{O}(D, \mathcal{V})$$

$\Updownarrow$ one to one

$K_{\mathcal{H}} \in \mathcal{O}(D \times D, \mathcal{H}om(\mathcal{V}^*, \mathcal{V}))$

positive definite operator-valued reproducing kernel

Operator-valued reproducing kernel

$$\mathcal{V} = \coprod_{x \in D} \mathcal{V}_x \longrightarrow D$$

$$\mathcal{H}om(\mathcal{V}^*, \mathcal{V}) = \coprod_{x,y} \mathrm{Hom}(\mathcal{V}_x^*, \mathcal{V}_y) \longrightarrow D \times D$$

$$\boxed{\mathcal{H} \underset{\text{Hilbert space}}{\hookrightarrow} \mathcal{O}(D, \mathcal{V})}$$

$\Updownarrow$ one to one

$K_{\mathcal{H}} \in \mathcal{O}(D \times D, \mathcal{H}om(\mathcal{V}^*, \mathcal{V}))$
positive definite operator-valued reproducing kernel

$\mathcal{H}$: unitarity, irreducibility, MF, $\ldots \Longleftrightarrow$ Properties on $K_{\mathcal{H}}$

Proof of $\widetilde{\mathcal{H}} = \mathcal{H}$ (Step 2)

$$\mathcal{O}(D, \mathcal{V}) \supset \mathcal{H} \rightsquigarrow \widetilde{\mathcal{H}} := \{\overline{f \circ \sigma} : f \in \mathcal{H}\} \subset \mathcal{O}(D, \mathcal{V})$$

Proof of $\widetilde{\mathcal{H}} = \mathcal{H}$ (Step 2)

$$\mathcal{O}(D, \mathcal{V}) \supset \mathcal{H} \rightsquigarrow \widetilde{\mathcal{H}} := \{\overline{f \circ \sigma} : f \in \mathcal{H}\} \subset \mathcal{O}(D, \mathcal{V})$$

For simplicity, consider the trivial line bundle $\mathcal{V} = D \times \mathbb{C}$.

Proof of $\widetilde{\mathcal{H}} = \mathcal{H}$ (Step 2)

$$\mathcal{O}(D, \mathcal{V}) \supset \mathcal{H} \rightsquigarrow \widetilde{\mathcal{H}} := \{\overline{f \circ \sigma} : f \in \mathcal{H}\} \subset \mathcal{O}(D, \mathcal{V})$$

For simplicity, consider the trivial line bundle $\mathcal{V} = D \times \mathbb{C}$.

<u>Assumption</u> $G \curvearrowright D$ strongly visible w.r.t. a slice S.

Proof of $\widetilde{\mathcal{H}} = \mathcal{H}$ (Step 2)

$$\mathcal{O}(D, \mathcal{V}) \supset \mathcal{H} \rightsquigarrow \widetilde{\mathcal{H}} := \{\overline{f \circ \sigma} : f \in \mathcal{H}\} \subset \mathcal{O}(D, \mathcal{V})$$

For simplicity, consider the trivial line bundle $\mathcal{V} = D \times \mathbb{C}$.

<u>Assumption</u> $G \curvearrowright D$ strongly visible w.r.t. a slice S.

Want to show $\mathcal{H} = \widetilde{\mathcal{H}}$

Proof of $\widetilde{\mathcal{H}} = \mathcal{H}$ (Step 2)

$$\mathcal{O}(D, \mathcal{V}) \supset \mathcal{H} \rightsquigarrow \widetilde{\mathcal{H}} := \{\overline{f \circ \sigma} : f \in \mathcal{H}\} \subset \mathcal{O}(D, \mathcal{V})$$

For simplicity, consider the trivial line bundle $\mathcal{V} = D \times \mathbb{C}$.

<u>Assumption</u> $G \curvearrowright D$ strongly visible w.r.t. a slice S.

Want to show $\mathcal{H} = \widetilde{\mathcal{H}}$

$\iff K_{\mathcal{H}}(z, w) = K_{\widetilde{\mathcal{H}}}(z, w), \quad {}^{\forall}z, {}^{\forall}w = D$

Proof of $\widetilde{\mathcal{H}} = \mathcal{H}$ (Step 2)

$$\mathcal{O}(D, \mathcal{V}) \supset \mathcal{H} \rightsquigarrow \widetilde{\mathcal{H}} := \{\overline{f \circ \sigma} : f \in \mathcal{H}\} \subset \mathcal{O}(D, \mathcal{V})$$

For simplicity, consider the trivial line bundle $\mathcal{V} = D \times \mathbb{C}$.

Assumption $G \curvearrowright D$ strongly visible w.r.t. a slice S.

Want to show $\mathcal{H} = \widetilde{\mathcal{H}}$

$$\iff K_{\mathcal{H}}(z, w) = K_{\widetilde{\mathcal{H}}}(z, w), \quad {}^{\forall}z, {}^{\forall}w = D$$

$$\iff K_{\mathcal{H}}(z, z) = K_{\widetilde{\mathcal{H}}}(z, z), \quad {}^{\forall}z = D$$

Proof of $\widetilde{\mathcal{H}} = \mathcal{H}$ (Step 2)

$$\mathcal{O}(D,\mathcal{V}) \supset \mathcal{H} \rightsquigarrow \widetilde{\mathcal{H}} := \{\overline{f \circ \sigma} : f \in \mathcal{H}\} \subset \mathcal{O}(D,\mathcal{V})$$

For simplicity, consider the trivial line bundle $\mathcal{V} = D \times \mathbb{C}$.

<u>Assumption</u> $G \curvearrowright D$ strongly visible w.r.t. a slice S.

Want to show $\mathcal{H} = \widetilde{\mathcal{H}}$

$$\iff K_{\mathcal{H}}(z,w) = K_{\widetilde{\mathcal{H}}}(z,w), \quad {}^{\forall}z, {}^{\forall}w = D$$

$$\iff K_{\mathcal{H}}(z,z) = K_{\widetilde{\mathcal{H}}}(z,z), \quad {}^{\forall}z = D$$

Write $z = g \cdot s \quad ({}^{\exists}g \in G,\ {}^{\exists}s \in S)$

- $K_{\mathcal{H}}(z,z) = K_{\mathcal{H}}(s,s)$.

Proof of $\widetilde{\mathcal{H}} = \mathcal{H}$ (Step 2)

$$\mathcal{O}(D, \mathcal{V}) \supset \mathcal{H} \rightsquigarrow \widetilde{\mathcal{H}} := \{\overline{f \circ \sigma} : f \in \mathcal{H}\} \subset \mathcal{O}(D, \mathcal{V})$$

For simplicity, consider the trivial line bundle $\mathcal{V} = D \times \mathbb{C}$.

<u>Assumption</u> $G \curvearrowright D$ strongly visible w.r.t. a slice S.

Want to show $\mathcal{H} = \widetilde{\mathcal{H}}$

$\Longleftrightarrow K_{\mathcal{H}}(z, w) = K_{\widetilde{\mathcal{H}}}(z, w), \quad {}^{\forall}z, {}^{\forall}w = D$

$\Longleftrightarrow K_{\mathcal{H}}(z, z) = K_{\widetilde{\mathcal{H}}}(z, z), \quad {}^{\forall}z = D$

Write $z = g \cdot s \quad ({}^{\exists}g \in G, {}^{\exists}s \in S)$

- $K_{\mathcal{H}}(z, z) = K_{\mathcal{H}}(s, s)$.

Note that $\sigma(z) = \sigma(g) \cdot s$ because $\sigma|_S = \mathrm{id}$.

- $K_{\widetilde{\mathcal{H}}}(z, z) = K_{\mathcal{H}}(\sigma(z), \sigma(z)) = K_{\mathcal{H}}(s, s)$.

‘Visible’ approach to multiplicity-free theorems

Theorem

$$\boxed{\text{fiber} \xrightarrow{\text{visible action}} \text{sections}}$$

‘Visible’ approach to multiplicity-free theorems

Thm (K– ’08) $\pi|_H$ is multiplicity-free if
π: highest wt. rep. of scalar type
(G, H): semisimple symmetric pair
(Hua, Kostant, Schmid, K– : explicit formula)

Fact (É. Cartan ’29, I. M. Gelfand ’50)
$L^2(G/K)$ is multiplicity-free

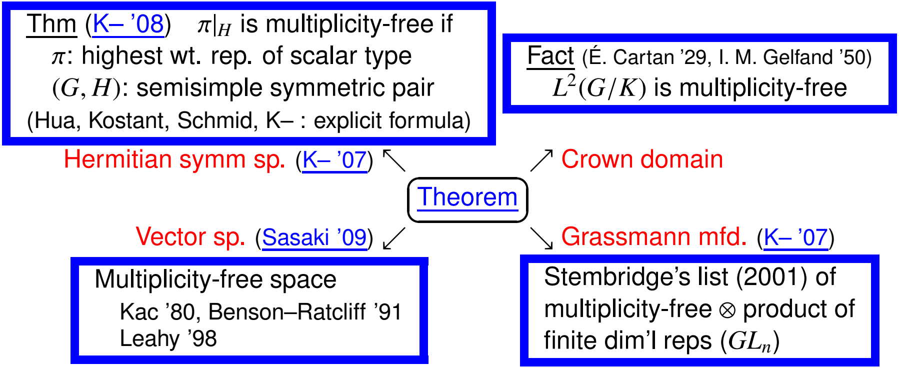
Theorem

Multiplicity-free space
Kac ’80, Benson–Ratcliff ’91
Leahy ’98

Stembridge’s list (2001) of multiplicity-free ⊗ product of finite dim’l reps (GL_n)

fiber $\xrightarrow{\text{visible action}}$ sections

‘Visible’ approach to multiplicity-free theorems



Thm (K– ’08) $\pi|_H$ is multiplicity-free if
π: highest wt. rep. of scalar type
(G, H): semisimple symmetric pair
(Hua, Kostant, Schmid, K– : explicit formula)

Fact (É. Cartan ’29, I. M. Gelfand ’50)
$L^2(G/K)$ is multiplicity-free

Hermitian symm sp. (K– ’07) ↖

↗ Crown domain

Theorem

Vector sp. (Sasaki ’09) ↙

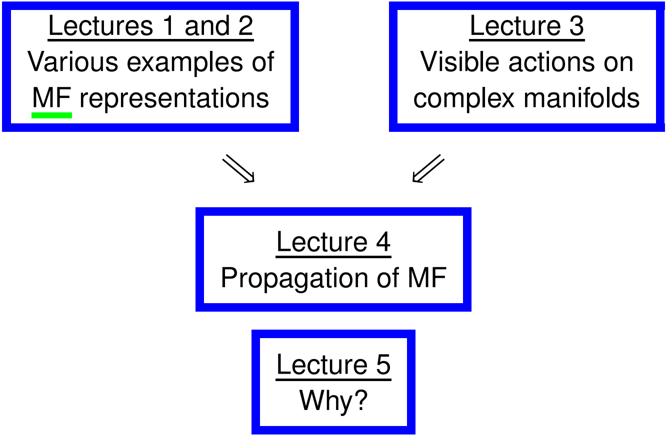
↘ Grassmann mfd. (K– ’07)

Multiplicity-free space
Kac ’80, Benson–Ratcliff ’91
Leahy ’98

Stembridge’s list (2001) of multiplicity-free ⊗ product of finite dim’l reps (GL_n)

$$\text{fiber} \xrightarrow{\text{visible action}} \text{sections}$$

Plan of Lecture 5



Lectures 1 and 2
Various examples of
MF representations

Lecture 3
Visible actions on
complex manifolds

⇘ ⇙

Lecture 4
Propagation of MF

Lecture 5
Why?

Thank you very much!

‘Visible’ approach

To give a simple principle that explains the property MF for both finite and infinite dimensional reps

MF (multiplicity-free) theorem

Propagation of MF property
from fiber to sections

$\Uparrow$

Visible actions on complex mfds

Analysis of group action with infinitely many orbits

References

[1] Proceedings of Representation Theory held at Saga, Kyushu, 1997 (K. Mimachi, ed.), 1997, pp. 9–17.

[2] Ann. of Math. (2) **147** (1998), 709–729.

[3] Proc. of ICM 2002, Beijing, vol. 2, 2002, pp. 615–627.

[4] (with S. Nasrin) Adv. in Math. Sci. 2 **210** (S. Gindikin, ed.) (2003), 161–169, special volume in memory of Professor F. Karpelevič.

[5] Acta Appl. Math. **81** (2004), 129–146.

[6] Publ. Res. Inst. Math. Sci. **41** (2005), 497–549, special issue commemorating the 40th anniversary of the founding of RIMS.

[7] J. Math. Soc. Japan **59** (2007), 669–691.

[8] Progr. Math., vol. 255, pp. 45–109, Birkhäuser, 2008.

[9] Progr. Math (2013).

[10] Transformation Groups **12** (2007), 671–694.

Thank you !!

(Short story by Soseki, 1908)

“He uses the hammer and chisel without any forethought, and he can make the eyebrows and nose as live.”

“Ah, they are not made by hammer and chisel. The eyebrows and nose are buried inside the tree. It is exactly the same as digging a rock out of the earth — there is no way to mistake.”

$\overset{\exists 1}{\hookleftarrow}$