Higher symmetries of the conformal Laplacian

Joint work with J.-P. Michel and J. Silhan

Varna, June 2014

Introduction

- On $(\mathbb{R}^2, \mathrm{g}_0)$, we consider the Helmholtz equation

$$\Delta\phi = E\phi,$$

where

$$\Delta = \partial_x^2 + \partial_y^2, \quad E \in \mathbb{R}.$$

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- Coordinates (u, v) separate this equation $\iff \exists$ solution of the form $f(u)g(v)$
- Coordinates (u, v) orthogonal $\iff \mathrm{g}_0(\partial_u, \partial_v) = 0$

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  4. Elliptic coordinates (α,β) :

$$\left\{\begin{array}{lcl} x & = & \sqrt{d}\cos(\alpha)\cosh(\beta) \\ y & = & \sqrt{d}\sin(\alpha)\sinh(\beta) \end{array}\right.$$

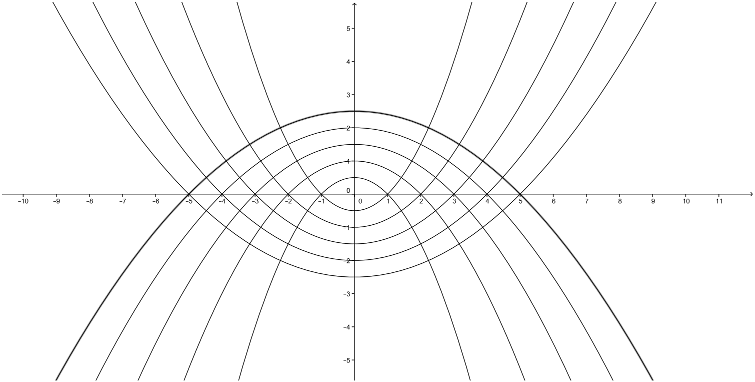

Figure: Coordinates lines corresponding to the parabolic coordinates system

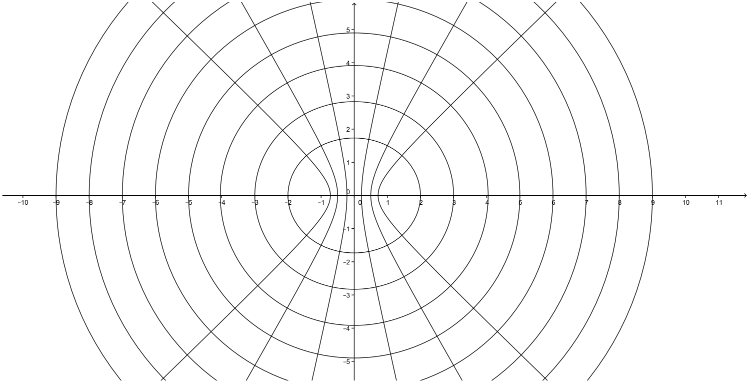

Figure: Coordinates lines corresponding to the elliptic coordinates system

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- Example : in cartesian coordinates (x, y), $f(x)g(y)$ is a solution of $\Delta\phi = E\phi$ iff

$$\left\{ \begin{array}{l} \partial_x^2 f - E_1 f = 0 \\ \partial_y^2 g - (E - E_1) g = 0 \end{array} \right.$$

- Bijective correspondence

$$\{\text{Separating coordinates systems}\}$$

$$\longleftrightarrow$$

$$\{\text{Second order symmetries of } \Delta : \text{second order differential operators } D \text{ such that } [\Delta, D] = 0\}$$

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Coordinates system	Symmetry
(x, y)	∂_x^2
(r, θ)	L_θ^2
(ξ, η)	$\frac{1}{2}(\partial_x L_\theta + L_\theta \partial_x)$
(α, β)	$L_\theta^2 + d\partial_x^2$

with $L_\theta = x\partial_y - y\partial_x$

- Link between the symmetry and the coordinates system : if the second-order part of D reads as

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  eigenvectors of A in this case :

$$\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} -y \\ x \end{pmatrix}$$

- On a n-dimensional pseudo-Riemannian manifold (M, g),

$$\Delta_Y := \nabla_i \mathrm{g}^{ij} \nabla_j - \frac{n-2}{4(n-1)} \mathrm{Sc},$$

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- Symmetry of Δ_Y : $D \in \mathcal{D}(M)$ such that $[\Delta_Y, D] = 0$
- Conformal symmetry of Δ_Y : $D_1 \in \mathcal{D}(M)$ such that $\exists D_2 \in \mathcal{D}(M)$ such that $\Delta_Y \circ D_1 = D_2 \circ \Delta_Y$

- (M, g) conformally flat : for each $x \in M$, there exist a neighborhood U of x and a function f on U such that $e^{2f}g$ is flat on U

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  Conformal symmetries of Δ_Y known (M. Eastwood, J.-P. Michel)
- (M, g) Einstein : $\mathrm{Ric} = \frac{1}{n}\mathrm{Sc}\ \mathrm{g}$
  Existence of a second order symmetry (B. Carter)

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$$\sigma(D) = \sum_{|\alpha|=k} D^\alpha p_1^{\alpha_1} \dots p_n^{\alpha_n},$$

where (x^i, p_i) are the canonical coordinates on T^*M

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- Is this condition sufficient ?

Definition

A quantization on M is a linear bijection $\mathcal{Q}^M$ from the space of symbols $\mathrm{Pol}(T^*M)$ to the space of differential operators $\mathcal{D}(M)$ such that

$$\sigma(\mathcal{Q}^M(S)) = S, \quad \forall S \in \mathrm{Pol}(T^*M)$$

Definition

A natural and conformally invariant quantization $\mathcal{Q}^M(\mathrm{g})$ *:*

- $\mathcal{Q}^M(\Phi^*\mathrm{g})(\Phi^* S) = \Phi^*(\mathcal{Q}^N(\mathrm{g})(S))$
- $\mathcal{Q}^M(\mathrm{g}) = \mathcal{Q}^M(\tilde{\mathrm{g}})$ *whenever* $\tilde{\mathrm{g}} = e^{2\Upsilon}\mathrm{g}$

- Proof of the existence of $\mathcal{Q}^M$:
  1. Work by A. Cap, J. Silhan
  2. Work by P. Mathonet, R.

- If K is a (conformal) Killing tensor of degree 2, there exists a (conformal) symmetry of Δ_Y with K as principal symbol iff $\mathrm{Obs}(K)^\flat$ is an exact one-form, where

$$\mathrm{Obs} = \frac{2(n-2)}{3(n+1)} p_i \partial_{p_j} \partial_{p_l} \left(C^k{}_{jl}{}^i \nabla_k - 3A_{jl}{}^i \right)$$

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- C : Weyl tensor :

$$C_{abcd} = R_{abcd} - \frac{2}{n-2}(\mathrm{g}_{a[c}\mathrm{Ric}_{d]b} - \mathrm{g}_{b[c}\mathrm{Ric}_{d]a}) + \frac{2}{(n-1)(n-2)}\mathrm{Sc}\ \mathrm{g}_{a[c}\mathrm{g}_{d]b}$$

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- A : Cotton-York tensor :

$$A_{ijk} = \nabla_k \mathrm{Ric}_{ij} - \nabla_j \mathrm{Ric}_{ik} + \frac{1}{2(n-1)} \left(\nabla_j \mathrm{Sc}\ g_{ik} - \nabla_k \mathrm{Sc}\ g_{ij}\right)$$

- If $\mathrm{Obs}(K)^\flat = 2df$, the (conformal) symmetries of Δ_Y whose the principal symbol is given by K are of the form

$$\mathcal{Q}(K) - f + L_X + c,$$

where X is a (conformal) Killing vector field, where $c \in \mathbb{R}$ and where $\mathcal{Q}$ denotes the natural and conformally invariant quantization

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  Hamiltonian $H = \mathrm{g}^{ij} p_i p_j$:

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  Killing tensor K :

$$\frac{c(x_3)a(x_1, x_2)p_1^2 + c(x_3)b(x_1, x_2)p_2^2 - \gamma(x_1, x_2)p_3^2}{\gamma(x_1, x_2) + c(x_3)},$$

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- In this situation, $\mathrm{Obs}(K)^\flat$ exact$\Rightarrow$existence of symmetries.

- Conformal Stäckel metric g : g s.t. the Hamilton-Jacobi equation

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- Coordinate x ignorable for g : ∂_x is a conformal Killing vector field for g.
- If g admits one ignorable coordinate x_1, then
$$\mathrm{g} = Q\left(dx_1^2 + (u(x_2) + v(x_3))(dx_2^2 + dx_3^2)\right).$$

- ∂_{x_1} is a conformal Killing vector field and

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- In general, $\mathrm{Obs}(K)^\flat$ not closed$\Rightarrow$no conformal symmetries with principal symbol K.

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- Reduction of g : $h = dr^2 + \frac{r^2 z^2}{z^2 - a^2 r^2} d\phi^2 + dz^2$
- Reduction of the Killing tensor $K = p_r^2 + \frac{1}{r^2} p_\phi^2$:
  $K' = p_r^2 + \frac{1}{r^2} p_\phi^2$

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- Schrödinger equation at zero energy : $(\Delta_Y + V)\psi = 0$, $V \in C^\infty(M)$ is a fixed potential

- Schrödinger equation R-separable in an orthogonal coordinates system (x^i) ($\mathrm{g}_{ij} = 0$ if $i \neq j$)

$$\Longleftrightarrow$$

$\forall\ E \in \mathbb{R}$, $\exists\ n+1$ functions $R, h_i \in C^\infty(M)$ and n differential operators $L_i := \partial_i^2 + l_i(x^i)\partial_i + m_i(x^i)$ such that

$$R^{-1}(\Delta_Y + V)R - E = \sum_{i=1}^{n} h_i L_i .$$

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- $R\prod_{i=1}^n \phi_i(x^i)$ solution of one of the two previous equations

$$\Longleftrightarrow$$

$$L_i\phi_i = 0 \quad \forall i$$

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  (a) $\exists$ a n-dimensional linear space of (resp. conformal) Killing 2-tensors $\mathcal{I}$ such that
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  (b) For all $K \in \mathcal{I}$, $\exists$ second order (resp. conformal) symmetry D, i.e. an operator such that $[\Delta_Y + V, D] = 0$ (resp. $\in (\Delta_Y + V)$), with principal symbol $\sigma_2(D) = K$.

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- Leaves of the corresponding foliations$\longleftrightarrow$Coordinate hyperplans of the R-separating coordinate systems