# Recursion Systems and Recursion Operators for the Soliton Equations Related to Rational Linear Problem with Reductions

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# Systems on $\mathfrak{sl}(3)$

It is well known the most important property of the soliton eqs is that they can be written in the Lax form $[L, A] = 0$ and fixing $L$ (auxiliary problem) one obtains hierarchy of equations for which the Inverse Scattering Method and its modifications could be applied. The auxiliary linear problems

$$(\mathrm{i}\partial_x + q(x) - \lambda J)\,\psi = L\psi = 0,$$

called Generalized Zakharov-Shabat system (GZS) when $J$ is real and and Caudrey - Beals - Coifman (CBC) system when $J$ complex, together with its gauge equivalent system

$$(\mathrm{i}\partial_x - \lambda S(x))\,\tilde{\psi} = \tilde{L}\tilde{\psi} = 0$$

are systems paradigms for the theory. In the above the potential $q(x)$ and $J$ belong to the fixed simple Lie algebra $\mathfrak{g}$ in some finite dimensional irreducible representation.

The element $J$ is regular, that is the kernel of $\mathrm{ad}_J$ is a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$. the 'potential' $q(x)$ belongs to the orthogonal completion $\mathfrak{h}^\perp$ of $\mathfrak{h}$ with respect to the Killing form:

$$\langle X, Y \rangle = \mathrm{tr}\,(\mathrm{ad}_X \, \mathrm{ad}_Y)\,, \qquad X, Y \in \mathfrak{g}, \tag{1}$$

and tends to zero as $x \to \pm\infty$. The potential $S(x)$ in $\tilde{L}$ takes values in the orbit of $J$ in the coadjoint representation of $\mathfrak{g}$ and usually satisfies $\lim_{x\to\pm\infty} S(x) = J$ but this is a matter of convenience. Bearing in mind the classical case (which we obtain for $\mathfrak{g} = \mathfrak{sl}(2)$ ) if one considers $A = \mathrm{i}\partial_t + V(x, \lambda)$ and polynomial dependence of $V$ on $\lambda$ the first nontrivial systems in the hierarchies of soliton equations corresponding to $L$ and $\tilde{L}$ are called NLS and HF type equations respectively.

This talk will be about the systems of equations that are obtained from the condition of the type $[L, A] = 0$ on the coefficients of $A$ for certain linear problems $L$ similar to CBC. It happens that they can be resolved recursively and this permit to introduce an operator which is called Recursion Operator (RO). It is a central object in a theory, called AKNS approach. ROs permit to treat the issue of the so called gauge-equivalent equations and for fixed $L$ through the ROs could be obtained

- the nonlinear equations (NLEEs) integrable through $L$
- the integrals of motion of the NLEEs
- the hierarchy of symplectic structures for the NLEEs
- ROs play central role in the so-called expansions over adjoint solutions of $L$

Gerdjikov V., Vilasi G. and Yanovski A., *Integrable Hamiltonian Hierarchies – Spectral and Geometric Methods*, Springer, Heidelberg 2008.

- The CBC in pole gauge on $\mathfrak{sl}(3)$ has been studied as an application of the general results of the ROs related to $L$ and $\tilde{L}$. The Generating Operator has been calculated and systems of HF type with possible physical applications have been found.

Yanovski A., Generating operators for the generalized Zakharov-Shabat system and its gauge equivalent system in $\mathrm{sl}(3, \mathbb{C})$ case. Preprint: Universität Leipzig, Naturwissenchaftlich Theoretisches Zentrum Report N20 (1993) http://cdsweb.cern.ch/record/256804/files/P00019754.pdf

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- The interest for systems on $\mathfrak{sl}(3)$ was renewed after the systems we call polynomial and rational GMV systems were introduced. Both have been studied in relation to Mikhailov-type reductions. Their spectral properties were investigated and Generating Operators were calculated.

1. Gerdjikov V., Mikhailov A. and Valchev T., *Reductions of Integrable Equations on* ***A.III***-*Symmetric Spaces*, J. Phys. A: Math Theor. **43** (2010) 434015.

2. Gerdjikov V., Mikhailov A. and Valchev T., *Recursion Operators and reductions of integrable equations on symmetric spaces*, J. Geom. Symm. Phys. (JGSP) **20** (2010) 1–34.

- For the polynomial GMV it was pointed out that could be treated as $\mathfrak{sl}(3)$ GZS system in pole gauge with reductions. The geometric interpretation of the Recursion Operators has been clarified.

1. Yanovski A., *On the Recursion Operators for the Gerdjikov, Mikhailov and Valchev system*, J. Math. Phys. **52** (8) (2011) 082703.

2. Yanovski A.,Vilasi G., *Geometry of the Recursion Operators for the GMV system*, J. Nonlinear Math. Phys. (JNMP), **19** N3 (2012) 1250023-1/18

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2. Yanovski A.,Vilasi G., *Geometry of the Recursion Operators for the GMV system*, J. Nonlinear Math. Phys. (JNMP), **19** N3 (2012) 1250023-1/18

- The rational GMV turned out to be more complicated, the Recursion Operator theory is not yet complete.

V. S. Gerdjikov., G. G. Grahovski., A. V. Mikhailov A. and T. I. Valchev, Rational Bundles and Recursion Operators for Integrable Equations on A. III-type Symmetric Spaces, Theor. Math. Phys. (TMF) **167** N3 (2011) 740–750.

- In this talk we shall try to address the issue of the Recursion Relations and Recursion Operators for the systems associated with the rational GMV.

# Polynomial GMV

The polynomial GMV system $L_{S_1}$ is defined as

$$L_{S_1}\psi = (\mathrm{i}\partial_x + \lambda S_1)\psi = 0, \quad S_1 = \begin{pmatrix} 0 & u & v \\ u^* & 0 & 0 \\ v^* & 0 & 0 \end{pmatrix}.$$

In the above $u, v$ (the potentials) are smooth complex valued functions on $x$ belonging to the real line and by $^*$ is denoted the complex conjugation. In addition, the functions $u$ and $v$ satisfy the relation:

$$|u|^2 + |v|^2 = 1$$

Natural asymptotic conditions for $u$ and $v$ are

$$\lim_{x\to\pm\infty} u(x) = 0, \quad \lim_{x\to\pm\infty} v(x) = v_\pm = e^{\mathrm{i}\eta_\pm}$$

or

$$\lim_{x\to\pm\infty} u(x) = u_\pm = e^{\mathrm{i}\mu_\pm}, \quad \lim_{x\to\pm\infty} v(x) = 0$$

where $\eta_\pm$, $\mu_\pm$ are real constants.

The polynomial GMV arises naturally when one looks for integrable system having Lax representation $[L, A] = 0$ with $L$ of the form $\mathrm{i}\partial_x + \lambda S$, $S \in \mathfrak{sl}(3, \mathbb{C})$ and $L, A$ subject to Mikhailov-type reduction requirements. In this particular case the reduction group $G_0$ is generated by the two elements $g_0$ and $g_1$ acting on the fundamental solutions of the system as

$$g_0(\psi)(x, \lambda) = \left[\psi(x, \lambda^*)^\dagger\right]^{-1}$$

$$g_1(\psi)(x, \lambda) = H_1\psi(x, -\lambda)H_1, \quad H_1 = \mathrm{diag}\,(-1, 1, 1).$$

Since $g_0 g_1 = g_1 g_0$ and $g_0^2 = g_1^2 = \mathrm{id}$ then $G_0 = \mathbb{Z}_2 \times \mathbb{Z}_2$. $G_0$ induces an action on the loop algebra $\mathfrak{sl}(3, \mathbb{C}) \otimes [\lambda]$.
If $X = \sum_{k=0}^N \lambda^k X_k$ then this action is given by
$g_0(X)(\lambda) = \sum_{k=0}^N \sigma(X_k)\lambda^k$, $g_1(X(\lambda) = \sum_{k=0}^N \mathcal{H}_1(X_k)\lambda^k$
where $\mathcal{H}_1 : X \mapsto H_1 X H_1 = H_1 X H_1^{-1}$ and $\sigma$ is the complex conjugation that defines the real form $\mathfrak{su}(3)$ of $\mathfrak{sl}(3, \mathbb{C})$.

# Rational GMV

Consider a wider reduction group, now defined by the elements

$$g_0(\psi)(x,\lambda) = \left[\psi(x,\lambda^*)^\dagger\right]^{-1}$$
$$g_1(\psi)(x,\lambda) = H_1\psi(x,-\lambda)H_1, \quad H_1 = \mathrm{diag}\,(-1,1,1)$$
$$g_2(\psi)(x,\lambda) = H_2\psi(x,\frac{1}{\lambda})H_2, \quad H_2 = \mathrm{diag}\,(1,-1,1)$$

which is isomorphic to $\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2$. Then $L_{S_1}$ cannot admit such such reduction group, for it rational dependence on $\lambda$ is needed and the loop algebra $\mathfrak{sl}(n,\mathbb{C})\otimes[\lambda,\frac{1}{\lambda}]$ is involved. This leads to another linear problem which we call rational GMV:

$$L_{S_{\pm1}} = \mathrm{i}\partial_x + \lambda S_1 + \lambda^{-1}S_{-1}$$

The reduction group forces $S_{-1} = \mathcal{H}_2(S_1) = H_2S_1H_2$.

# GZS pole gauge recursion system

Given some $L$ we find the equations associated with it resolving some systems we call $L$-recursion systems. For example, consider $[L_{S_1}, A] = 0$, where the pair is on $\mathfrak{sl}(3)$, $S_1(x)$ is in the adjoint representation orbit of some regular element and

$$L_{S_1} = \mathrm{i}\partial_x + \lambda S_1, \quad A = \mathrm{i}\partial_t + \sum_{k=0}^{N} \lambda^k A_k.$$

Then the condition $[L_{S_1}, A] = 0$ is equivalent to

$$\begin{aligned}
&\mathrm{i}A_{0;x} = 0\\
&\mathrm{i}A_{1;x} - \mathrm{i}S_{1;t} = 0\\
&\mathrm{i}A_{k;x} + [S_1, A_{k-1}] = 0, \qquad k = 2, 3, \ldots N-1\\
&\mathrm{i}A_{N;x} + [S_1, A_{N-1}] = 0\\
&[S_1, A_N] = 0.
\end{aligned}$$

We see that basically we need to find $A_{k+1}$ if $A_k$ is known. In dealing with systems of the above type it is useful to use the Lemma we introduce below but we need some notation first. Let $\mathfrak{h}_{S_1} = \ker \mathrm{ad}_{S_1}$ be the Cartan subalgabra defined by $S_1$ (it is $x$-dependent), $\pi_+$ is the projector on the orthogonal complement $\mathfrak{h}_{S_1}^{\perp} \equiv \mathfrak{g}_{S_1}$ of $\mathfrak{h}_{S_1}$ with respect to the Killing form. For $X \in \mathfrak{sl}(3)$ we set $X^{+a} = \pi_+ X$, $X^{+d} = (\mathrm{id} - \pi_+)X$ and we also put $S_2 = S_1^2 - \frac{1}{3}\mathrm{tr}\, S_1^2 \mathbf{1}$. Then we have

## Lemma

*Suppose we need to solve with respect to $X$ the equation $\mathrm{i}\partial_x R + T = -[S_1, X]$ where $R, T, X$ are functions with values in $\mathfrak{sl}(3)$. Suppose the compatibility condition $(\mathrm{id} - \pi_+)(\mathrm{i}\partial_x R + T) = (\mathrm{i}\partial_x R + T)^{+a} = 0$ holds. Then the general solution of the above equation is $X^{+a} + D^{+d}$ where*

1. *$D^{+d}$ is arbitrary function with values in $\mathfrak{h}_{S_1}$.*
2. *$X^{+a} = \Lambda_{S_1} R^{+a} + \Theta_{S_1}(T)$.*

In the above $\Theta_{S_1}$ and $\Lambda_{S_1}$ are the operators defined by

$$\Theta_{S_1}(T) = \mathrm{ad}\,_{S_1}^{-1}(T^{+a}) +$$

$$\mathrm{ad}\,_{S_1}^{-1}(\frac{\mathrm{i}}{12}\partial_x^{-1}(\langle T^{+d}, S_1\rangle)S_{1;x} + \frac{\mathrm{i}}{4}\partial_x^{-1}(\langle T^{+d}, S_2\rangle)S_{2;x})$$

$$\Lambda_{S_1}(X) =$$

$$-\,\mathrm{ad}\,_{S_1}^{-1}\pi_+(\mathrm{i}\partial_x X + \frac{\mathrm{i}}{12}S_{1;x}\partial_x^{-1}\langle X, S_{1;x}\rangle + \frac{\mathrm{i}}{4}S_{2;x}\partial_x^{-1}\langle X, S_{2;x}\rangle) =$$

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$\Lambda_{S_1}$ is the Recursion Operator for $L_{S_1}$ (without reductions).

Indeed, the Lemma permits to resolve the $L_{S_1}$ recursion system and to obtain that the soliton equation is of the form

$$\mathrm{i}S_{1;t} = -\mathrm{ad}\,_{S_1}\Lambda_{S_1}^{N} A_N.$$

## Polynomial GMV recursion system

In order to find the Recursion Operator for the polynomial GMV system one needs to take into account the reductions. For the first one is easy to do, it simply means that all $A_k$ are Hermitian, that is belong to $\mathfrak{g} = \mathrm{i}\mathfrak{su}$. Next, the automorphism $\mathcal{H}_1$ splits the spaces $\mathfrak{g}, \mathfrak{h}_{S_1}, \mathfrak{h}_{S_1}$ into

$$\mathfrak{g} = \mathfrak{g}^{[0]} \oplus \mathfrak{g}^{[1]}$$

$$\mathfrak{h}_{S_1} = \mathfrak{h}_{S_1}^{[0]} \oplus \mathfrak{h}_{S_1}^{[1]}$$

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corresponding to eigenvalues $+1$ and $-1$ respectively. If $\pi^{[0]}, \pi^{[1]}$ are the corresponding projectors for $X \in \mathfrak{sl}(3)$ we write $\pi^{[0]}X = X^{[0]}$, $\pi^{[1]}X = X^{[1]}$.

We must have also $\mathcal{H}_1(A_l) = (-1)^l A_l$ for $l = 0, 1, 2, \ldots N$. Now, if $\mathfrak{f}_{S_1}$ is a field of spaces each defined for $S_1(x)$ (that is for each rel $x$ we have the fixed linear subspace $\mathfrak{f}_{S_1}(x) \subset \mathfrak{sl}(3, \mathbb{C})$ we shall put $\mathfrak{F}(\mathfrak{f}_{S_1})$ for the space of functions $x \mapsto \mathfrak{f}_{S_1}(x)$. With the above notation we have

$$\Lambda_{S_1}\mathfrak{F}(\mathfrak{g}^{[0]}_{S_1}) \subset \mathfrak{F}(\mathfrak{g}^{[1]}_{S_1}), \qquad \Lambda_{S_1}\mathfrak{F}(\mathfrak{g}^{[1]}_{S_1}) \subset \mathfrak{F}(\mathfrak{g}^{[0]}_{S_1}).$$

We deduce that NLEE will be consistent with the reduction if for even $N$, $A_N$ is taken into the form $\alpha S_2$ ($\alpha = \mathrm{const}$) and for odd $N$ we have $A_N = \beta S_1$, $\beta = \mathrm{const}$.
Thus effectively the hierarchy of the soliton equations related to GMV system is obtained by the action of $\Lambda^2_{S_1}$ and this is one of the reasons the Recursion Operator for the GMV system is considered to be $\Lambda^2_{S_1}$.
Stronger reasons are provided by the expansions on the adjoint solutions and the geometric picture.

# Rational GMV recursion system

The rational GMV recursion system is obtained when we consider $[L, A] = 0$ with

$$L = \mathrm{i}\partial_x + \lambda S_1 + \lambda^{-1} S_{-1}$$

$$A = \mathrm{i}\partial_t + A_0 + \sum_{k=1}^{N} (\lambda^k A_k + \lambda^{-k} A_{-k}).$$

($L$ is as in the GMV system.) Except the conditions of the type we had before, i.e.

- (i) $A_l^{\dagger} = A_l$ for $l = 0, \pm 1, \pm 2, \ldots \pm N$
  (ii) $\mathcal{H}_1(A_l) = (-1)^l A_l$ for $l = 0, \pm 1, \pm 2, \ldots \pm N$

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  (ii) $\mathcal{H}_1(A_l) = (-1)^l A_l$ for $l = 0, \pm 1, \pm 2, \ldots \pm N$

we have also

- $\mathcal{H}_2(A_l) = A_{-l}$ for $l = 0, \pm 1, \pm 2, \ldots \pm N$ and
  $\mathcal{H}_2(S_l) = S_{-l}$ for $l = \pm 1$ where $\mathcal{H}_2$ is the involution
  $\mathcal{H}_2(X) = H_2 X H_2$, $H_2 = \mathrm{diag}\,(1, -1, 1)$.

Thus half of the equations become consequences of the other half and we obtain the rational GMV recursion system

$$\begin{aligned}
&\mathrm{i}A_{0;x} + (\mathrm{id} + \mathcal{H}_2)[S_{-1}, A_1] = 0 \\
&\mathrm{i}A_{1;x} - \mathrm{i}S_{1;t} + [S_1, A_0] + [S_{-1}, A_2] = 0 \\
&\mathrm{i}A_{k;x} + [S_1, A_{k-1}] + [S_{-1}, A_{k+1}] = 0, \quad k = 2, 3, \dots N-1 \\
&\mathrm{i}A_{N;x} + [S_1, A_{N-1}] = 0 \\
&[S_1, A_N] = 0.
\end{aligned}$$

The system can be resolved recursively starting from $A_N$, but we were unsuccessful to find an operator playing the same role as $\Lambda_{S_1}$ plays for the polynomial GMV system and acting on the coefficient functions $A_k$. The main difficulty is that when we resolve for $A^{+a}_{k-1}$ we find it through $A^{+a}_k$ and $A^{-a}_k$, where by $X^{-a}$ is denoted the orthogonal projection of $X$ onto the space $\mathfrak{g}_{S_{-1}}$, orthogonal to $\mathfrak{h}_{S_{-1}} = \ker \mathrm{ad}_{S_{-1}}$. So we decided to try another idea, to see whether we can find a 'recursion' operator on some linear combination of the $A_k$'s.

Let us introduce:

$$P_k = A_k + A_{-k} = A_k + \mathcal{H}_2(A_k) = (\mathrm{id} + \mathcal{H}_2)(A_k)$$
$$Q_k = A_{k-1} + A_{-(k+1)} = A_{k-1} + \mathcal{H}_2(A_{k+1}).$$

We extend the definition of the matrices $P_k$ and $Q_k$ for arbitrary $k \in \mathbb{Z}$ assuming that $A_k = 0$ if $|k| > N$. $P_k$ and $Q_k$ have the properties

$$\mathcal{H}_1 P_k = (-1)^k P_k, \quad \mathcal{H}_1 Q_k = (-1)^{k+1} Q_k, \quad \mathcal{H}_2 P_k = P_k$$
$$P_k = P_{-k}, \quad Q_k = Q_{-k}.$$

Further, for $k \geq 1$

$$A_k = Q_{k+1} - \mathcal{H}_2(Q_{k+3}) + Q_{k+5} - \mathcal{H}_2(A_{k+7}) + \dots$$

(since $Q_s = 0$ for $s > N + 1$ the above series is finite). In particular, since $Q_0 = 2\mathcal{H}_2(A_1)$ we have that

$$\frac{1}{2}Q_0 = \mathcal{H}_2(A_1) = \mathcal{H}_2(Q_2) - Q_4 + \mathcal{H}_2(Q_6) - \dots \equiv \frac{1}{2}F(Q).$$

## Summarizing

### Proposition

*The set of $Q_k$, $k = 0, 1, \ldots N+1$ determines uniquely the quantities $A_k$. $Q_0$ (and $A_1$) is a linear combination $F(Q)$ of the $Q_{2s}, \mathcal{H}_2(Q_{2s})$ for $s \geq 1$.*

Using the rational GMV system recursion relations it is easy to check that for $|k| > 2$ we have

$$\begin{aligned} &\mathrm{i}\partial_x P_k + (\mathrm{id} + \mathcal{H}_2)[S_1, Q_k] = 0 \\ &\mathrm{i}\partial_x Q_k - [S_1 - S_{-1}, P_k] + [S_1, Q_{k-1} + Q_{k+1}] = 0. \end{aligned}$$

We solve the first equation for $P_k$ and introduce into the second one. We then get

$$\mathrm{i}\partial_x Q_k - [S_1 - S_{-1}, \mathrm{i}\partial_x^{-1}(\mathrm{id} + \mathcal{H}_2)[S_1, Q_k]] = -[S_1, Q_{k+1} + Q_{k-1}].$$

This system is of the type that is considered in our Lemma but in order to write the things in a concise way we shall introduce some notation. First, we introduce the operator $\Omega_{S_1}$ by

$$\Omega_{S_1}(Z) = [S_1 - S_{-1}, \mathrm{i}\partial_x^{-1}(\mathrm{id} + \mathcal{H}_2)[S_1, Z]]$$

In fact $\Omega_{S_1}(Z)$ depends only on the projection of $Z$ on the space $\mathfrak{g}_{S_1}$, that is $\Omega_{S_1}(Z) = \Omega_{S_1}(Z^{+a})$. We also note that for $\Omega_{S_1}$ and for $\Theta_{S_1}$ introduced earlier we have:

$$\Omega_{S_1}(\mathfrak{F}(\mathfrak{g}_{S_1}^{[0]}) \subset \mathfrak{F}(\mathfrak{g}_{S_1}^{[0]}), \qquad \Omega_{S_1}(\mathfrak{F}(\mathfrak{g}_{S_1}^{[1]}) \subset \mathfrak{F}(\mathfrak{g}_{S_1}^{[1]}).$$

$$\Theta_{S_1}(\mathfrak{F}(\mathfrak{g}_{S_1}^{[0]}) \subset \mathfrak{F}(\mathfrak{g}_{S_1}^{[1]}), \qquad \Theta_{S_1}(\mathfrak{F}(\mathfrak{g}_{S_1}^{[1]}) \subset \mathfrak{F}(\mathfrak{g}_{S_1}^{[0]}).$$

Now, using our Lemma we get

$$(\Lambda_{S_1} - \Theta_{S_1} \circ \Omega_{S_1})Q_k^{+a} = Q_{k+1}^{+a} + Q_{k-1}^{+a}.$$

It is obvious that if we put $\mathbf{R}_{S_1} = \Lambda_{S_1} - \Theta_{S_1} \circ \Omega_{S_1}$ the above equation will have even nicer form

$$\mathbf{R}_{S_1}(Q_k^{+a}) = Q_{k+1}^{+a} + Q_{k-1}^{+a}.$$

$\mathbf{R}_{S_1}$ has the properties

$$\mathbf{R}_{S_1}(\mathfrak{F}(\mathfrak{g}^{[0]}_{S_1}) \subset \mathfrak{F}(\mathfrak{g}^{[1]}_{S_1}), \qquad \mathbf{R}_{S_1}(\mathfrak{F}(\mathfrak{g}^{[1]}_{S_1}) \subset \mathfrak{F}(\mathfrak{g}^{[0]}_{S_1}).$$

In order to resolve (2) we have the compatibility condition

$$(\mathrm{id} - \pi_+)(\mathrm{i}\partial_x Q_k + [S_{-1}, P_k]) = 0.$$

One checks that the compatibility equation remains true even for $|k| \leq 2$ so it is valid for all $k$). It is easily put into the form

$$\langle S_1, \mathrm{i}\partial_x Q_k\rangle + \langle S_1, [S_{-1}, P_k]\rangle = 0, \quad k \text{ even}$$
$$\langle S_2, \mathrm{i}\partial_x Q_k\rangle + \langle S_2, [S_{-1}, P_k]\rangle = 0 \quad k \text{ odd.}$$

Then for example if $k$ is even we have already that $\langle S_1, P_k\rangle = 0$ and further

$$\mathrm{i}\partial_x\langle S_1, Q_k\rangle = \mathrm{i}\langle S_{1;x}, Q_k^{+a}\rangle + \langle S_1, \mathrm{i}\partial_x Q_k\rangle =$$
$$\mathrm{i}\langle S_{1;x}, Q_k^{+a}\rangle - \langle [S_{-1}, S_1], P_k]\rangle$$

showing that we can recover $Q_k^{+d}$ from $P_k$ and $Q_k^{+a}$, that is from $Q_k^{+a}$. Skipping the technical details we obtain

## Proposition

*The rational GMV recursion system could be resolved in the following way: First we resolve*

$$\mathbf{R}_{S_1} Q_k = Q^a_{k+1} + Q^{+a}_{k-1}, \quad P_k = \Omega_{S_1}(Q^{+a}_k) \quad N+1 \leq k \leq 2$$

*to find $Q_2$ as a function of $S_1, S_2$ and their $x$-derivatives. Then using $Q_2$ and $Q_0 = F(Q)$ we resolve*

$$\mathrm{i}\partial_x Q_1 + \frac{1}{2}(\mathrm{id} + \mathcal{H}_2)[S_1, Q_0] + \frac{1}{2}(\mathrm{id} - \mathcal{H}_2)[S_{-1}, Q_0] + [S_1, Q_2] = 0$$

*to find $Q_1$ and obtain the corresponding soliton equation*

$$\mathrm{i}\,\mathrm{ad}^{-1}_{S_1} S_{1;t} = Q^{+a}_3 - \mathbf{R}_{S_1} Q^{+a}_2 + Q^{+a}_1$$

*Finally $P_0 = \Omega_{S_1}(Q^{+a}_0)$ and $P_1 = \frac{1}{2}(\mathrm{id} + \mathcal{H}_2)Q_0$ are also determined thus finding all the functions $A_k$ needed for the Lax representation of the NLEE.*

- The above suggests that $\mathbf{R}_{S_1}$ could be the wanted Recursion Operator because as one can see we can obtain recursively all $Q_k^{+a}$ though the recursion procedure looks now different and more complicated comparing with the one we had in the polynomial GMV case.

- The above suggests that $\mathbf{R}_{S_1}$ could be the wanted Recursion Operator because as one can see we can obtain recursively all $Q_k^{+a}$ though the recursion procedure looks now different and more complicated comparing with the one we had in the polynomial GMV case.
- There is additional reason why $\mathbf{R}_{S_1}$ (or rather its square) is the Recursion Operator. It is known that the Recursion Operators appear in several roles:
  1. Resolve the recursion systems and in the case of polynomial GMV we had the square of $\Lambda_{S_1}$.
  2. For them some projections of the adjoint solutions to the auxiliary problems are eigenfunctions and as we will see here appears exactly $\mathbf{R}^2_{S_1}$.
  3. They are related with the bi-Hamiltonian properties of the soliton equations and for the polynomial GMV we again have the square of $\Lambda_{S_1}$.

# Adjoint solutions approach

We discuss now a result about the Recursion Operator based on (2). In such an approach we start from Wronskian type relations to what to consider as 'adjoint solutions', then we find the operators for which these solutions are eigenfunctions and finally we must find completeness relations. Expanding on them the potential, as coefficients we obtain the scattering data. (This is known as the AKNS approach). For the rational GMV system the above is done only partially, though we have information from private communication that the final step – the completeness relations is also done.

Gerdjikov V., Grahovski G., Mikhailov A. and Valchev T., *Rational Bundles and Recursion Operators for Integrable Equations on A. III-type Symmetric Spaces*, Theor. Math. Phys. (TMF) **167** N3 (2011) 740–750.

However, the Recursion Operator presented is given in a form that is difficult to relate directly with the one we introduced so we give a sketch of it.

Let $\chi$ be a fundamental solution to $L_{S_{\pm 1}}\chi = 0$. Then if $A$ is a fixed constant matrix, the function $\Phi_A = \chi A \hat{\chi}$ (where $\hat{\chi} = \chi^{-1}$ will satisfy the equation

$$\mathrm{i}\partial_x \Phi_A + [\lambda S_1 + \lambda^{-1} S_{-1}, \Phi_A] = 0.$$

Let us introduce the functions

$$\Phi_{A;k} = \lambda^k \Phi_A + \lambda^{-k}\mathcal{H}_2(\Phi_A), \quad k = 0, \pm 1, \pm 2, \ldots.$$

The attention to these functions is motivated by the following Wronskian identities

$$\mathrm{i}\langle[\chi^{-1}A\chi](x,\lambda) - A, E_\alpha\rangle\big|_{x=-\infty}^{+\infty} = \int_{-\infty}^{+\infty} \langle [S_1(x), A], \Phi_{A;1}(x,\lambda)\rangle \mathrm{d}x$$

$$\mathrm{i}\langle \chi^{-1}\delta\chi(x,\lambda), E_\alpha\rangle\big|_{x=-\infty}^{+\infty} = \int_{-\infty}^{+\infty} \langle \delta S_1(x), \Phi_{A;1}(x,\lambda)\rangle \mathrm{d}x.$$

In these relations $A$ is a constant diagonal matrix; $E_\alpha$ is any of the Cartan generators for the canonical choice of the Cartan subalgebra, that is when it is the subalgebra of the diagonal matrices and $\delta\chi$ is the variation of the fundamental solution $\chi$ corresponding to variation $\delta S_1$ of $S_1$.

Note that not only in these relations enter only the projections $\pi_+\Phi_{A;1}(x,\lambda) = \Phi^{+a}_{A;1}(x,\lambda)$ on the orthogonal complement $\mathfrak{g}_{S_1}$ to the Cartan subalgebra $\mathfrak{h}_{S_1}$ but because all diagonal matrices in $\mathfrak{g}$ belong to $\mathfrak{g}^{[0]}$ in fact in the Wronskian relations enters only the projection of $\pi_+\Phi^{[1]}_{A;1}(x,\lambda)$ on $\mathfrak{g}^{[0]}$ which we denote by $\Phi^{[1]+a}_{A;1}(x,\lambda)$ (note that $\pi_+$, the projector on the orthogonal complement $\mathfrak{g}_{S_1}$ to the Cartan subalgebra $\mathfrak{h}_{S_1}$. commutes with $\pi^{[0]}$ and $\pi^{[1]}$).

The experience one has from the study of GZS, CBC and polynamial GMV systems suggests that the functions $\Phi^{[1]+a}_{A;1}(x,\lambda)$ involved into these relations are eigenfunctions for the Generating Operators.
The functions $\Phi_{A;-k}$ have some basic properties. On the first place

$$\mathcal{H}_2(\Phi_{A;k}) = \Phi_{A;-k}, \qquad \mathcal{H}_2(\Phi_{A;0}) = \Phi_{A;0}.$$

Also, the functions $\Phi_{A;k}$ satisfy the equations

$$\mathrm{i}\partial_x \Phi_{A;k} + [S_1, \Phi_{A;k+1}] + [S_{-1}, \Phi_{A;k-1}] = 0.$$

In particular, since $S_{-1} = \mathcal{H}_2(S_1)$ the equation for $\Phi_{A;0}$ can be written into the form

$$\mathrm{i}\partial_x \Phi_{A;0} + (\mathrm{id} + \mathcal{H}_2)[S_1, \Phi_{A;1}] = 0.$$

In order to obtain a closed system the idea carried out in the paper we cited was to write the equation for $\Phi_{A;1}$ into the following equivalent form

$$\mathrm{i}\partial_x \Phi_{A;1} - [S_1 - S_{-1}, \Phi_{A;0}] = -(\lambda + \lambda^{-1})[S_1, \Phi_{A;1}].$$

Using our Lemma this permits us to write immediately

$$\mathbf{R}_{S_1}\Phi^{+a}_{A;1} = (\lambda + \lambda^{-1})\Phi^{+a}_{A;0}.$$

Let us introduce the splittings:

$$\Phi_{A;0} = \Phi^{[0]}_{A;0} + \Phi^{[1]}_{A;0}, \quad \Phi_{A;1} = \Phi^{[0]}_{A;1} + \Phi^{[1]}_{A;1}$$

corresponding to the splitting $\mathfrak{g} = \mathfrak{g}^{[0]} \oplus \mathfrak{g}^{[1]}$ and let us put

$$\Phi^{[0]+a}_{A;1} = (\Phi^{[0]}_{A;1})^{+a} = \pi_+\Phi^{[0]}_{A;1}, \quad \Phi^{[1]+a}_{A;1} = (\Phi^{[1]}_{A;1})^{+a} = \pi_+\Phi^{[1]}_{A;1}.$$

(recall that $\pi_+$ and the projectors $\pi^{[0]}$ and $\pi^{[1]}$ commute). Then using we immediately get

$$\mathbf{R}_{S_1}\Phi^{[0]+a}_{A;1} = (\lambda + \lambda^{-1})\Phi^{[1]+a}_{A;1}, \quad \mathbf{R}_{S_1}\Phi^{[1]+a}_{A;1} = (\lambda + \lambda^{-1})\Phi^{[0]+a}_{A;1}.$$

As a consequence,

$$\mathbf{R}^2_{S_1}\Phi^{[0]+a}_{A;1} = (\lambda+\lambda^{-1})^2\Phi^{[0]+a}_{A;1}, \quad \mathbf{R}^2_{S_1}\Phi^{[1]+a}_{A;1} = (\lambda+\lambda^{-1})^2\Phi^{[1]+a}_{A;1}.$$

There are number of terms that cancel when one calculates explicitly the action of $\mathbf{R}^2_{S_{\pm 1}}$ on $(\Phi^{[0]}_{A;1})^{+a}$ and $(\Phi^{[0]}_{A;1})^{+a}$. If you do those cancellations it is hard to notice that it is the same operator involved so in the work we cited

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were introduced two operators $\Lambda_1, \Lambda_2$ such that

$$\mathbf{R}^2_{S_1}\Phi^{[0]+a}_{A;1} = \Lambda_2\Lambda_1\Phi^{[0]+a}_{A;1}, \quad \mathbf{R}^2_{S_1}\Phi^{[1]+a}_{A;1} = \Lambda_1\Lambda_2\Phi^{[1]+a}_{A;1}.$$

As we explained already this situation is rather typical when one has $\mathbb{Z}_2$ reductions. What one has is that the operator $\mathbf{R}_{S_1}$ restricted to functions taking values in $\mathfrak{g}^{[0]}$ is equal to $-\Lambda_1$ and restricted to functions taking values in $\mathfrak{g}^{[1]}$ is equal to $-\Lambda_2$.

# Conclusion

As we have mentioned already the Recursion Operators associated with an auxiliary linear problem $L$ appear in several different roles:

1. Resolve the recursion systems for the soliton equations associated with $L$.
2. For them some projections of the adjoint solutions to the auxiliary problems are eigenfunctions.
3. Their adjoint relate two Hamiltonian structures for the NLEEs associated with $L$.

We believe we established already the property (1). The first part of (2) is as we have seen true, according to a private communication its second part is also true. The geometric properties of $\mathbf{R}_{S_1}$ are not yet treated and it will be an interesting task to study them.

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Thank you for your attention!