Parallel second order tensors on Vaisman manifolds

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1. Introduction

The Theorem 2 of [8] states that a parallel second order tensor field in a non-flat complex space form is a linear combination (with constant coefficients) of the underlying Kähler metric and Kähler 2-form. The aim of this paper is to consider the symmetric part of this result in the non-Kähler setting provided by locally conformal Kähler (lcK) geometry, more precisely Vaisman geometries. These are introduced in [9] under the name of *generalized Hopf manifolds* or *PK*-manifolds.

Our main result, namely Theorem 3.1, asserts that the above statement holds again in this framework for symmetric and J-skew-symmetric tensor fields of $(0, 2)$-type; here J denotes the complex structure of the given Hermitian geometry. As application, we obtain a reduction result for a special type if holomorhic vector fields in a subclass of Vaisman manifolds, usually denoted P_0K-manifolds and given by the flatness of the local Kähler metrics of our structure. This reduction result is on nature of Theorem 3 from [8, p. 789] and states that certain holomorphic vector field is in fact a homothetic one. Another reduction result of this type, but for conformal Killing vector fields on a special class of compact Vasiman manifolds, is the Theorem 3.2 of [5, p. 99]. Recently, the compact lck manifolds with parallel vector fields are completely classified in [4].

2.Vaisman Manifolds

Let (M^{2n}, J, g) be a complex n-dimensional Hermitian manifold and Ω its fundamental 2-form given by $\Omega(X, Y) = g(X, JY)$ for any vector fields $X, Y \in \Gamma(TM)$. Recall from [2, p. 1] that (M, J, g, Ω) is a *locally conformal Kähler manifold* (l.c.K) if there exists a closed 1-form $\omega \in \Gamma(T_1^0(M))$ such that: $d\Omega = \omega \wedge \Omega$. In particular, M is called *strongly non-Kähler* if ω is without singularities i.e. $\omega \neq 0$ everywhere; hence we consider $2c = \|\omega\|$ and $u = \omega/2c$ the corresponding 1-form. Since ω is called *the Lee form* of M the vector field $U = u^\sharp$ will be called *the Lee vector field*. Consider also the unit vector field $V = JU$, *the anti-Lee vector field*, as well as its dual form $v = V^\flat$, so:
$u(V) = v(U) = 0, v = -u \circ J, u = v \circ J$.

Our setting is provided by the particular case of strongly non-Kähler l.c.K. manifolds, called *Vaisman manifolds*, and given by the parallelism of ω with respect to the Levi-Civita connection ∇ of g. Hence c is a positive constant and the Lemma 2 of [6] gives the covariant derivative of V with respect to any $X \in \Gamma(TM)$:

$$\nabla_X V = c[u(X)V - v(X)U - JX] \tag{2.1}$$

which yields the dual:

$$(\nabla_X v)Y = c[u(X)v(Y) - u(Y)v(X) + \Omega(X, Y)] \tag{2.2}$$

and the curvature:

$$R(X, Y)V = c^2\{[u(X)v(Y) - u(Y)v(X)]U + v(X)Y - v(Y)X\}. \tag{2.3}$$

Hence:

$$R(X,V)V = c^2[u(X)U + v(X)V - X] \tag{2.4}$$

and for an unitary X, orthogonal to V we derive the sectional curvature:

$$K(X,V) = c^2[u(X)^2 - 1]. \tag{2.5}$$

In particular: $K(U,V) = 0$.

The class of Vaisman manifolds was introduced in [9] and their old notation is that of *PK-manifolds*. A main subclass of Vaisman manifolds, denoted P_0K, is provided by the flatness of the local Kähler metrics generated by g and the local exactness of ω; see details in [9]. For these manifolds it is known the express of the Ricci tensor of g; with formula (2.10) of [3, p. 125] one obtains:

$$Ric = 2c^2(n-1)[g - u \otimes u] \tag{2.6}$$

which means that the triple (M, g, U) is an *eta-Einstein manifold*.

3. Parallel second order tensors in a Vaisman geometry

The purpose of this Section is to prove the main result of the paper:

Theorem (3.1)

Let (M, J, g, Ω) be a Vaisman manifold.
i) Fix a tensor field $\alpha \in \Gamma(T_2^0(M))$ which is symmetric and J-skew-symmetric i.e.:

$$\alpha(JX, Y) + \alpha(X, JY) = 0 \tag{3.1}$$

for all $X, Y \in \Gamma(TM)$. If α is parallel with respect to ∇ then it is a constant multiple of the metric tensor g.
ii) Let the 2-form $\beta \in \bigwedge^2(M)$ which is J-skew-symmetric and satisfies:

$$\begin{aligned}(\nabla_Z\beta)(X, Y) &= c[g(X, Z)\beta(U, Y) - \Omega(X, Z)\beta(V, Y) \\ &\quad -v(X)\beta(Y, JZ) + u(X)\beta(Y, Z)]\end{aligned} \tag{3.2}$$

for all $X, Y, Z \in \Gamma(TM)$. Then β is a constant multiple of the fundamental form Ω.

Proof.

i) Applying the Ricci commutation identity [1, p. 14] and $\nabla^2_{X,Y}\alpha(Z,W) - \nabla^2_{X,Y}\alpha(W,Z) = 0$ for all vector fields X, Y, Z, W we obtain the relation (1.1) of [8, p. 787]:

$$\alpha(R(X,Y)Z,W) + \alpha(Z,R(X,Y)W) = 0 \tag{3.3}$$

which is fundamental in all papers treating this subject. Replacing $Z = W = V$ and using (2.3) it results, by the symmetry of α:

$$v(X)\alpha(Y,V) = v(Y)\alpha(X,V). \tag{3.4}$$

With $X = V$ we get:

$$\alpha(Y,V) = v(Y)\alpha(V,V). \tag{3.5}$$

□

Proof.

[Proof (continuation)] From the symmetry and J-symmetry of α we have:

$$\alpha(U, V) = 0 \tag{3.6}$$

and then the parallelism of α and formulae $(3.5) - (3.6)$ imply that $\alpha(V, V)$ is a constant. Applying X to (3.5) and using (2.2) we have:

$$\begin{aligned} X(\alpha(Y, V)) &= \alpha(\nabla_X Y, V) + \alpha(Y, \nabla_X V) \\ &= X(v(Y))\alpha(V, V) + 2v(Y)\alpha(\nabla_X V, V) \end{aligned}$$

which means that:

$$\begin{aligned} c\alpha(Y, u(X)V - v(X)U - JX) &= (\nabla_X v)(Y)\alpha(V, V) \\ &\quad +2cv(Y)\alpha(u(X)V - JX, V). \end{aligned}$$

□

Proof.

[Proof (continuation)] Due to (3.5) and $v \circ J = u$ the last term above is zero. With (2.2) and (3.5) again it results:

$$\begin{aligned} -v(X)\alpha(U,Y) + \alpha(X,JY) &= -u(Y)v(X)\alpha(V,V) \\ &\quad + \Omega(X,Y)\alpha(V,V). \end{aligned} \tag{3.7}$$

We have a relation similar to (3.5) but in terms of U:

$$\begin{aligned} \alpha(Y,U) &= \alpha(Y,-JV) = \alpha(JY,V) = v(JY)\alpha(V,V) \\ &= u(Y)\alpha(V,V) \end{aligned} \tag{3.8}$$

and then, returning to (3.7) we get:

$$\alpha(X,JY) = \alpha(V,V)\Omega(X,Y) \tag{3.9}$$

and a transformation $Y \to JY$ gives the conclusion.

□

Proof.

[Proof (continuation)] ii) Let $\alpha \in \Gamma(T^0_2(M))$ be given by a relation dual to that defining Ω through g:

$$\alpha(X, Y) := \beta(JX, Y). \tag{3.10}$$

Hence: $\alpha(Y, X) = \beta(JY, X) = -\beta(X, JY)$ which by J-skew-symmetry means $\beta(JX, Y)$ and consequently α is symmetric. Also:

$$\begin{aligned} \alpha(JX, Y) + \alpha(X, JY) &= -\beta(X, Y) + \beta(JX, JY) \\ &= -\beta(X, Y) - \beta(J^2X, Y) = 0. \end{aligned}$$

Finally, (3.2) express the parallelism of α by using the following covariant derivative of J resulting from Proposition 1 of [6, p. 338]:

$$(\nabla_Z J)X = c[\Omega(X, Z)U + g(X, Z)V - u(X)JZ - v(X)Z]. \tag{3.11}$$

Therefore we apply i) for α and (3.9) is exactly the conclusion: $\beta(X, Y) = \alpha(V, V)\Omega(X, Y)$ with $\alpha(V, V) = -\beta(U, V)$. □

Remarks (3.2) i) The reduction of a covariant second order tensor field to a multiple of the metric holds generally under the hypothesis of irreducibility of the holonomy group/algebra, see for example the Theorem 57 of [7. p. 254]. Our result above implies weaker conditions for the l.c.K. metric in the Vaisman framework.

ii) The parallel forms of compact connected Vaisman manifolds are completely treated in Theorem 7.7. of [2, p. 78].

As an application of Theorem 3.1, we obtain the following result which is similar to Theorem 3 of [8]:

Corollary (3.3)

Let ξ be a holomorphic vector field on a Vaisman manifold such that $\nabla_\xi J$ is skew-symmetric with respect to g and $\mathcal{L}_\xi g$ is parallel. Then ξ is a homothetic vector field. Moreover, if (M, g, J) is a P_0K-manifold then ξ is a Killing vector field.

Proof.

For the second order covariant tensor field $\alpha = \mathcal{L}_\xi g$ we can apply the previous theorem if the skew-symmetry (3.1) is satisfied. We have:

$$\begin{aligned} \alpha(JX, Y) + \alpha(X, JY) &= g(\nabla_{JX}\xi - J(\nabla_X \xi), Y) \\ &\quad + g(X, \nabla_{JY}\xi - J(\nabla_Y \xi)) \end{aligned} \tag{3.12}$$

and the holomorphic hypothesis $\mathcal{L}_\xi J = 0$ yields:

$$\alpha((\nabla_\xi J)X, Y) + \alpha(X, (\nabla_\xi J)Y). \tag{3.13}$$

□

Proof.

[Proof (continuation)] Hence the claimed skew-symmetry holds and consequently:

$$\mathcal{L}_{\xi} g = \alpha(V, V) g \tag{3.14}$$

is exactly the first conclusion regarding ξ. This relation implies $\mathcal{L}_{\xi} Ric = 0$ and in the P_0K setting the equation (2.6) gives:

$$\mathcal{L}_{\xi} g = \mathcal{L}_{\xi}(u \otimes u). \tag{3.15}$$

The right hand side of (3.15) applied to (V, V) gives that $\alpha(V, V) = 0$ and then (3.14) gives the second conclusion. □

Examples (3.4)

i) The vector field U is a holomorphic ([2, p. 37]) and Killing one in a Vaisman manifold since it is parallel: $\nabla U = 0$. Then $\alpha := u \otimes u$ is symmetric and parallel while the condition (3.1) means the Kählerian setting $\omega = 0$. Indeed, with $X = Y$, the equation (3.1) reads $u(X)u(JX) = 0$ for all X i.e. $u = 0$.

ii) By using again [2, p. 37], the vector field V is holomorphic and Killing.

The locally conformal Kähler geometry can be studied in terms of Weyl structures and their associated Weyl connections conform Theorem 1.4 of [2, p. 5]. The expression of the Weyl connection of (M, g, J, ω) is formula (2) of [5, p. 94] which for our notation becomes:

$$D = \nabla - c(u \otimes I + I \otimes u - g \otimes U) \tag{3.16}$$

with the Kronecker tensor field I. Hence, the symmetric tensor field $\alpha \in \Gamma(T^0_2(M))$ is ∇-parallel if and only if its Weyl derivative is:

$$\begin{aligned} D_Z\alpha(X, Y) \;=\; & c[2u(Z)\alpha(X, Y) \\ & +u(X)\alpha(Y, Z) + u(Y)\alpha(X, Z) \\ & -g(X, Z)\alpha(U, Y) - g(Y, Z)\alpha(U, X)] \end{aligned} \tag{3.17}$$

for all vector fields X, Y, Z.

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