NORMAL TRANSPORT SURFACES IN EUCLIDEAN SPACES

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1. Introduction

✓ The 3D offsets or parallel surfaces are very widely used in many applications;

■ tool path generation for 3N machining (Mechawa, 1999) and (Pham,1992),

■ pre-process modifications to CAD geometry (Farouki,1985), (Forsyth, 1995).

✓ Focal surfaces are known in the field of line congruences (Hagen and Pottmann. Focal surfaces are used in;

- visualization (Hagen and Pottmann,1991).
- surface interrogation tool in a NC-milling operation (Hagen and Hahman, 1992).

✓ The normal transport surfaces are the generalization of offset surfaces in 4-dimensional Euclidean space $\mathbb{E}^4$ (Frohlich,2013).

2. Basic Concepts

We recall definitions and results of (Fröhlich,2013).
Let M be a local surface in $\mathbb{E}^{n+2}$ given with the regular patch $x(u,v) : (u,v) \in D \subset \mathbb{E}^2$.
The tangent space $T_p(M)$ to M at an arbitrary point $p = x(u,v)$ of M is spanned by $\{x_u, x_v\}$.
For the coefficients

$$g_{11} = \langle x_u, x_u \rangle, g_{12} = \langle x_u, x_v \rangle, g_{22} = \langle x_v, x_v \rangle, \tag{2.1}$$

the first fundamental form of M is given by

$$ds^2 = \sum_{i,j=1}^{2} g_{ij} du^i du^j. \tag{2.2}$$

The **Gauss equation** of the surface M is given by

$$x_{u^i u^j} = \widetilde{\nabla}_{x_{u^i}} x_{u^j} = \sum_{k=1}^{2} \Gamma_{ij}^k x_{u^k} + \sum_{\alpha=1}^{n} c_{ij}^\alpha N_\alpha, \tag{2.3}$$

where

$$c_{ij}^\alpha = \langle x_{u^i u^j}, N_\alpha \rangle ; \; c_{ij}^\alpha = c_{ji}^\alpha, \tag{2.4}$$

are the **coefficients of the second fundamental form** and

$$\Gamma_{ij}^k = \sum_{l=1}^{2} g^{kl} \left(\frac{\partial g_{jl}}{\partial u^i} + \frac{\partial g_{li}}{\partial u^j} - \frac{\partial g_{ij}}{\partial u^l} \right), \tag{2.5}$$

Christoffel symbols corresponding to $x(u, v)$.

The **Weingarten equation** of the surface M is given by

$$(N_\alpha)_{u^i} = \widetilde{\nabla}_{x_{u^i}} N_\alpha = -\sum_{k=1}^{2} c_\alpha^{ik} x_{u^k} + \sum_{\beta=1}^{n} T_i^{\alpha\beta} N_\beta, \tag{2.6}$$

where

$$c_\alpha^{ik} = \sum_{j=1}^{2} c_{ij}^\alpha g^{jk}; \; c_\alpha^{ik} = c_\alpha^{ki}, \tag{2.7}$$

are the **Weingarten forms** of M with respect to N_α

$$T_i^{\alpha\beta} = \langle (N_\alpha)_{u^i}, N_\beta \rangle ; T_i^{\alpha\beta} = -T_i^{\beta\alpha}, \; i = 1, 2, \tag{2.8}$$

torsion coefficients and

$$\left(g^{ij}\right)_{i,j=1,2} = \frac{1}{g} \begin{bmatrix} g_{22} & -g_{12} \\ -g_{21} & g_{11} \end{bmatrix}. \tag{2.9}$$

The **Gaussian curvature** of the surface M is defined by

$$K = \sum_{\alpha=1}^{n} K_\alpha, \quad K_\alpha = \frac{c_{11}^{\alpha} c_{22}^{\alpha} - (c_{12}^{\alpha})^2}{g}. \tag{2.10}$$

The Gaussian curvature vanishes M is called **flat surface**.
Observe that

$$K_\alpha = c_\alpha^{11} c_\alpha^{22} - (c_\alpha^{12})^2. \tag{2.11}$$

The **mean curvature vector field** $\overrightarrow{H}$ of the surface M is defined by

$$\overrightarrow{H} = \sum_{\alpha=1}^{n} H_\alpha N_\alpha, \tag{2.12}$$

where

$$H_\alpha = \frac{1}{2} \sum_{i,j=1}^{2} g^{ij} c_{ij}^{\alpha} = \frac{g_{22} c_{11}^{\alpha} + g_{11} c_{22}^{\alpha} - 2 g_{12} c_{12}^{\alpha}}{2g}, \tag{2.13}$$

The **mean curvature** H of M is defined by $H = \left\| \overrightarrow{H} \right\|$.
The mean curvature (vector) vanishes M is called **minimal**.
Observe that

$$H_\alpha = \frac{c_\alpha^{11} + c_\alpha^{22}}{2}. \tag{2.14}$$

The **curvature tensor of the normal bundle** NM of the surface M is defined by

$$S_{ij}^{\alpha\beta} = \left(T_i^{\alpha\beta}\right)_{u^j} - \left(T_j^{\alpha\beta}\right)_{u^i} + \sum_{\sigma=1}^{n}\left(T_i^{\alpha\sigma}T_j^{\sigma\beta} - T_j^{\alpha\sigma}T_i^{\sigma\beta}\right),$$
$$= \sum_{m,n=1}^{2}\left(c_{1m}^{\alpha}c_{n2}^{\beta} - c_{2m}^{\alpha}c_{n1}^{\beta}\right) g^{mn}; 1 \leq \alpha, \beta \leq n. \tag{2.16}$$

The equality

$$S_N^{\alpha\beta} = \frac{1}{\sqrt{g}}S_{12}^{\alpha\beta}, \tag{2.17}$$

is called the **normal sectional curvature** with respect to the plane $\Pi = span\left\{x_u, x_v\right\}$.

For the case $n = 2$ the scalar curvature of its normal bundle is defined as

$$K_N = S_N^{12} = \frac{1}{\sqrt{g}} S_{12}^{12}. \tag{2.18}$$

which is also called **normal curvature of the surface** M in $\mathbb{E}^4$. Observe that

$$K_N = \frac{1}{\sqrt{g}} \left(\left(T_2^{12} \right)_u - \left(T_1^{12} \right)_v \right). \tag{2.19}$$

We observe that the **normal connection** D of M **is flat** if and only if $K_N = 0$, and by a result of Cartan, this equivalent to the diagonalisability of all shape operators A_{N_α} of M, which means that M is a **totally umbilical** surface in $\mathbb{E}^{n+2}$.

3. Generalized Focal Surfaces in E^3

Given a set of unit vectors $E(u,v)$ one can define a **line congruence**:

$$C(u,v) = x(u,v) + D(u,v)E(u,v) \tag{3.1}$$

where $D(u,v)$ is called the signed distence between $X(u,v)$ and $E(u,v)$.
If $E(u,v) = N(u,v)$, then C is normal congruence.
A **focal surface** $C_F(u,v)$ is a special normal congruence with $D(u,v) = k_1^{-1}(u,v)$ or $D(u,v) = k_2^{-1}(u,v)$:

$$C_F(u,v) = x(u,v) + k_i^{-1}(u,v)N(u,v), \quad i = 1,2. \tag{3.2}$$

The generalization of this classical concept leads to the **generalized focal surfaces**:

$$y(u,v) = x(u,v) + F(k_1,k_2)N(u,v), \tag{3.3}$$

where N is the unit normal vector of the surface $x(u,v)$ and F is a real valued function (**offset function**) in the parameter values u and v (Hahmann,1999).

If the offset function F depends on the principal curvatures k_1 and k_2 of M then one can choose the variable offset function as;

1. $F = k_1 k_2$, Gaussian curvature,
2. $F = \frac{1}{2}(k_1 + k_2)$, mean curvature,
3. $F = k_1^2 + k_2^2$, energy functional,
4. $F = |k_1| + |k_2|$, absolute functional,
5. $F = k_i, 1 \leq i \leq 2$, principal curvature,
6. $F = \frac{1}{k_i}$, focal points,
7. $F = const.$, parallel surface.

The different offset functions listed above can now be used to interrogate and visualize surfaces with respect to the following criteria:

- convexity test,
- detection of flat points,
- detection of surface integration,
- visualization of curvature behaviour,
- visualization of technical smoothness,
- visualization of C^2 and C^3 discontinuities,
- test of technical aspects.

In (Özdemir and ——,2008) the following offset functions are used to construct general focal surfaces;

a. $F = K$, Gaussian curvature,
b. $F = H$, mean curvature,
c. $F = K - H$, diference of the curvatures,
d. $F = H + \sqrt{H^2 - K}$,or
e. $F = H - \sqrt{H^2 - K}$ principal curvature,
f. $F = 4H^2 - 2K$, energy functional,
g. $F = \left|H + \sqrt{H^2 - K}\right| + \left|H - \sqrt{H^2 - K}\right|$, absolute curvature,
h. $F = c$ offset surfaces.

3.1. Visualization

Translation surfaces in 3-dimensional Euclidean space $\mathbb{E}^3$ *are given with the paremetrization*

$$x(u, v) = (u, v, h(u, v)); \; h(u, v) = f(u) + g(v).$$

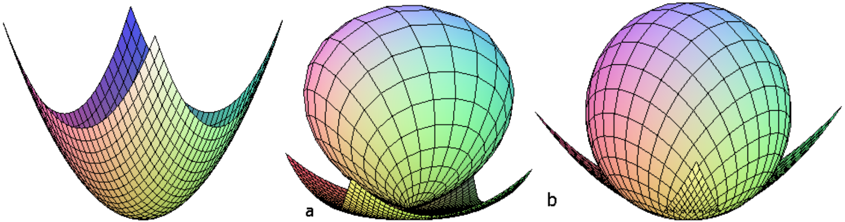
In (Özdemir and ——,2008) we give some examples of generalized focal surfaces of traslation surfaces;

Example (1)

Consider the **paraboloid** $h(u, v) = u^2 + v^2$.
Gaussian and mean curvatures;

$$K = \frac{4}{(4u^2 + 4v^2 + 1)^2},$$

$$H = \frac{2(1 + 2v^2 + 2u^2)}{(4u^2 + 4v^2 + 1)^{\frac{3}{2}}}.$$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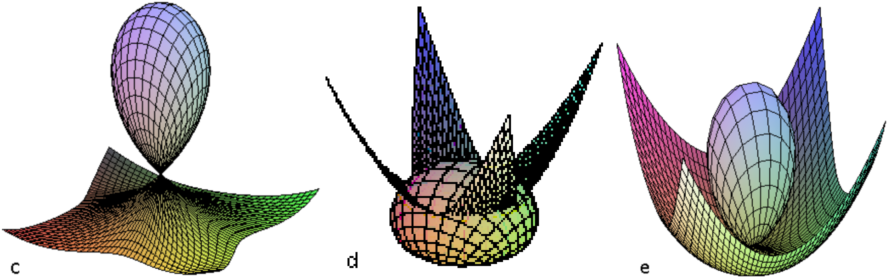
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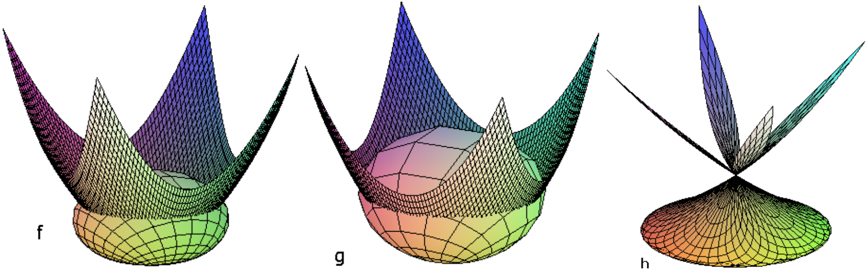

f
g
h

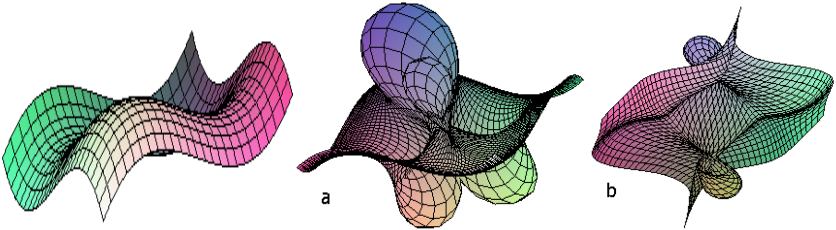
Example (2)

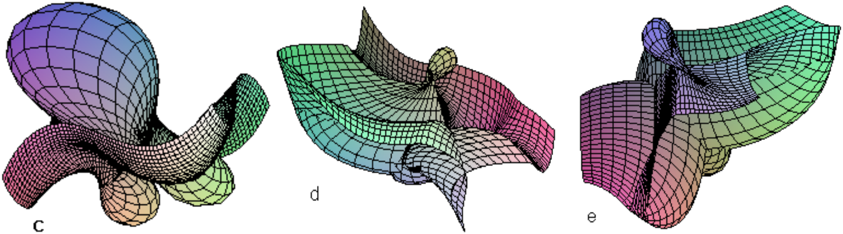
Consider the **cubic function** $h(u,v) = u^3 + v^3$. We can calculate th Gaussian and mean curvatures;

$$K = \frac{36uv}{(9u^4 + 9v^4 + 1)^2},$$

$$H = \frac{3(u + 9uv^4 + v + 9vu^4).}{(9u^4 + 9v4 + 1)^{\frac{3}{2}}}$$

3.1. Visualization

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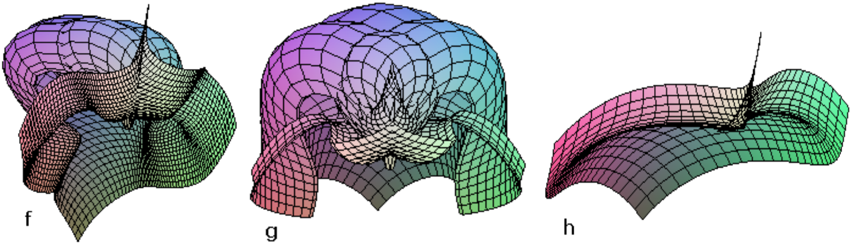
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4. Normal transport surfaces in E4

The **normal transport surface** $\widetilde{M}$ of M are generalization of offset surfaces to 4-dimensional Euclidean space $\mathbb{E}^4$ (Fröhlich, 2013). Observe that, **evolute surfaces** and **parallel type surfaces** in $\mathbb{E}^4$ are the special type normal transport surfaces (Krivonosov, 1970), (Cheshkova, 2001), (Fröhlich, 2013).

Parallel type surface are widely used in geometry and mathematical physics (Fröhlich, 2013).

4.1 Surfaces with Flat Normal Bundle

Definition (1)

Let M be a local surface in $\mathbb{E}^{n+2}$. The mean curvature vector $\overrightarrow{H}$ is **parallel in the normal bundle** if and only if

$$(H_\alpha)_u^\perp = 0, (H_\alpha)_v^\perp = 0, \tag{4.1}$$

holds (Fröhlich, 2013). Equivalently

$$(H_\alpha)_{u^i} = \sum_{\beta=1}^{n} H_\beta T_i^{\alpha\beta}. \tag{4.2}$$

The following result due to (Fröhlich, 2013).

Theorem (1)

The mean curvature vector $\overrightarrow{H}$ is called parallel in the normal bundle if and only if the squared mean curvature $\left\|\overrightarrow{H}\right\|^2$ of M is a constant function.

Definition (2)

A local surface of $\mathbb{E}^{n+2}$ is said to have **flat normal bundle** if and only if the orthonormal frame $N_1, ...N_n$ of M is of torsion free.

Fact

The existence of flat normal bundle of M is equivalent to say that normal curvature K_N of M vanishes identically.

The following classification result due to Chen from (Chen, 1972).

Theorem (2)

Let M be an immersed surface in $\mathbb{E}^{n+2}$. If $\overrightarrow{H} \neq 0$ is parallel in the normal bundle then either M is a minimal surface of a hypersphere of $\mathbb{E}^{n+2}$, or it has flat normal bundle.

4.2 Normal transport Surfaces

Let M and $\widetilde{M}$ be two smooth surfaces in Euclidean 4-space $\mathbb{E}^4$ and let $\varphi : M \to \widetilde{M}$ be a diffeomorphism. Then the surface $\widetilde{M}$ enveloping family of normal 2-planes to M is called the **normal transport** of M in $\mathbb{E}^4$ (Fröhlich, 2013).
Further, let $\overrightarrow{x}$ be a position (radius) vector of $p \in M$, and $\widetilde{x}$ be the position (radius) vector of the point $\varphi(p) \in \widetilde{M}$.
Then the mapping $\varphi : M \to \widetilde{M}$ has the form

$$\widetilde{x} = x + \overrightarrow{w}, \quad \overrightarrow{w} \in T_p^{\perp} M.$$

where, $\overrightarrow{p\varphi(p)} = \overrightarrow{w}(p)$, $\overrightarrow{w}(p) \in T_p^{\perp} M$ is the normal vector to M.

For the case

$$\overrightarrow{w}(p) = \sum_{i=1}^{2} f_i(u,v) N_i(u,v),$$

the normal transport surface $\widetilde{M}$ of M given by

$$\widetilde{M} : \widetilde{x}(u,v) = x(u,v) + \sum_{i=1}^{2} f_i(u,v) N_i(u,v), \tag{4.3}$$

where $f_i \quad (i = 1, 2)$ are offset functions (Fröhlich, 2013).

The tangent space to $\widetilde{M}$ at an arbitrary point $p = \widetilde{x}(u, v)$ of $\widetilde{M}$ is spanned by

$$\begin{aligned} \widetilde{x}_u &= x_u + f_1 (N_1)_u + f_2 (N_2)_u + (f_1)_u N_1 + (f_2)_u N_2, \\ \widetilde{x}_v &= x_v + f_1 (N_1)_v + f_2 (N_2)_v + (f_1)_v N_1 + (f_2)_v N_2. \end{aligned} \tag{4.4}$$

Further, using the Weingarten equation (2.6) we get

$$\begin{aligned} (N_1)_u &= -\left(c_1^{11} x_u + c_1^{12} x_v\right) + T_1^{12} N_2 \\ (N_2)_u &= -\left(c_2^{11} x_u + c_2^{12} x_v\right) - T_1^{12} N_1 \\ (N_1)_v &= -\left(c_1^{21} x_u + c_1^{22} x_v\right) + T_2^{12} N_2 \\ (N_2)_v &= -\left(c_2^{21} x_u + c_2^{22} x_v\right) - T_2^{12} N_2. \end{aligned} \tag{4.5}$$

So, substituting (4.5) into (4.4) we get

$$\begin{aligned} \widetilde{x}_u = & \left(1 - f_1 c_1^{11} - f_2 c_2^{11}\right) x_u - \left(f_1 c_1^{12} + f_2 c_2^{12}\right) x_v \\ & + \left((f_1)_u - f_2 T_1^{12}\right) N_1 + \left((f_2)_u + f_1 T_1^{12}\right) N_2, \end{aligned} \tag{4.6}$$

$$\begin{aligned} \widetilde{x}_v = & -\left(f_1 c_1^{21} + f_2 c_2^{21}\right) x_u + \left(1 - f_1 c_1^{22} - f_2 c_2^{22}\right) x_v \\ & + \left((f_1)_v - f_2 T_2^{12}\right) N_1 + \left((f_2)_v + f_1 T_2^{12}\right) N_2. \end{aligned} \tag{4.7}$$

Definition (3)

i) The normal transport surface $\widetilde{M}_H$ given with the parametrization

$$\widetilde{M}_H : \widetilde{x}(u,v) = x(u,v) + H_1(u,v)\ N_1(u,v) + H_2(u,v)\ N_2(u,v),$$

is called **normal transport surface of H-type**.

ii) The normal transport surface $\widetilde{M}_K$ given with the parametrization

$$\widetilde{M}_K : \widetilde{x}(u,v) = x(u,v) + K_1(u,v)\ N_1(u,v) + K_2(u,v)\ N_2(u,v),$$

is called **normal transport surface of K-type**.

4.3.Parallel surfaces in E4

Definition (4)

The normal transport surface $\widetilde{M}$ of M is called **parallel surface** of M in $\mathbb{E}^4$ if the equality

$$\langle \widetilde{x}_{u_i}, N_\alpha \rangle = 0, \ 1 \leq i, \alpha \leq 2, \tag{4.8}$$

holds for all $N_\alpha \in T_p^\perp M$ (Fröhlich, 2013).

Let $\widetilde{M}$ be a parallel surface of M in $\mathbb{E}^4$ with non-zero offset functions f_1 and f_2. Then by use of (4.6) and (4.7) with (4.8) one can get

$$\begin{aligned} 0 &= \langle \widetilde{x}_u, N_1 \rangle = (f_1)_u - f_2 T_1^{12}, \\ 0 &= \langle \widetilde{x}_v, N_1 \rangle = (f_1)_v - f_2 T_2^{12}, \\ 0 &= \langle \widetilde{x}_u, N_2 \rangle = (f_2)_u + f_1 T_1^{12}, \\ 0 &= \langle \widetilde{x}_v, N_2 \rangle = (f_2)_v + f_1 T_2^{12}. \end{aligned} \quad (4.9)$$

Differentiating the first two equations and making use of the other equations shows us

$$\begin{aligned} (f_1)_{uv} + f_1 T_2^{12} T_1^{12} - f_2 \left(T_1^{12}\right)_v &= 0, \\ (f_1)_{vu} + f_1 T_1^{12} T_2^{12} - f_2 \left(T_2^{12}\right)_u &= 0. \end{aligned} \quad (4.10)$$

Thus a computation of the left hand sides of (4.10) brings

$$-f_2 \left\{ \left(T_1^{12} \right)_v - \left(T_2^{12} \right)_u \right\} = 0.$$

So, by the use of (2.19) we can conclude that the normal curvature K_N of M vanishes identically.
Consequently, we obtain the following result of (Fröhlich, 2013).

Theorem (3)

The normal transport surface $\widetilde{M}$ of M is parallel if and only if M has flat normal bundle.

We obtain the following result.

Corollary (1)

The normal transport surface $\widetilde{M}$ of M is parallel if and only if the squared sum of the offset functions is constant, i.e.,

$$\sum_{i=1}^{2} f_i^2(u,v) = const.$$

Proof.

From the expressions in (4.9) we get

$$\begin{aligned}(f_1)_u f_1 + (f_2)_u f_2 &= 0,\\ (f_1)_v f_1 + (f_2)_v f_2 &= 0.\end{aligned} \tag{4.11}$$

which completes the proof. □

We give the following examples.

Example (3)

The normal transport surface $\widetilde{M}$ of M is given with the patch

$$\widetilde{X}(u,v) = X(u,v) + r\cos u\ N_1(u,v) + r\sin u\ N_2(u,v),$$

is a parallel surface of M in $\mathbb{E}^4$.

Example (4)

Rotation surfaces are defined by the following parametrization

$$\begin{aligned} M \quad : \quad X(s,t) = & (r(s)\cos s\cos t, r(s)\cos s\sin t, \\ & r(s)\sin s\cos t, r(s)\sin s\sin t) \end{aligned}$$

where $r(s)$ is a real valued non-zero function (Vranceanu, 1977).

Example (Continue)

We choose a moving frame $\{e_1, e_2, e_3, e_4\}$ (Yoon, 2001):

$$\begin{aligned} e_1 &= \frac{1}{r}\frac{\partial}{\partial t} \\ &= (-\cos s \sin t, \cos s \cos t, -\sin s \sin t, \sin s \cos t), \\ e_2 &= \frac{1}{A}\frac{\partial}{\partial s} \\ &= \frac{1}{A}(B\cos t, B\sin t, C\cos t, C\sin t), \\ e_3 &= \frac{1}{A}(-C\cos t, -C\sin t, B\cos t, B\sin t), \\ e_4 &= (-\sin s \sin t, \sin s \cos t, \cos s \sin t, -\cos s \cos t), \end{aligned}$$

Example (Continue)

where

$$\begin{aligned} A &= \sqrt{r^2(s)+(r'(s))^2}, \quad B = r'(s)\cos s - r(s)\sin s, \\ C &= r'(s)\sin s + r(s)\cos s. \end{aligned}$$

The Gauss and mean curvatures of M are given by

$$K = K_N = \frac{(r')^2 - rr''}{\left(r^2+(r')^2\right)^2}.$$

The normal transport surface $\widetilde{M}$ of M is parallel if and only if $r(s) = \alpha e^{(\beta s)}$, for some constants $\alpha \neq 0$ and β.

I) Let M be a non-minimal local surface in $\mathbb{E}^4$ and $\widetilde{M}_H$ its normal transport surface.
If $\widetilde{M}_H$ is a parallel surface of M in $\mathbb{E}^4$ then by Theorem 3 M has vanishing normal curvature.
Furthermore, by the use of (4.11) we get

$$\begin{aligned} (H_1)_u H_1 + (H_2)_u H_2 &= 0, \\ (H_1)_v H_1 + (H_2)_v H_2 &= 0. \end{aligned}$$

Thus, $\left\| \overrightarrow{H} \right\|^2 = \sum_{\alpha=1}^{2} H_\alpha^2$ is a constant function.
So, by Theorem 1 we conclude that the mean curvature vector $\overrightarrow{H}$ of M is parallel in the normal bundle.

Thus, we have proved the following result.

Theorem (4)

Let M be a non-minimal local surface in $\mathbb{E}^4$. Then the normal transport surface $\widetilde{M}_H$ of M in $\mathbb{E}^4$ is parallel if and only if the mean curvature vector $\overrightarrow{H}$ of M is parallel in the normal bundle.

II) Let M be a non-flat local surface in $\mathbb{E}^4$ and $\widetilde{M}_K$ its normal transport surface. If $\widetilde{M}_K$ is a parallel surface of M in $\mathbb{E}^4$ then by Theorem 3 $\widetilde{M}_K$ has vanishing normal curvature. Furthermore, by the use of (4.11) we get

$$\begin{aligned} (K_1)_u K_1 + (K_2)_u K_2 &= 0, \\ (K_1)_v K_1 + (K_2)_v K_2 &= 0. \end{aligned}$$

Thus, we conclude that $K = \sum_{\alpha=1}^{2} K_\alpha^2$ is a constant function, i.e., M has constant Gauss curvature.

Thus, we have proved the following result.

Theorem (5)

Let M be a non-flat local surface in $\mathbb{E}^4$. Then the normal transport surface $\widetilde{M}_K$ of M in $\mathbb{E}^4$ is parallel if and only if the Gaussian curvature of M is a non-zero constant.

4.4 Evolute surfaces in E4

Definition (5)

The normal transport surface $\widetilde{M}$ of M is called **evolute surface** of M in $\mathbb{E}^4$ if the equality

$$\left\langle \widetilde{x}_{u_i}, x_{u_j} \right\rangle = 0,\ 1 \leq i,j \leq 2, \qquad (4.12)$$

holds for all $x_{u_j} \in T_pM$ (Cheshkova, 2001).

Let $\widetilde{M}$ be a evolute surface of M in $\mathbb{E}^4$. Then by use of (4.6) with (4.12) we can get

$$\begin{aligned}
0 &= \langle \widetilde{x}_u, x_u \rangle = \left(1\text{-}f_1 c_1^{11}\text{-}f_2 c_2^{11}\right) g_{11}\text{-} \left(f_1 c_1^{12} + f_2 c_2^{12}\right) g_{21}, \\
0 &= \langle \widetilde{x}_u, x_v \rangle = \left(1\text{-}f_1 c_1^{11}\text{-}f_2 c_2^{11}\right) g_{12}\text{-} \left(f_1 c_1^{12} + f_2 c_2^{12}\right) g_{22}, \quad (4.13) \\
0 &= \langle \widetilde{x}_v, x_u \rangle = \text{-} \left(f_1 c_1^{12} + f_2 c_2^{12}\right) g_{11} + \left(1\text{-}f_1 c_1^{22}\text{-}f_2 c_2^{22}\right) g_{21}, \\
0 &= \langle \widetilde{x}_v, x_v \rangle = \text{-} \left(f_1 c_1^{12} + f_2 c_2^{12}\right) g_{12} + \left(1\text{-}f_1 c_1^{22}\text{-}f_2 c_2^{22}\right) g_{22}.
\end{aligned}$$

From now on we assume that the surface patch $x(u, v)$ satisfies the metric condition $g_{12} = 0$. So the equations in (4.13) turn into

$$\begin{aligned} f_1 c_1^{11} + f_2 c_2^{11} &= 1, \\ f_1 c_1^{22} + f_2 c_2^{22} &= 1, \\ f_1 c_1^{12} + f_2 c_2^{12} &= 0. \end{aligned} \tag{4.14}$$

Consequently by the use of (4.14) with (2.14) we get

$$f_1 H_1 + f_2 H_2 = 1. \tag{4.15}$$

So, we obtain the following result.

Theorem (6)

Let M be local surface in $\mathbb{E}^4$ with $g_{12} = 0$. Then the normal transport surface $\widetilde{M}$ in $\mathbb{E}^4$ is evolute surface of M if and only if the first and second mean curvatures H_1, H_2 satisfies the condition $f_1 H_1 + f_2 H_2 = 1$.

M. A. Cheshkova gave the following results;

Theorem (7)

Let M be local surface in $\mathbb{E}^4$. If the normal transport surface $\widetilde{M}$ in $\mathbb{E}^4$ is evolute surface of M then M has flat normal bundle.

Theorem (8)

The minimal surfaces have no evolutes.

Example (5)

Let M is a translation surface $x(u,v) = \alpha(u) + \beta(v)$ in $\mathbb{E}^4$, then the translation curves $\alpha(u) = (\alpha_1(u), \alpha_2(u), 0, 0)$ and $\beta(v) = (0, 0, \beta_1(v), \beta_2(v))$ are plane curves of mutually orthogonal 2-planes. The surface $\widetilde{M} = \widetilde{\alpha}(u) + \widetilde{\beta}(v)$ is a translation surface, and its translation curves $\widetilde{\alpha}(u)$, $\widetilde{\beta}(v)$ are the evolutes of the curves $\alpha(u)$, $\beta(v)$

$$\begin{aligned} \widetilde{x}(u,v) &= \alpha(u) + \frac{1}{\kappa_\alpha} n_\alpha(u) + \beta(v) + \frac{1}{\kappa_\beta} n_\beta(v) \\ &= x(u,v) + \frac{1}{\kappa_\alpha} n_\alpha(u) + \frac{1}{\kappa_\beta} n_\beta(v). \end{aligned}$$

Example (Continue)

The tangent space to $\widetilde{M}$ at an arbitrary point $p = \widetilde{x}(u, v)$ of $\widetilde{M}$ is spanned by

$$\widetilde{x}_u = \left(\frac{1}{\kappa_\alpha}\right)' n_\alpha(u),$$

$$\widetilde{x}_v = \left(\frac{1}{\kappa_\beta}\right)' n_\beta(v).$$

Consequently, the normal transport surface $\widetilde{M}$ of M satisfies the equality

$$\left\langle \widetilde{x}_{u_i}, x_{u_j} \right\rangle = 0.$$

Hence, $\widetilde{M}$ is the evolute of M (Cheshkova, 2001).

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