Four-dimensional rigid body and the related two-gyroscopic problems

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When dealing with the low-dimensional Lie groups and their algebras, one is faced with various mutual identifications. Some are obvious, some not directly visible. There are no counterparts in higher dimensions. Are those identifications “accidental” or just “mysterious”? The question has to do with the anthropic principle.

The universal covering groups of $\mathrm{SO}(3,\mathbb{R}) \subset \mathrm{GL}(3,\mathbb{R})$ and $\mathrm{SO}(1,3)^{\uparrow} \subset \mathrm{GL}(4,\mathbb{R})$ are isomorphic with $\mathrm{SU}(2) \subset \mathrm{GL}(2,\mathbb{C})$ and $\mathrm{SL}(2,\mathbb{C}) \subset \mathrm{GL}(2,\mathbb{C})$. The groups $\mathrm{SL}(2,\mathbb{R}) \subset \mathrm{GL}(2,\mathbb{R})$, $\mathrm{SO}(1,2) \subset \mathrm{GL}(3,\mathbb{R})$, $\mathrm{SU}(1,1) \subset \mathrm{GL}(2,\mathbb{C})$ have the same Lie algebras. The special pseudounitary group $\mathrm{SU}(2,2) \subset \mathrm{GL}(4,\mathbb{C})$ is isomorphic with the universal covering group of the Minkowskian conformal group $\mathrm{CO}(1,3)$.

The special orthogonal group in four dimensions, $\mathrm{SO}(4,\mathbb{R})$, and the Cartesian product $\mathrm{SO}(3,\mathbb{R}) \times \mathrm{SO}(3,\mathbb{R})$ have isomorphic Lie algebras.

IMPORTANT: $n = 4$ is the only exceptional case among all $\mathrm{SO}(n, \mathbb{R})$ with $n > 2$ when the semi-simplicity breaks down.

$$\mathrm{SO}(4, \mathbb{R})' \simeq \mathrm{SO}(3, \mathbb{R})' \times \mathrm{SO}(3, \mathbb{R})'$$

but

$$\mathrm{SO}(4, \mathbb{R}) \neq \mathrm{SO}(3, \mathbb{R}) \times \mathrm{SO}(3, \mathbb{R}).$$

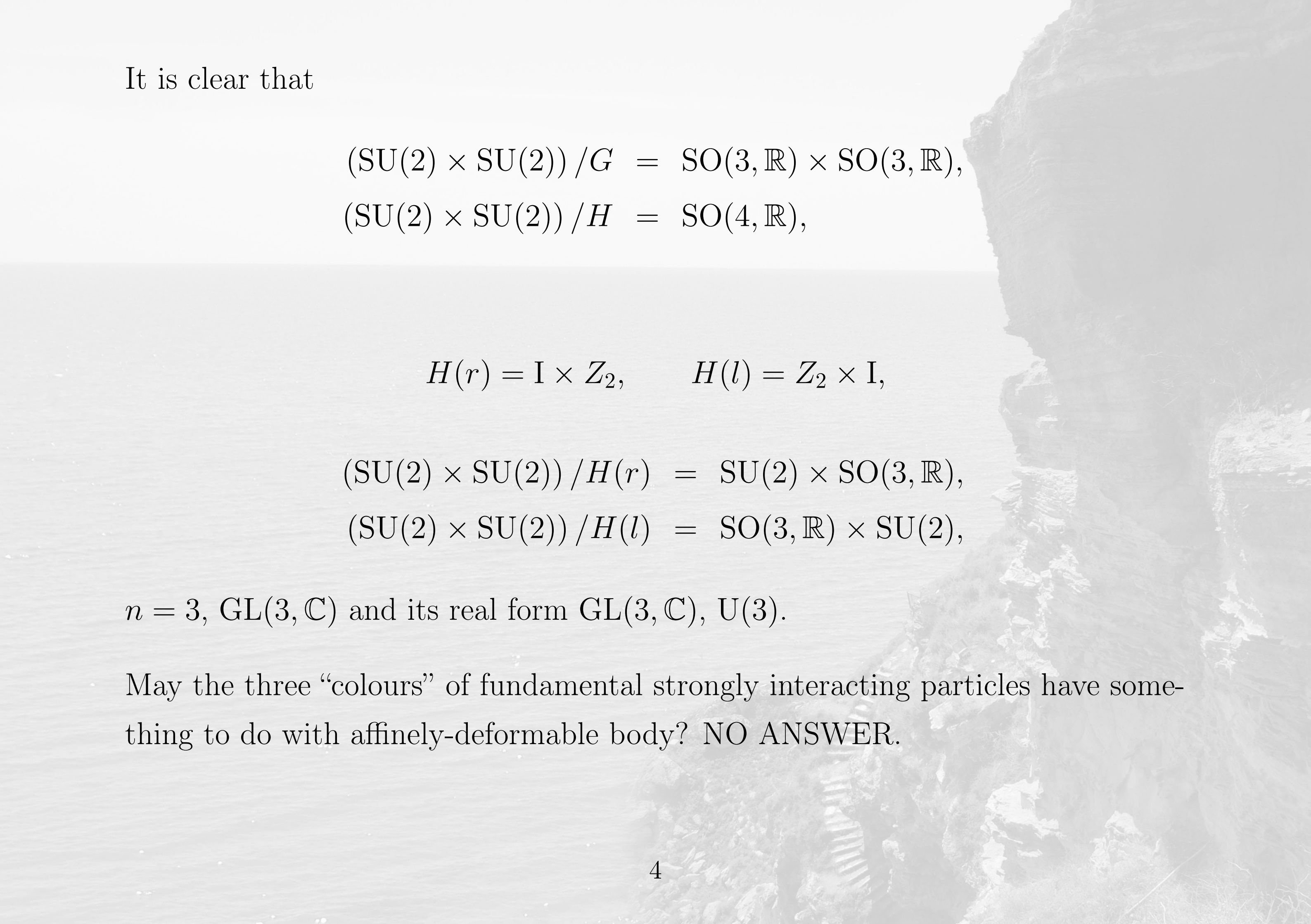
$\mathbf{Z}_2 = \{\mathrm{I}, -\mathrm{I}\}$ - the two-element center of $\mathrm{SU}(2)$.

$$\mathrm{SO}(3, \mathbb{R}) \simeq \mathrm{SU}(2)/Z_2$$

$$G = \mathbf{Z}_2 \times \mathbf{Z}_2 = \{(\mathrm{I}, \mathrm{I}), (\mathrm{I}, -\mathrm{I}), (-\mathrm{I}, \mathrm{I}), (-\mathrm{I}, -\mathrm{I})\}$$

center of $\mathrm{SU}(2) \times \mathrm{SU}(2)$. It contains three two-element subgroups, in particular

$$H = \{(\mathrm{I}, \mathrm{I}), (-\mathrm{I}, -\mathrm{I})\}\,.$$



It is clear that

$$\begin{aligned}(\mathrm{SU}(2)\times \mathrm{SU}(2))\,/G &= \mathrm{SO}(3,\mathbb{R})\times \mathrm{SO}(3,\mathbb{R}),\\ (\mathrm{SU}(2)\times \mathrm{SU}(2))\,/H &= \mathrm{SO}(4,\mathbb{R}),\end{aligned}$$

$$H(r) = \mathrm{I}\times Z_2, \qquad H(l) = Z_2 \times \mathrm{I},$$

$$\begin{aligned}(\mathrm{SU}(2)\times \mathrm{SU}(2))\,/H(r) &= \mathrm{SU}(2)\times \mathrm{SO}(3,\mathbb{R}),\\ (\mathrm{SU}(2)\times \mathrm{SU}(2))\,/H(l) &= \mathrm{SO}(3,\mathbb{R})\times \mathrm{SU}(2),\end{aligned}$$

$n = 3$, $\mathrm{GL}(3,\mathbb{C})$ and its real form $\mathrm{GL}(3,\mathbb{C})$, $\mathrm{U}(3)$.

May the three “colours” of fundamental strongly interacting particles have something to do with affinely-deformable body? NO ANSWER.

The basic 2×2 Pauli matrices:

$$\sigma_0 = \mathrm{I}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \ \sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \ \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \ \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

"Relativistic" notation: σ_μ, $\mu = 0, 1, 2, 3$, $i = 1, 2, 3$.

Remark: they are basic in double sense:

1. Basic Hermitian forms in $\mathrm{Herm}(\mathbb{C}^{2*} \otimes \mathbb{C}^{2*})$, twice covariant.

2. Basic Hermitian linear mappings of $\mathbb{C}^2$, elements of $\mathrm{L}(\mathbb{C}^2)$, Hermitian in the sense of scalar product $\delta\,(u, v) = \delta_{ab}\overline{u}^{\bar{a}} v^b$.

One often uses another bases:

$$\tau_0 = \frac{1}{2}\,\sigma_0 = \frac{1}{2}\,\mathrm{I}_2, \qquad \tau_a = \frac{1}{2i}\,\sigma_a, \qquad a = 1, 2, 3.$$

or, in some problems

$$\theta_\mu = \frac{1}{2i}\,\sigma_\mu.$$

Remark:

- when σ_μ are linear mappings, then the “relativistic” conventions are misleading:

$$x \to a x a^{-1}$$

preserves σ_0 and there is no Lorentz mixing.

- relativistic conventions are satisfied in the manifold of Hermitian forms:

$$a \sigma_\mu a^+ = |\det a|\, \sigma_\nu L^\nu{}_\mu, \qquad L \in \mathrm{SO}(1,3).$$

- $$[\tau_a, \tau_b] = \varepsilon_{ab}{}^c \tau_c.$$

Nevertheless we use this convention to simplify notation, e.g., when SU(2) is parametrized:

$$u\left(\overline{k}\right) = \exp\left(k^a \tau_a\right) = x^\mu \left(\overline{k}\right)\left(2\,\tau_\mu\right),$$

$$x^0 = \cos\frac{k}{2}, \qquad x^a = \frac{k^a}{k}\,\sin\frac{k}{2} = n^a \sin\frac{k}{2},$$

$\overline{k}$ - rotation vector, $k \in [0,\pi]$ on $\mathrm{SO}(3,\mathbb{R})$, $k \in [0, 2\,\pi]$ on $\mathrm{SU}(2)$.

The covering $\mathrm{SU}(2) \to \mathrm{SO}(3, \mathbb{R})$ is given by:

$$\mathrm{SU}(2) \ni v \mapsto R \in \mathrm{SO}(3, \mathbb{R}), \qquad \text{where} \qquad v u(\bar{k}) v^{-1} = u(R\bar{k}).$$

On $\mathrm{SU}(2)$:

$$u(\overline{O}) = u(O\bar{n}) = \mathrm{I}_2, \qquad u(2\pi\bar{n}) = -\mathrm{I}_2, \qquad \bar{n} \cdot \bar{n} = 1.$$

On $\mathrm{SO}(3, \mathbb{R})$ for any $\bar{n}$ we have

$$R\,(\pi\bar{n})\,R\,(\pi\bar{n}) = \mathrm{I}_3.$$

It is clear that the parameters x^μ are constrained by:

$$(x^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2 = 1.$$

Description is also unique, therefore

$$\mathrm{SU}(2) \simeq S^3(0,1) \subset \mathbb{R}^4,$$

the unit sphere.

One uses also spherical coordinates in the space of $\bar{k}$,

$$k^1 = k \sin\vartheta \cos\varphi, \qquad k^2 = k \sin\vartheta \sin\varphi, \qquad k^3 = k \cos\vartheta.$$

k^a are canonical coordinates of the first kind.

Strange, canonical coordinates of the second kind are only exceptionally used:

$$u\{\alpha, \beta, \gamma\} = \exp(\alpha\tau_1)\exp(\beta\tau_2)\exp(\gamma\tau_3).$$

Unlike this, Euler angles are commonly used:

$$u[\phi, \vartheta, \psi] = \exp(\phi\tau_3)\ \exp(\vartheta\tau_1)\ \exp(\psi\tau_3).$$

The Killing tensor multiplied by (-2) is positively definite and given by:

$$ds^2 = dk^2 + 4\sin^2\frac{k}{2}\left(d\vartheta^2 + \sin^2\vartheta\right)d\phi^2 = dk^2 + 4\sin^2\frac{k}{2}\,d\bar{n}\cdot d\bar{n},$$

or more geometrically:

$$g = dk \otimes dk + 4\sin^2\frac{k}{2}\,\delta_{AB}\,dn^A \otimes dn^B.$$

It is invariant under translations:

$$\mathrm{SU}(2) \ni x \mapsto kxl \in \mathrm{SU}(2), \qquad k, l \in \mathrm{SU}(2).$$

ds^2 is (-2) restriction of

$$dS^2 = (dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2,$$

to $S^3(0,1)$. The isometry group:

$$\mathrm{SO}(4,\mathbb{R}) \simeq \left(\mathrm{SU}(2)\times\mathrm{SU}(2)\right)/H.$$

The common Lie algebra:

$$\mathrm{SU}(2)' \times \mathrm{SU}(2)' = \mathrm{SO}(4,\mathbb{R})' = \mathrm{SO}(3,\mathbb{R})' \times \mathrm{SO}(3,\mathbb{R})'.$$

Indeed, denoting the standard basis of $\mathrm{SO}(4,\mathbb{R})'$ by $\mathcal{E}^{\mu\nu}$ and introducing:

$$M_1 = \mathcal{E}^{32}, \qquad M_2 = \mathcal{E}^{13}, \qquad M_3 = \mathcal{E}^{21},$$

$$N_1 = \mathcal{E}^{01}, \qquad N_2 = \mathcal{E}^{02}, \qquad N_3 = \mathcal{E}^{03},$$

$$X_i = \frac{1}{2}\left(M_i + N_i\right), \qquad Y_i = \frac{1}{2}\left(M_i - N_i\right),$$

we obtain:

$$[X_i, X_j] = \varepsilon_{ij}{}^{k} X_k, \qquad [Y_i, Y_j] = \varepsilon_{ij}{}^{k} Y_k, \qquad [X_i, Y_j] = 0.$$

Inverse metric:

$$g^{-1} = \frac{\partial}{\partial k} \otimes \frac{\partial}{\partial k} + \frac{1}{4\sin^2 \frac{k}{2}}\ \delta^{AB} D_A \otimes D_B,$$

$D_A = \varepsilon_{AB}{}^{C} k^B \frac{\partial}{\partial k^C}$ - generators of $u \mapsto vuv^{-1}$.

Duality (modified):

$$\left\langle dk, \frac{\partial}{\partial k} \right\rangle = 1 \quad , \quad \langle dk, D_A \rangle = 0,$$

$$\left\langle dn^A, \frac{\partial}{\partial k} \right\rangle = 0 \quad , \quad \langle dn^A, D_B \rangle = \varepsilon^A{}_{BC}\, n^C.$$

Coordinate expression of the Killing metric:

$$g^{ij} = \frac{k^2}{4\sin^2 \frac{k}{2}}\ \delta^{ij} + \left(1 - \frac{k^2}{4\sin^2 \frac{k}{2}}\right) n^i n^j,$$

$$g_{ij} = \frac{4}{k^2} \sin^2 \frac{k}{2}\ \delta_{ij} + \left(1 - \frac{4}{k^2} \sin^2 \frac{k}{2}\right) n_i n_j.$$

Vector fields generating left and right translations (right-invariant and left-invariant):

$$
\begin{aligned}
{}^{l}E_A &= n_A \frac{\partial}{\partial k} - \frac{1}{2}\cot\frac{k}{2}\varepsilon_{ABC} n^B D^C + \frac{1}{2} D_A, \\
{}^{r}E_A &= n_A \frac{\partial}{\partial k} - \frac{1}{2}\cot\frac{k}{2}\varepsilon_{ABC} n^B D^C - \frac{1}{2} D_A, \\
D_A &= {}^{l}E_A - {}^{r}E_A.
\end{aligned}
$$

Commutation rules:

$$
\left[{}^{l}E_A, {}^{l}E_B\right] = -\varepsilon_{AB}{}^{C}\, {}^{l}E_C, \qquad \left[{}^{r}E_A, {}^{r}E_B\right] = \varepsilon_{AB}{}^{C}\, {}^{r}E_C,
$$

$$
\left[{}^{l}E_A, {}^{r}E_B\right] = 0, \qquad \left[D_A, D_B\right] = -\varepsilon_{AB}{}^{C}\, D_C.
$$

The dual Maurer-Cartan forms ${}^{l}E^{A}$, ${}^{r}E^{B}$:

$$\left\langle {}^{l}E^{A}, {}^{l}E_{B} \right\rangle = \delta^{A}{}_{B}, \qquad \left\langle {}^{r}E^{A}, {}^{r}E_{B} \right\rangle = \delta^{A}{}_{B}.$$

$$\begin{aligned}
{}^{l}E^{A} &= n^{A} dk + 2\sin^{2}\frac{k}{2}\,\varepsilon^{ABC} n_{B} dn_{C} + \sin k\, dn^{A}, \\
{}^{r}E^{A} &= n^{A} dk - 2\sin^{2}\frac{k}{2}\,\varepsilon^{ABC} n_{B} dn_{C} + \sin k\, dn^{A}, \\
g &= \delta_{AB}\, {}^{l}E^{A} \otimes {}^{l}E^{B} = \delta_{AB}\, {}^{r}E^{A} \otimes {}^{r}E^{B}. \\
g^{-1} &= \delta^{AB}\, {}^{l}E_{A} \otimes {}^{l}E_{B} = \delta^{AB}\, {}^{r}E_{A} \otimes {}^{r}E_{B}
\end{aligned}$$

Peter-Weyl theorem on SU(2), SU(2) × SU(2), etc.:

$$\Psi(u) = \sum_{jmk} c^j{}_{km} D^j{}_{mk}(u) = \sum_j Tr\left(c^j D^j(u)\right),$$
$$\Psi(u,v) = \sum_{\substack{ls \\ mk\ rn}} c^l{}_{km}{}^s{}_{nr} D^l{}_{mk}(u) D^s{}_{rn}(v).$$

D^j - $(2j+1)$ - dimensional irreducible representation of SU(2).

On SU(2) × SU(2) - C-coefficients arbitrary.

On $\mathrm{SO}(4,\mathbb{R}) \simeq (\mathrm{SU}(2) \times \mathrm{SU}(2))/H$ - C-coefficients vanish when l, s have different halfness, i.e., $2l$, $2s$ have a different parity.

The group property:

$$D^j(u_1 u_2) = D^j(u_1) D^j(u_2), \qquad D^j(\mathrm{I}_2) = \mathrm{I}_{2j+1},$$

implies that

$$D^j\left(u(\bar{k})\right) = \exp\left(\frac{i}{\hbar} k^a \mathcal{S}^j{}_a\right),$$

where $\mathcal{S}^j{}_a$ - $(2j+1) \times (2j+1)$ matrices of the j-th angular momentum

$$\frac{1}{\hbar i}\left[\mathcal{S}_a, \mathcal{S}_b\right] = \varepsilon_{ab}{}^c \mathcal{S}_c.$$

D^j satisfy differential equations:

$$\frac{\hbar}{i}\,{}^l E_A D^j = \mathcal{S}^j{}_A D^j \quad , \quad \frac{\hbar}{i}\,{}^r E_A D^j = D^j \mathcal{S}^j{}_A,$$

$$\frac{\hbar}{i} D_A D^j \;=\; \left[\mathcal{S}^j{}_A, D^j\right],$$

$$-\hbar^2 \sum_A {}^l E_A {}^l E_A D^j = -\hbar^2 \sum_A {}^r E_A {}^r E_A D^j = \hbar^2 j(j+1) D^j.$$

There are a few mechanical problems based on $\mathrm{SU}(2) \times \mathrm{SU}(2)$:

- Two-gyroscopic system with $\mathrm{SO}(3,\mathbb{R}) \times \mathrm{SO}(3,\mathbb{R})$ as a configuration space, or some spinorial modifications in $\mathrm{SU}(2) \times \mathrm{SU}(2)$ or its quotients

- Rigid body in Einstein Universe $S^3(0,R)$ with the internal space ruled by $\mathrm{SO}(3,\mathbb{R})$ or $\mathrm{SU}(2)$.

General remarks:

(M, g) - Riemann space of translational motion.
$F(M, g)$ - the connected component of the bundle of orthonormal frames - total configuration space.

$F(M, g) \ni e = (\ldots, e_A, \ldots)$ at $x \in M$, where:

$$g_x\left(e_A, e_B\right) = g(x)_{ij}\ e^i{}_A e^j{}_B = \delta_{AB}.$$

Motions: $\mathbb{R} \ni t \mapsto \gamma(t) \in F(M, g)$, $\big(dx^i(t), e^i{}_A(t)\big)$.

Generalized velocity:

$$\left(\frac{dx^i}{dt}, \frac{d\,e^i{}_A}{dt}\right) \qquad \frac{d\,e^i{}_A}{dt} \text{ – not vectors.}$$

Covariant velocity vectors:

$$\left(\frac{dx^i}{dt}, \frac{D}{Dt}e^i{}_A\right) = \left(\frac{dx^i}{dt}, \frac{de^i{}_A}{dt} + \Gamma^i{}_{jk}(x(t))\, e^j{}_A(t)\, \frac{dx^k}{dt}\right).$$

Notation:

$$\mathbf{V}^i{}_A = \frac{D\ e^i{}_A}{Dt}.$$

Kinetic energy:

$$
\begin{aligned}
T &= T_{tr} + T_{int} = \frac{m}{2}\, g_{ij} \frac{dx^i}{dt}\frac{dx^j}{dt} + \frac{1}{2}\, g_{ij} \left(\frac{D}{Dt}\; e^i{}_A \right) \left(\frac{D}{Dt}\; e^j{}_B \right) J^{AB} = \\
&= \frac{m}{2}\, g_{ij} v^i v^j + \frac{1}{2}\, \delta_{KL} \widehat{\Omega}^K{}_A \widehat{\Omega}^L{}_B J^{AB} = \\
&= \frac{m}{2}\, \delta_{AB} \hat{v}^A \hat{v}^B + \frac{1}{2}\, \delta_{KL} \widehat{\Omega}^K{}_A \widehat{\Omega}^L{}_B J^{AB}.
\end{aligned}
$$

$$\Omega^i{}_j = \left(\frac{D}{Dt} \, e^i{}_A \right) e^A{}_j, \qquad \widehat{\Omega}^A{}_B = e^A{}_i \frac{D}{Dt} \, e^i{}_B = e^A{}_i \, \Omega^i{}_j \, e^j{}_B,$$

$$\hat{v}^A = e^A{}_i v^i = e^A{}_i \frac{dx^i}{dt},$$

$$\Omega^i{}_j = -g_{jk} \Omega^k{}_l g^{li} = -\Omega_j{}^i, \qquad \widehat{\Omega}^A{}_B = -\delta_{BC} \delta^{AD} \widehat{\Omega}^C{}_D = -\widehat{\Omega}_B{}^A.$$

$e^i{}_A$ are not independent variables. The best solution to fix some non-holonomic reference frame in M, $(\ldots, E_A, \ldots)$ and to express the moving orthonormal bases is as follows:

$$e_A \left(x(t) \right) = E_B \left(x(t) \right) L^B{}_A(t),$$

were $\left[L^B{}_A \right]$ - an orthogonal $n \times n$ matrix parameterized in terms of some fixed coordinates in $\mathrm{SO}(n, \mathbb{R})$.

If M is an n-dimensional semi-simple Lie group G with the Killing metric g, then $F(M, g)$ becomes identified with $G \times \mathrm{SO}(n, \mathbb{R})$. There are two identifications based on the left- and right-invariant fields:

$$e_A = {}^l E_{Bx} \; {}^l L^B{}_A, \qquad e_A = {}^r E_{Bx} \; {}^r L^B{}_A.$$

$F(G, g)$ becomes the Cartesian product $G \times \mathrm{SO}(n, \mathbb{R})$. In our case $F(M, g)$ becomes $M \times \mathrm{SO}(3, \mathbb{R})$.

We assume the simplest isotropic case:

$$J^{AB} = \mathrm{I}\delta^{AB}.$$

In a manifold M the angular velocity splits:

$$\widehat{\Omega}^{A}{}_{B} = \widehat{\Omega}(rl)^{A}{}_{B} + \widehat{\Omega}(dr)^{A}{}_{B},$$

the sum of relative (internal) $\widehat{\Omega}(rl)$ and the drive $\widehat{\Omega}(dr)$ terms. They are given respectively by:

$$\begin{aligned}
\Omega(rl)^{C}{}_{D} &= L^{-1C}{}_{E}\,\frac{dL^{E}{}_{D}}{dt}, \\
\widehat{\Omega}(dr)^{A}{}_{B} &= L^{-1A}{}_{K}\,\Gamma^{K}{}_{LM}\,L^{L}{}_{B}\,L^{M}{}_{N}\,\hat{v}^{N}, \\
\hat{v}^{N} &= e^{N}{}_{i}v^{i}.
\end{aligned}$$

$\Gamma^A{}_{BC}$ are non-holonomic components of the Levi-Civita connection:

$$\begin{aligned}\Gamma^A{}_{BC} &= E^A{}_i \left(\Gamma^i{}_{jk} - \Gamma_{tel}(E)^i{}_{jk}\right) E^j{}_B\, E^k{}_C,\\ \Gamma^i{}_{jk} &= \frac{1}{2}\, g^{im} \left(g_{mj,k} + g_{mk,j} - g_{jk,m}\right).\end{aligned}$$

On SU(2) the Killing-Levi-Civita connection is:

$$\Gamma^A{}_{BC} = -\frac{1}{2}\, \varepsilon^A{}_{BC}.$$

In SU(2)×SU(2) we replace the first factor by $S^3(0, R)$ - three-dimensional sphere with radius R. We perform the rescaling:

$$r := R\,k/2, \qquad \frac{\bar{r}}{r} = \frac{\bar{k}}{k}$$

“North Pole” $r = 0$

“South Pole” $r = R\pi$.

Metric tensor:

$$\begin{aligned} ds^2 &= dr^2 + R^2 \sin^2 \frac{r}{R} \left(d\vartheta^2 + \sin^2 \vartheta d\varphi^2 \right) = \\ &= dr^2 + R^2 \sin^2 \frac{r}{R}\, d\bar{n} \cdot d\bar{n}. \end{aligned}$$

Geometrically:

$$\begin{aligned} g(R) &= dr \otimes dr + R^2 \sin^2 \frac{r}{R}\, \delta_{AB}\, dn^A \otimes dn^B = \\ &= \delta_{AB}\, {}^{l}E(R)^A \otimes {}^{l}E(R)^B = \delta_{AB}\, {}^{r}E(R)^A \otimes {}^{r}E(R)^B. \end{aligned}$$

Renormalized co-bases and bases are given by:

$$
\begin{aligned}
{}^{l}E(R)^{A} &= n^{A}dr + R\sin^{2}\frac{r}{R}\varepsilon^{A}{}_{BC}n^{B}dn^{C} + \frac{R}{2}\sin\frac{2r}{R}dn^{A}, \\
{}^{r}E(R)^{A} &= n^{A}dr - R\sin^{2}\frac{r}{R}\varepsilon^{A}{}_{BC}n^{B}dn^{C} + \frac{R}{2}\sin\frac{2r}{R}dn^{A}, \\
{}^{l}E(R)_{A} &= n^{A}\frac{\partial}{\partial r} - \frac{1}{R}\cot\frac{r}{R}\;\varepsilon_{ABC}n^{B}D^{C} + \frac{1}{R}D^{A}, \\
{}^{r}E(R)_{A} &= n^{A}\frac{\partial}{\partial r} - \frac{1}{R}\cot\frac{r}{R}\;\varepsilon_{ABC}n^{B}D^{C} - \frac{1}{R}D^{A},
\end{aligned}
$$

$$
D_{A} = \frac{R}{2}\left({}^{l}E_{A} - {}^{r}E_{A}\right) = \varepsilon_{AB}{}^{C}r^{B}\frac{\partial}{\partial r^{C}}
$$

Index-free form:

$$\begin{aligned}
{}^{l}\underline{E} &= \bar{n}dr + R\sin^2\frac{r}{R}\bar{n}\times d\bar{n} + \frac{R}{2}\sin\frac{2r}{R}d\bar{n},\\
{}^{r}\underline{E} &= \bar{n}dr - R\sin^2\frac{r}{R}\bar{n}\times d\bar{n} + \frac{R}{2}\sin\frac{2r}{R}d\bar{n},\\
{}^{l}\overline{E} &= \bar{n}\frac{\partial}{\partial r} - \frac{1}{R}\cot\frac{r}{R}\bar{n}\times\overline{D} + \frac{1}{R}\overline{D},\\
{}^{r}\overline{E} &= \bar{n}\frac{\partial}{\partial r} - \frac{1}{R}\cot\frac{r}{R}\bar{n}\times\overline{D} - \frac{1}{R}\overline{D}.
\end{aligned}$$

The metric on $S^3(0,R)$ follows from:

$$dS^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2 + (dx^4)^2,$$

by substitution:

$$x^1 = R\sin\frac{r}{R}\sin\vartheta\cos\varphi \quad , \quad x^2 = R\sin\frac{r}{R}\sin\vartheta\sin\varphi,$$
$$x^3 = R\sin\frac{r}{R}\cos\vartheta \quad , \quad x^4 = R\cos\frac{r}{R}.$$

The R-gauged fields ${}^{l}E(R)$, ${}^{r}E(R)$ satisfy:

$$\begin{aligned}
\left[{}^{l}E(R)_A, {}^{l}E(R)_B\right] &= -\frac{2}{R}\,\varepsilon_{AB}{}^{C}\;{}^{l}E(R)_C, \\
\left[{}^{r}E(R)_A, {}^{r}E(R)_B\right] &= \frac{2}{R}\,\varepsilon_{AB}{}^{C}\;{}^{r}E(R)_C, \\
\left[{}^{l}E(R)_A, {}^{r}E(R)_B\right] &= 0.
\end{aligned}$$

When $R \to \infty$, we obtain Euclidean relationships:

$$\begin{aligned}
\lim_{R\to\infty} {}^{l}E(R)_A &= \lim_{R\to\infty} {}^{r}E(R)_A = \frac{\partial}{\partial x^A}, \\
\lim_{R\to\infty} {}^{l}E(R)^A &= \lim_{R\to\infty} {}^{r}E(R)^A = dx^A, \\
\lim_{R\to\infty} g(R)_{ij} &= \delta_{ij}.
\end{aligned}$$

On the fibers of $F(M, g)$, all identified with $\mathrm{SO}(3, \mathbb{R})$, or its covering $\mathrm{SU}(2)$ we are given vector fields ${}^{l}E_A(\bar{\varkappa})$ and duals ${}^{l}E^A(\bar{\varkappa})$ obtained from ${}^{l}E_A(\bar{k})$ and ${}^{l}E^A(\bar{k})$ by replacing $\bar{k} \mapsto \bar{\varkappa}$. The same for ${}^{r}E_A(\bar{\varkappa})$, ${}^{r}E^A(\bar{\varkappa})$.

Angular velocities:

$$\widehat{\Omega}_{tr}(R)^D = {}^r E^D{}_j(R,\bar{r})\,\frac{dr^j}{dt}, \qquad \widehat{\Omega}^D_{int} = {}^r E^D{}_j(\bar{\varkappa})\,\frac{d\varkappa^j}{dt}.$$

The total kinetic energy becomes:

$$T = \frac{1}{2}\left(m + \frac{\mathrm{I}}{R^2}\right)\delta_{AB}\,\Omega_{tr}(R)^A\,\Omega_{tr}(R)^B - \frac{\mathrm{I}}{R}\,\delta_{AB}\,\Omega^A_{int}\,\Omega_{tr}(R)^B +$$

$$+ \frac{\mathrm{I}}{2}\,\delta_{AB}\,\Omega^A_{int}\,\Omega^B_{int},$$

$$L = T - V(\bar{r},\bar{\varkappa}), \qquad \text{-Lagrangian.}$$

Legendre:

$$
\begin{aligned}
S_{tr}(R)_A &= \frac{\partial T}{\partial \Omega_{tr}(R)^A} = {}^{l}E^{i}{}_{A}(R,\bar{r})p_i, \\
S_{int\ A} &= \frac{\partial T}{\partial \Omega^A_{int}} = {}^{l}E^{i}{}_{A}(\bar{\varkappa})\pi_i,
\end{aligned}
$$

or, equivalently:

$$
\begin{aligned}
\widehat{S}_{tr}(R)_A &= \frac{\partial T}{\partial \widehat{\Omega}_{tr}(R)^A} = {}^{r}E^{i}{}_{A}(R,\bar{r})p_i, \\
\widehat{S}_{int\ A} &= \frac{\partial T}{\partial \widehat{\Omega}_{int}{}^A} = {}^{r}E^{i}{}_{A}(\bar{\varkappa})\pi_i.
\end{aligned}
$$

Duality:

$$
\begin{aligned}
{}^{l}E_A(R,\bar{r}) \sim {}^{l}E^A(R,\bar{r}) \quad &, \quad {}^{r}E_A(R,\bar{r}) \sim {}^{r}E^A(R,\bar{r}),\\
\left(S_{tr}(R)_A, S_{int\ A}\right) \ &\sim \ \left(\Omega_{tr}(R)^A, \Omega_{int}{}^A\right),\\
\left(\widehat{S}_{tr}(R)_A, \widehat{S}_{int\ A}\right) \ &\sim \ \left(\widehat{\Omega}_{tr}(R)^A, \widehat{\Omega}_{int}{}^A\right),\\
(p_i, \pi_i) \ &\sim \ \left(r^i, \varkappa^i\right).
\end{aligned}
$$

Poisson brackets:

$$
\begin{aligned}
\{S_{tr}(R)_A, S_{tr}(R)_B\} &= \frac{2}{R}\varepsilon_{AB}{}^{C} S_{tr}(R)_C, \\
\left\{\widehat{S}_{tr}(R)_A, \widehat{S}_{tr}(R)_B\right\} &= -\frac{2}{R}\varepsilon_{AB}{}^{C} \widehat{S}_{tr}(R)_C, \\
\left\{S_{tr}(R)_A, \widehat{S}_{tr}(R)_B\right\} &= 0, \\
\{S_{int\ A}, S_{int\ B}\} &= \varepsilon_{AB}{}^{C} S_{int\ C}, \\
\left\{\widehat{S}_{int\ A}, \widehat{S}_{int\ B}\right\} &= -\varepsilon_{AB}{}^{C} \widehat{S}_{int\ C}, \\
\left\{S_{int\ A}, \widehat{S}_{int\ B}\right\} &= 0, \\
\{S_{tr}(R)_A, S_{int\ B}\} &= 0, \qquad \textit{etc.}
\end{aligned}
$$

Legendre explicitly:

$$
\begin{aligned}
S_{tr}(R)_A &= \left(m + \frac{\mathrm{I}}{R^2}\right)\Omega_{tr}(R)_A - \frac{\mathrm{I}}{R}\,\Omega_{int\ A}, \\
S_{int\ A} &= -\frac{\mathrm{I}}{R}\,\Omega_{tr}(R)_A + \mathrm{I}\,\Omega_{int\ A}.
\end{aligned}
$$

Inverse Legendre:

$$
\begin{aligned}
\Omega_{tr}(R)^A &= \frac{1}{m}\,S_{tr}(R)^A + \frac{1}{mR}\,S_{int}{}^A, \\
\Omega_{int}{}^A &= \frac{1}{mR}\,S_{tr}(R)^A + \frac{\mathrm{I} + mR^2}{\mathrm{I}mR^2}\,S_{int}{}^A.
\end{aligned}
$$

Hamilton equation:

$$\frac{dF}{dt} = \{F, H\},$$

i.e., in the potential case:

$$\begin{aligned} \frac{d}{dt} S_{tr}(R)_A &= \frac{2}{mR^2}\, \varepsilon_A{}^{BC} S_{int\ B} S_{tr}(R)_C + F_A, \\ \frac{d}{dt} S_{int\ A} &= \frac{1}{mR}\, \varepsilon_A{}^{BC} S_{tr}(R)_B S_{int\ C} + N_A, \end{aligned}$$

$$\begin{aligned} F_A &= \{S_{tr}(R)_A, V\} = -{}^{l}E(R, \bar{r})_A V(\bar{r}, \bar{\varkappa}), \\ N_A &= \{S_{int\ A}, V\} = -{}^{l}E(\bar{\varkappa})_A V(\bar{r}, \bar{\varkappa}). \end{aligned}$$

Similarly in the non-potential case.

Kinetic energy in canonical terms:

$$\mathcal{T} = \frac{1}{2m}\overline{S}_{tr}(R)\cdot\overline{S}_{tr}(R) + \frac{1}{mR}\overline{S}_{tr}(R)\cdot\overline{S}_{int} + \frac{\mathrm{I} + mR^2}{2\mathrm{I}mR^2}\overline{S}_{int}\cdot\overline{S}_{int},$$

$$\begin{aligned} \frac{d}{dt}\,\overline{S}_{tr}(R) &= \frac{2}{mR^2}\,\overline{S}_{int}\times\overline{S}_{tr}(R), \\ \frac{d}{dt}\,\overline{S}_{int} &= \frac{1}{mR}\,\overline{S}_{tr}(R)\times\overline{S}_{int}. \end{aligned}$$

Equation of motion independent on I, but the total system for $\left(\bar{r}, \bar{\varkappa}, \overline{S}_{tr}, \overline{S}_{int}\right)$ depends on I.

Constant of motion:

$$\overline{J} := \frac{R}{2}\,\overline{S}_{tr}(R) + \overline{S}_{int},$$

$$\overline{S}_{tr}(R)\cdot\overline{S}_{tr}(R), \qquad \overline{S}_{int}\cdot\overline{S}_{int}.$$

The only time-dependent variable: the angle between the plane spanned by $\overline{S}_{tr}(R)$, $\overline{S}_{int}$ and a fixed plane containing $\overline{J}$.

Non-geodetic problems - very difficult.

Quantized problems. Wave mechanics on $\mathrm{SU}(2) \times \mathrm{SU}(2)$, $S^3(0,R) \times \mathrm{SU}(2)$, $S^3(0,R) \times \mathrm{SO}(3,\mathbb{R})$, etc.

Angular momenta:

$$\begin{aligned} \boldsymbol{S}_{tr}\,(R)_A &= \frac{\hbar}{\imath}\,{}^{l}E^{m}{}_{A}\,(R,\overline{r})\,\frac{\partial}{\partial r^m}, \\ \boldsymbol{S}_{int\,A} &= \frac{\hbar}{\imath}\,{}^{l}E^{m}{}_{A}\,(\overline{\varkappa})\,\frac{\partial}{\partial \varkappa^m}. \end{aligned}$$

The co-moving version:

$$\begin{aligned} \widehat{\boldsymbol{S}}_{tr}\,(R)_A &= \frac{\hbar}{\imath}\,{}^{r}E^{m}{}_{A}\,(R,\overline{r})\,\frac{\partial}{\partial r^m}, \\ \widehat{\boldsymbol{S}}_{int\,A} &= \frac{\hbar}{\imath}\,{}^{r}E^{m}{}_{A}\,(\overline{\varkappa})\,\frac{\partial}{\partial \varkappa^m}. \end{aligned}$$

Quantum Poisson brackets:

$$
\begin{aligned}
\frac{1}{\hbar\imath}\left[\boldsymbol{S}_{tr}\,(R)_A\,,\boldsymbol{S}_{tr}\,(R)_B\right] &= \frac{2}{R}\,\varepsilon_{AB}{}^{C}\boldsymbol{S}_{tr}\,(R)_C\,,\\
\frac{1}{\hbar\imath}\left[\widehat{\boldsymbol{S}}_{tr}\,(R)_A\,,\widehat{\boldsymbol{S}}_{tr}\,(R)_B\right] &= -\frac{2}{R}\,\varepsilon_{AB}{}^{C}\widehat{\boldsymbol{S}}_{tr}\,(R)_C\,,\\
\frac{1}{\hbar\imath}\left[\boldsymbol{S}_{tr}\,(R)_A\,,\widehat{\boldsymbol{S}}_{tr}\,(R)_B\right] &= 0,
\end{aligned}
$$

and without the $2/R$-multipliers for the internal angular momentum:

$$
\begin{aligned}
\frac{1}{\hbar\imath}\left[\boldsymbol{S}_{int\,A}\,,\boldsymbol{S}_{int\,B}\right] &= \varepsilon_{AB}{}^{C}\,\boldsymbol{S}_{int\,C}\,,\\
\frac{1}{\hbar\imath}\left[\widehat{\boldsymbol{S}}_{int\,A}\,,\widehat{\boldsymbol{S}}_{int\,B}\right] &= -\varepsilon_{AB}{}^{C}\,\widehat{\boldsymbol{S}}_{int\,C}\,,\\
\frac{1}{\hbar\imath}\left[\boldsymbol{S}_{int\,A}\,,\widehat{\boldsymbol{S}}_{int\,B}\right] &= 0.
\end{aligned}
$$

Kinetic energy operator is given by:

$$
\begin{aligned}
\mathbf{T} &= \frac{1}{2m}\,\delta^{AB}\,\boldsymbol{S}_{tr}(R)_A\,\boldsymbol{S}_{tr}(R)_B + \frac{1}{mR}\,\delta^{AB}\,\boldsymbol{S}_{tr}(R)_A\,\boldsymbol{S}_{int\,B} + \\
&+ \frac{1}{2}\left(\frac{1}{I} + \frac{1}{mR^2}\right)\delta^{AB}\,\boldsymbol{S}_{int\,A}\,\boldsymbol{S}_{int\,B}\,.
\end{aligned}
$$

The Hilbert space $\mathrm{L}^2(S^3(0,R)\times \mathrm{SU}(2))$ is meant in the sense of measure $\mu_R \otimes \mu$, where:

$$
\begin{aligned}
d\mu_R\left(u\left(R,\bar{r}\right)\right) &= R^2\sin^2\frac{r}{R}\sin\vartheta\, dr\, d\vartheta\, d\varphi = \frac{R^2}{r^2}\sin^2\frac{r}{R} d_3\bar{r}, \\
d\mu\left(v\left(\bar{\varkappa}\right)\right) &= 4\sin^2\frac{\varkappa}{2}\sin\vartheta\, d\varkappa\, d\vartheta\, d\varphi = \frac{4}{\varkappa^2}\sin^2\frac{\varkappa}{2} d_3\bar{\varkappa}.
\end{aligned}
$$

The scalar product is:

$$\langle \Psi_1 \mid \Psi_2 \rangle = \int \overline{\Psi_1(u,v)} \Psi_2(u,v) d\mu_R(u) d\mu(v).$$

The group volumes are then:

$$\mu\left(S^3(0,R)\right) = 2\pi^2 R^3, \quad \mu\left(\mathrm{SU}(2)\right) = 16\pi^2, \quad \mu\left(\mathrm{SO}(3,\mathbb{R})\right) = 8\pi^2.$$

There is a conflict with normalizing the measure on compact groups to unity.

When using the Peter-Weyl theorem, the action of $\boldsymbol{S}_{tr\ A}$ on wave functions is algebraically represented by the following action on expansion coefficients:

$$\left[C^{l}{}_{km}{}^{s}{}_{nr}\right] \mapsto \left[\frac{2}{R}\, C^{l}{}_{kp}{}^{s}{}_{nr}\ S^{l}{}_{pm}\right],$$

the summation over p is meant here.

Similarly, the action of spin operators $\boldsymbol{S}_{int\ A}$ is represented by:

$$\left[C^{l}{}_{km}{}^{s}{}_{nr}\right] \mapsto \left[C^{l}{}_{km}{}^{s}{}_{np}\ S^{s}{}_{pr}\right],$$

again the summation over the matrix index p is assumed.

The $\frac{2}{R}$ -factor is important. When $R \to \infty$, distances between energy levels tend to zero. The spectrum becomes “continuous”, just like in $\mathbb{R}^3$.

All terms of the kinetic energy operator $\mathbf{T}$ do commute with the operators:

$$(\mathbf{S}_{tr})^2 = \delta^{AB}\,\mathbf{S}_{tr\ A}\,\mathbf{S}_{tr\ B}, \qquad (\mathbf{S}_{int})^2 = \delta^{AB}\,\mathbf{S}_{int\ A}\,\mathbf{S}_{int\ B}.$$

Thus, s, j are "good quantum numbers" to label the stationary states of:

$$\mathbf{T}\Psi = E\Psi.$$

Those basic states, labeled partially by s, J satisfy the system of algebraic eigenequations:

$$\delta^{AB} C^{l}{}_{kp}{}^{s}{}_{nq}\; S_{A}{}^{l}{}_{pm} S_{B}{}^{s}{}_{qr} = \lambda C^{l}{}_{km}{}^{s}{}_{nr}.$$

There is the following relationship:

$$E = \frac{2}{mR^2}\left(\lambda + l(l+1)\hbar^2\right) + \frac{1}{2}\left(\frac{1}{\mathrm{I}} + \frac{1}{mR^2}\right) s(s+1)\hbar^2.$$

It is clear that λ and $l(l+1)\hbar^2$ are R-independent, and so is the first of $s(s+1)$-terms, proportional to $\frac{1}{\mathrm{I}}$. With any fixed numbers l, s, there is a complete degeneracy $(2k+1)(2n+1)$ with respect to k, n.

When $R \to \infty$, the spectrum of translational quantum numbers in $S^3(0, R)$ becomes "almost continuous". Unlike this, the internal spectrum remains discrete, just one of the spherical top.

When some potential is present, the problem becomes very difficult.

The pair of rigid bodies in Euclidean space

$$Q = \mathrm{SU}(2) \times \mathrm{SU}(2) \quad \text{or some quotient}$$

Kinetic energy of a single rigid body:

$$T = \frac{1}{2} \sum_{A=1}^{3} \mathrm{I}_A \, \widehat{\Omega}^{A2}, \qquad \widehat{\Omega}^A = {}^r E^A{}_j(\bar{\varkappa}) \frac{d\varkappa^j}{dt},$$

$\bar{\varkappa}$ - rotation vector.

The two-body system:

$$T = \frac{1}{2}\sum_{A=1}^{3} \mathrm{I}_A(1)\widehat{\Omega}[\bar{\varkappa}]^{A2} + \frac{1}{2}\sum_{A=1}^{3} \mathrm{I}_A(2)\widehat{\Omega}[\bar{\lambda}]^{A2},$$

$$\widehat{\Omega}[\bar{\varkappa}]^{A} = {}^{r}E^{A}{}_{j}(\bar{\varkappa})\frac{d\varkappa^{j}}{dt}, \qquad \widehat{\Omega}[\bar{\lambda}]^{A} = {}^{r}E^{A}{}_{j}(\bar{\lambda})\frac{d\lambda^{j}}{dt},$$

or, in canonical terms:

$$\mathcal{T} = \sum_{A=1}^{3} \frac{1}{2\mathrm{I}_A(1)}\widehat{S}[\bar{\varkappa}]_{A}{}^{2} + \frac{1}{2}\sum_{A=1}^{3} \frac{1}{2\mathrm{I}_A(2)}\widehat{S}[\bar{\lambda}]_{A}{}^{2},$$

$$\widehat{S}[\bar{\varkappa}]_{A} = \mathrm{I}_A(1)\widehat{\Omega}[\bar{\varkappa}]_{A}, \qquad \widehat{S}[\bar{\lambda}]_{A} = \mathrm{I}_A(2)\widehat{\Omega}[\bar{\lambda}]_{A}.$$

Poisson brackets are obviously:

$$\begin{aligned}
\left\{\widehat{S}[\bar{\varkappa}]_A, \widehat{S}[\bar{\varkappa}]_B\right\} &= -\varepsilon_{AB}{}^C \widehat{S}[\bar{\varkappa}]_C, \\
\left\{\widehat{S}[\bar{\lambda}]_A, \widehat{S}[\bar{\lambda}]_B\right\} &= -\varepsilon_{AB}{}^C \widehat{S}[\bar{\lambda}]_C.
\end{aligned}$$

In quantized theory:

$$\widehat{\mathbf{S}}[\bar{\varkappa}]_A = \frac{\hbar}{i}\, {}^r E^a{}_A(\bar{\varkappa}) \frac{\partial}{\partial \varkappa^a}, \qquad \widehat{\mathbf{S}}[\bar{\lambda}]_A = \frac{\hbar}{i}\, {}^r E^a{}_A(\bar{\lambda}) \frac{\partial}{\partial \lambda^a}.$$

$\widehat{\mathbf{S}}[\bar{\varkappa}]_A$, $\widehat{\mathbf{S}}[\bar{\lambda}]_A$ acting on coefficients are as follows:

$$\begin{aligned}
\left[C^l{}_{km}{}^s{}_{nr}\right] &\mapsto \left[S^l{}_{kp} C^l{}_{pm}{}^s{}_{nr}\right], \\
\left[C^l{}_{km}{}^s{}_{nr}\right] &\mapsto \left[S^s{}_{np} C^l{}_{km}{}^s{}_{pr}\right].
\end{aligned}$$

Kinetic energy operator acts on the C- matrices on the left, through multiplication by:

$$\sum_{A=1}^{3} \frac{1}{2\mathrm{I}_A(1)} \widehat{S}^l[\bar{\varkappa}]_A^2, \qquad \sum_{A=1}^{3} \frac{1}{2\mathrm{I}_A(2)} \widehat{S}^s[\bar{\lambda}]_A^2.$$

When the tops are spherical:

$$\left[C^l{}_{km}{}^s{}_{nr}\right] \mapsto \left[\hbar^2 \left(l(l+1) + s(s+1)\right) C^l{}_{km}{}^s{}_{nr}\right]. \tag{1}$$

Also some gyroscopic coupling is possible in $\mathcal{T}$:

$$\begin{aligned} \mathcal{T} \ &= \ \sum_{A=1}^{3} \frac{1}{2\mathrm{I}_A(1)} \widehat{S}[\bar{\varkappa}]_A{}^2 + \frac{1}{2} \sum_{A=1}^{3} \frac{1}{2\mathrm{I}_A(2)} \widehat{S}[\bar{\lambda}]_A{}^2 + \\ &+ \ \sum_{A,B=1}^{3} \frac{1}{2\mathrm{I}_{AB}(1,2)} \widehat{S}[\bar{\varkappa}]_A \widehat{S}[\bar{\lambda}]_B, \end{aligned}$$

In the four-dimensional language:

$$\mathcal{T} = \frac{1}{4\mathrm{I}} \left(\overline{M} \cdot \overline{M} + \overline{N} \cdot \overline{N}\right),$$

where

$$M_A = \widehat{S}[\bar{\varkappa}]_A + \widehat{S}[\bar{\lambda}]_A, \qquad N_A = \widehat{S}[\bar{\varkappa}]_A - \widehat{S}[\bar{\lambda}]_A.$$

But there is also fourth-order Casimir

$$\left(\overline{M} \cdot \overline{N}\right)^2$$

(in $SO(4, \mathbb{R})$ there are two of them).

The possibility:

$$\mathcal{T} = \frac{1}{4\mathrm{I}}\left(\overline{M}\cdot\overline{M} + \overline{N}\cdot\overline{N}\right) + \frac{1}{4\mathrm{K}}\overline{M}\cdot\overline{N},$$

is to be rejected, because $\overline{M}\cdot\overline{N}$ is a pseudoscalar.

But one can try to assume:

$$\mathcal{T} = \frac{1}{4\mathrm{I}}\left(\overline{M}\cdot\overline{M} + \overline{N}\cdot\overline{N}\right) + \frac{1}{4L}\left(\overline{M}\cdot\overline{N}\right)^2,$$

-forth order kinetic energy.

Appendix - Bertrand model

$$\xi = R \tan \frac{r}{2R}, \qquad \bar{n} = \frac{\bar{r}}{r} = \frac{\bar{\xi}}{\xi}.$$

Conformal mapping:

$$ds^2 = \frac{4}{\left(1 + \xi^2/R^2\right)^2} \left(d\xi^2 + \xi^2 \left(d\vartheta^2 + \sin^2\vartheta d\varphi^2\right)\right),$$

i.e.,

$$ds^2 = \frac{4}{\left(1 + \xi^2/R^2\right)^2} \left(d\xi^2 + \xi^2 d\bar{n} \cdot d\bar{n}\right),$$

or in tensorial language:

$$g = \frac{4}{\left(1 + \xi^2/R^2\right)^2} \left(d\xi \otimes d\xi + \xi^2 \delta_{AB}\, dn^A \otimes dn^B\right).$$

The projective mapping:

$$\theta = 2 \tan \frac{k}{2},$$

shows that there are two Bertrand potentials:

$$V_{osc} = 2\varkappa \tan^2 \frac{k}{2}, \qquad V_{co} = -\frac{\alpha}{2} \cot^2 \frac{k}{2},$$

and of course

$$V = \text{const.}$$

Thank you !