CR-structure and Levi form on real hypersurfaces in Kähler manifolds

Jong Taek Cho

Chonnam National University
Gwangju, Korea

XVIIth International Conference
Geometry, Integrability and Quantization
June 5-10, 2015 Varna, Bulgaria

Motivation I.

Let M be a real hypersurface in a Kähler manifold $(\widetilde{M}, \widetilde{J}, \widetilde{g})$ with Kähler structure $(\widetilde{J}, \widetilde{g})$. In particular, we consider a model space $\widetilde{M} = P_n\mathbb{C}$ or $H_n\mathbb{C}$.

Facts:

• There are no parallel ($\nabla A = 0$) real hypersurfaces in a non-flat complex space form, where A is the shape operator and ∇ is the induced Levi-Civita connection.

• There are no locally symmetric ($\nabla R = 0$) real hypersurfaces in a non-flat complex space form, where R denotes the Riemann curvature tensor on M.

A fundamental question:

Could we find a canonical connection (parallelism) other than Levi-Civita connection on real hypersurfces in a Kähler manifold?

Motivation I.

Let M be a real hypersurface in a Kähler manifold $(\widetilde{M}, \widetilde{J}, \widetilde{g})$ with Kähler structure $(\widetilde{J}, \widetilde{g})$. In particular, we consider a model space $\widetilde{M} = P_n\mathbb{C}$ or $H_n\mathbb{C}$.

Facts:
• There are no parallel ($\nabla A = 0$) real hypersurfaces in a non-flat complex space form, where A is the shape operator and ∇ is the induced Levi-Civita connection.
• There are no locally symmetric ($\nabla R = 0$) real hypersurfaces in a non-flat complex space form, where R denotes the Riemann curvature tensor on M.

A fundamental question:
Could we find a canonical connection (parallelism) other than Levi-Civita connection on real hypersurfces in a Kähler manifold?

Motivation I.

Let M be a real hypersurface in a Kähler manifold $(\widetilde{M}, \widetilde{J}, \widetilde{g})$ with Kähler structure $(\widetilde{J}, \widetilde{g})$. In particular, we consider a model space $\widetilde{M} = P_n\mathbb{C}$ or $H_n\mathbb{C}$.

Facts:

• There are no parallel ($\nabla A = 0$) real hypersurfaces in a non-flat complex space form, where A is the shape operator and ∇ is the induced Levi-Civita connection.

• There are no locally symmetric ($\nabla R = 0$) real hypersurfaces in a non-flat complex space form, where R denotes the Riemann curvature tensor on M.

A fundamental question:

Could we find a canonical connection (parallelism) other than Levi-Civita connection on real hypersurfces in a Kähler manifold?

Motivation I.

Let M be a real hypersurface in a Kähler manifold $(\widetilde{M}, \widetilde{J}, \widetilde{g})$ with Kähler structure $(\widetilde{J}, \widetilde{g})$. In particular, we consider a model space $\widetilde{M} = P_n\mathbb{C}$ or $H_n\mathbb{C}$.

Facts:

• There are no parallel $(\nabla A = 0)$ real hypersurfaces in a non-flat complex space form, where A is the shape operator and ∇ is the induced Levi-Civita connection.

• There are no locally symmetric $(\nabla R = 0)$ real hypersurfaces in a non-flat complex space form, where R denotes the Riemann curvature tensor on M.

A fundamental question:

Could we find a canonical connection (parallelism) other than Levi-Civita connection on real hypersurfces in a Kähler manifold?

Motivation I.

Let M be a real hypersurface in a Kähler manifold $(\widetilde{M}, \widetilde{J}, \widetilde{g})$ with Kähler structure $(\widetilde{J}, \widetilde{g})$. In particular, we consider a model space $\widetilde{M} = P_n\mathbb{C}$ or $H_n\mathbb{C}$.

Facts:

• There are no parallel ($\nabla A = 0$) real hypersurfaces in a non-flat complex space form, where A is the shape operator and ∇ is the induced Levi-Civita connection.

• There are no locally symmetric ($\nabla R = 0$) real hypersurfaces in a non-flat complex space form, where R denotes the Riemann curvature tensor on M.

A fundamental question:

Could we find a canonical connection (parallelism) other than Levi-Civita connection on real hypersurfces in a Kähler manifold?

Summary

• Tanaka-Webster connection is a canonical affine connection on a non-degenerate CR-manifold.
• A real hypersurface in a Kähler manifold has an integrable CR-structure (η, J) which is associated with an almost contact structure (η, ϕ, ξ), but the Levi form is not guaranteed to be non-degenerate, in general.
• In this context, the author defined the generalized Tanaka-Webster connection (in short, the g.-Tanaka-Webster connection) $\hat{\nabla}^{(k)}$, $k \neq 0$ for real hypersurfaces in a Kähler manifold. If the shape operator A of a real hypersurface satisfies $\phi A + A\phi = 2k\phi$, then its associated CR-structure is strongly pseudo-convex, and further the g.-Tanaka-Webster connection $\hat{\nabla}^{(k)}$ coincides with the Tanaka-Webster connection (see Proposition 3).

In this talk, all manifolds are assumed to be connected and of class C^∞ and the real hypersurfaces are supposed to be oriented.

Summary

• Tanaka-Webster connection is a canonical affine connection on a non-degenerate CR-manifold.

• A real hypersurface in a Kähler manifold has an integrable CR-structure (η, J) which is associated with an almost contact structure (η, ϕ, ξ), but the Levi form is not guaranteed to be non-degenerate, in general.

• In this context, the author defined the generalized Tanaka-Webster connection (in short, the g.-Tanaka-Webster connection) $\hat{\nabla}^{(k)}$, $k \neq 0$ for real hypersurfaces in a Kähler manifold. If the shape operator A of a real hypersurface satisfies $\phi A + A\phi = 2k\phi$, then its associated CR-structure is strongly pseudo-convex, and further the g.-Tanaka-Webster connection $\hat{\nabla}^{(k)}$ coincides with the Tanaka-Webster connection (see Proposition 3).

In this talk, all manifolds are assumed to be connected and of class C^∞ and the real hypersurfaces are supposed to be oriented.

Summary

• Tanaka-Webster connection is a canonical affine connection on a non-degenerate CR-manifold.

• A real hypersurface in a Kähler manifold has an integrable CR-structure (η, J) which is associated with an almost contact structure (η, ϕ, ξ), but the Levi form is not guaranteed to be non-degenerate, in general.

• In this context, the author defined the generalized Tanaka-Webster connection (in short, the g.-Tanaka-Webster connection) $\hat{\nabla}^{(k)}$, $k \neq 0$ for real hypersurfaces in a Kähler manifold. If the shape operator A of a real hypersurface satisfies $\phi A + A\phi = 2k\phi$, then its associated CR-structure is strongly pseudo-convex, and further the g.-Tanaka-Webster connection $\hat{\nabla}^{(k)}$ coincides with the Tanaka-Webster connection (see Proposition 3).

In this talk, all manifolds are assumed to be connected and of class C^∞ and the real hypersurfaces are supposed to be oriented.

Summary

• Tanaka-Webster connection is a canonical affine connection on a non-degenerate CR-manifold.
• A real hypersurface in a Kähler manifold has an integrable CR-structure (η, J) which is associated with an almost contact structure (η, ϕ, ξ), but the Levi form is not guaranteed to be non-degenerate, in general.
• In this context, the author defined the generalized Tanaka-Webster connection (in short, the g.-Tanaka-Webster connection) $\hat{\nabla}^{(k)}$, $k \neq 0$ for real hypersurfaces in a Kähler manifold. If the shape operator A of a real hypersurface satisfies $\phi A + A\phi = 2k\phi$, then its associated CR-structure is strongly pseudo-convex, and further the g.-Tanaka-Webster connection $\hat{\nabla}^{(k)}$ coincides with the Tanaka-Webster connection (see Proposition 3).

In this talk, all manifolds are assumed to be connected and of class C^∞ and the real hypersurfaces are supposed to be oriented.

Summary

- Tanaka-Webster connection is a canonical affine connection on a non-degenerate CR-manifold.
- A real hypersurface in a Kähler manifold has an integrable CR-structure (η, J) which is associated with an almost contact structure (η, ϕ, ξ), but the Levi form is not guaranteed to be non-degenerate, in general.
- In this context, the author defined the generalized Tanaka-Webster connection (in short, the g.-Tanaka-Webster connection) $\hat{\nabla}^{(k)}$, $k \neq 0$ for real hypersurfaces in a Kähler manifold. If the shape operator A of a real hypersurface satisfies $\phi A + A\phi = 2k\phi$, then its associated CR-structure is strongly pseudo-convex, and further the g.-Tanaka-Webster connection $\hat{\nabla}^{(k)}$ coincides with the Tanaka-Webster connection (see Proposition 3).

In this talk, all manifolds are assumed to be connected and of class C^∞ and the real hypersurfaces are supposed to be oriented.

Pseudo-Hermitian geometry

Let TM be the tangent bundle of a manifold M and $TM^{\mathbb{C}}$ be its complexification. Let $\mathcal{H}$ be a subbundle of $TM^{\mathbb{C}}$ and suppose that $\mathcal{H} \bigcap \bar{\mathcal{H}} = \{0\}$, where $\bar{\mathcal{H}}$ denotes the complex conjugate of $\mathcal{H}$. Then there is a unique subbundle D of TM such that $D^{\mathbb{C}} = \mathcal{H} \oplus \bar{\mathcal{H}}$. Also we have a unique homomorphism $J : D \to D$ such that $J^2 = -I$, where I denotes the identity and

$$\mathcal{H} = \{X - iJX : X \in D\}.$$

$\{D, J\}$ is called the real expression of $\mathcal{H}$.

Now we suppose that M is an m-dimensional contact manifold with a contact form η, that is $\eta \wedge (d\eta)^{n-1} \neq 0$ everywhere on M, where $m = 2n - 1$. Define D by the kernel of η. If the form L (the Levi form) defined by

$$L(X, Y) = d\eta(X, JY), \quad X, Y \in \Gamma(D),$$

is hermitian, then (M, η, J) is called a non-degenerate pseudo-Hermitian manifold. If the Levi form L is positive definite, then "non-degenerate" is replaced by "strongly pseudoconvex". Here, $\Gamma(D)$ denotes the space of all sections of D.

Pseudo-Hermitian geometry

Let TM be the tangent bundle of a manifold M and $TM^{\mathbb{C}}$ be its complexification. Let $\mathcal{H}$ be a subbundle of $TM^{\mathbb{C}}$ and suppose that $\mathcal{H} \bigcap \bar{\mathcal{H}} = \{0\}$, where $\bar{\mathcal{H}}$ denotes the complex conjugate of $\mathcal{H}$. Then there is a unique subbundle D of TM such that $D^{\mathbb{C}} = \mathcal{H} \oplus \bar{\mathcal{H}}$. Also we have a unique homomorphism $J : D \to D$ such that $J^2 = -I$, where I denotes the identity and

$$\mathcal{H} = \{X - iJX : X \in D\}.$$

$\{D, J\}$ is called the real expression of $\mathcal{H}$.

Now we suppose that M is an m-dimensional contact manifold with a contact form η, that is $\eta \wedge (d\eta)^{n-1} \neq 0$ everywhere on M, where $m = 2n - 1$. Define D by the kernel of η. If the form L (the Levi form) defined by

$$L(X, Y) = d\eta(X, JY), \quad X, Y \in \Gamma(D),$$

is hermitian, then (M, η, J) is called a non-degenerate pseudo-Hermitian manifold. If the Levi form L is positive definite, then "non-degenerate" is replaced by "strongly pseudoconvex". Here, $\Gamma(D)$ denotes the space of all sections of D.

Pseudo-Hermitian geometry

Let TM be the tangent bundle of a manifold M and $TM^{\mathbb{C}}$ be its complexification. Let $\mathcal{H}$ be a subbundle of $TM^{\mathbb{C}}$ and suppose that $\mathcal{H} \bigcap \bar{\mathcal{H}} = \{0\}$, where $\bar{\mathcal{H}}$ denotes the complex conjugate of $\mathcal{H}$. Then there is a unique subbundle D of TM such that $D^{\mathbb{C}} = \mathcal{H} \oplus \bar{\mathcal{H}}$. Also we have a unique homomorphism $J : D \to D$ such that $J^2 = -I$, where I denotes the identity and

$$\mathcal{H} = \{X - iJX : X \in D\}.$$

$\{D, J\}$ is called the real expression of $\mathcal{H}$.

Now we suppose that M is an m-dimensional contact manifold with a contact form η, that is $\eta \wedge (d\eta)^{n-1} \neq 0$ everywhere on M, where $m = 2n - 1$. Define D by the kernel of η. If the form L (the Levi form) defined by

$$L(X, Y) = d\eta(X, JY), \quad X, Y \in \Gamma(D),$$

is hermitian, then (M, η, J) is called a non-degenerate pseudo-Hermitian manifold. If the Levi form L is positive definite, then "non-degenerate" is replaced by "strongly pseudoconvex". Here, $\Gamma(D)$ denotes the space of all sections of D.

We note that

$$L(X,Y) = L(JX,JY) \Leftrightarrow [\Gamma(\mathcal{H}),\Gamma(\mathcal{H})] \subset \Gamma(D^{\mathbb{C}})$$

(the partial integrability of $\mathcal{H}$).
If the integrability condition:

$$[\Gamma(\mathcal{H}),\Gamma(\mathcal{H})] \subset \Gamma(\mathcal{H})$$

is satisfied, then (η, J) is said to be integrable.
A non-degenerate (strongly pseudoconvex, resp.) integrable pseudo-Hermitian manifold is called a non-degenerate (strongly pseudoconvex, resp.) integrable CR manifold.

We note that

$$L(X,Y) = L(JX,JY) \Leftrightarrow [\Gamma(\mathcal{H}),\Gamma(\mathcal{H})] \subset \Gamma(D^{\mathbb{C}})$$

(the partial integrability of $\mathcal{H}$).
If the integrability condition:

$$[\Gamma(\mathcal{H}),\Gamma(\mathcal{H})] \subset \Gamma(\mathcal{H})$$

is satisfied, then (η, J) is said to be integrable.
A non-degenerate (strongly pseudoconvex, resp.) integrable pseudo-Hermitian manifold is called a non-degenerate (strongly pseudoconvex, resp.) integrable CR manifold.

We note that

$$L(X,Y) = L(JX,JY) \Leftrightarrow [\Gamma(\mathcal{H}), \Gamma(\mathcal{H})] \subset \Gamma(D^{\mathbb{C}})$$

(the partial integrability of $\mathcal{H}$).
If the integrability condition:

$$[\Gamma(\mathcal{H}), \Gamma(\mathcal{H})] \subset \Gamma(\mathcal{H})$$

is satisfied, then (η, J) is said to be integrable.
A non-degenerate (strongly pseudoconvex, resp.) integrable pseudo-Hermitian manifold is called a non-degenerate (strongly pseudoconvex, resp.) integrable CR manifold.

We note that

$$L(X, Y) = L(JX, JY) \Leftrightarrow [\Gamma(\mathcal{H}), \Gamma(\mathcal{H})] \subset \Gamma(D^{\mathbb{C}})$$

(the partial integrability of $\mathcal{H}$).
If the integrability condition:

$$[\Gamma(\mathcal{H}), \Gamma(\mathcal{H})] \subset \Gamma(\mathcal{H})$$

is satisfied, then (η, J) is said to be integrable.
A non-degenerate (strongly pseudoconvex, resp.) integrable pseudo-Hermitian manifold is called a non-degenerate (strongly pseudoconvex, resp.) integrable CR manifold.

Define ϕ by $\phi|_D = J$ and $\phi\xi = 0$. Then ϕ is a $(1,1)$-tensor field on M and the notion of strongly pseudoconvex pseudo-Hermitian structure (η, J) is equivalent to the notion of contact Riemannian structure (η, g) by the relation:

$$g = L + \eta \otimes \eta,$$

where the same L denotes the natural extension to a $(0,2)$-tensor field on M.

Define ϕ by $\phi|_D = J$ and $\phi\xi = 0$. Then ϕ is a $(1,1)$-tensor field on M and the notion of strongly pseudoconvex pseudo-Hermitian structure (η, J) is equivalent to the notion of contact Riemannian structure (η, g) by the relation:

$$g = L + \eta \otimes \eta,$$

where the same L denotes the natural extension to a $(0,2)$-tensor field on M.

Tanaka-Webster connection

Definition 1. ([Tanaka],[Webster])

Tanaka-Webster connection $\hat{\nabla}$ on a non-degenerate integrable CR manifold $M = (M; \eta, J)$ is the unique linear connection satisfying the following conditions:

(i) $\hat{\nabla}\eta = 0$, $\hat{\nabla}\xi = 0$;

(ii) $\hat{\nabla}g = 0$, $\hat{\nabla}\phi = 0$;

$(iii-1)$ $\hat{T}(X, Y) = 2L(X, JY)\xi$, $X, \ Y \in \Gamma(D)$;

$(iii-2)$ $\hat{T}(\xi, \phi Y) = -\phi\hat{T}(\xi, Y)$, $Y \in \Gamma(D)$.

Tanno defined the generalized Tanaka-Webster connection by replacing the condition $\hat{\nabla}\phi = 0$ by a $(\hat{\nabla}_X\phi)Y = \Omega(X, Y)$ on a contact Riemannian manifold (whose associated pseudo-Hermitian structure is not necessarily CR-integrable), where Ω is a $(1, 2)$-tensor field.

Tanaka-Webster connection

Definition 1. ([Tanaka],[Webster])

Tanaka-Webster connection $\hat{\nabla}$ on a non-degenerate integrable CR manifold $M = (M; \eta, J)$ is the unique linear connection satisfying the following conditions:

(i) $\hat{\nabla}\eta = 0$, $\hat{\nabla}\xi = 0$;
(ii) $\hat{\nabla}g = 0$, $\hat{\nabla}\phi = 0$;
$(iii-1)$ $\hat{T}(X, Y) = 2L(X, JY)\xi$, X, $Y \in \Gamma(D)$;
$(iii-2)$ $\hat{T}(\xi, \phi Y) = -\phi\hat{T}(\xi, Y)$, $Y \in \Gamma(D)$.

Tanno defined the generalized Tanaka-Webster connection by replacing the condition $\hat{\nabla}\phi = 0$ by a $(\hat{\nabla}_X\phi)Y = \Omega(X, Y)$ on a contact Riemannian manifold (whose associated pseudo-Hermitian structure is not necessarily CR-integrable), where Ω is a (1, 2)-tensor field.

Tanaka-Webster connection

Definition 1. ([Tanaka],[Webster])

Tanaka-Webster connection $\hat{\nabla}$ on a non-degenerate integrable CR manifold $M = (M; \eta, J)$ is the unique linear connection satisfying the following conditions:

(i) $\hat{\nabla}\eta = 0$, $\hat{\nabla}\xi = 0$;
(ii) $\hat{\nabla}g = 0$, $\hat{\nabla}\phi = 0$;
$(iii-1)$ $\hat{T}(X, Y) = 2L(X, JY)\xi$, X, $Y \in \Gamma(D)$;
$(iii-2)$ $\hat{T}(\xi, \phi Y) = -\phi\hat{T}(\xi, Y)$, $Y \in \Gamma(D)$.

Tanno defined the generalized Tanaka-Webster connection by replacing the condition $\hat{\nabla}\phi = 0$ by a $(\hat{\nabla}_X\phi)Y = \Omega(X, Y)$ on a contact Riemannian manifold (whose associated pseudo-Hermitian structure is not necessarily CR-integrable), where Ω is a $(1, 2)$-tensor field.

N. Tanaka, On non-degenerate real hypersurfaces, graded Lie algebras and Cartan connections, Japan J. Math. 2 (1976), 131–190.

S. M. Webster, Pseudohermitian structures on a real hypersurface, J. Diff. Geometry 13 (1978), 25–41.

S. Tanno, Variational problems on contact Riemannian manifolds, Trans. Amer. Math. Soc., 314 (1989), 349–379.

Almost contact structure and the associated CR-structure

A $(2n+1)$-dimensional manifold M is said to be an *almost contact manifold* if its structure group of the linear frame bundle is reducible to $U(n) \times \{1\}$. This is equivalent to the existence of a $(1,1)$-tensor field ϕ, a vector field ξ and a 1-form η satisfying

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1. \tag{1}$$

Then one can find always a compatible Riemann metric, namely which satisfies

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{2}$$

for all vector fields X, Y on M. We call (η, ϕ, ξ, g) an *almost contact metric structure* of M and $M = (M; \eta, \phi, \xi, g)$ an *almost contact metric manifold*. The *fundamental 2-form* Φ is defined by $\Phi(X, Y) = g(\phi X, Y)$. In particular, $d\eta = \Phi$, then M is called an *contact metric manifold.* From (1) and (2) we easily get

$$\phi\xi = 0, \quad \eta \circ \phi = 0, \quad \eta(X) = g(X, \xi). \tag{3}$$

For more details about the general theory of almost contact metric manifolds, we refer to [Blair].

D. E. Blair, Riemannian geometry of contact and symplectic

Almost contact structure and the associated CR-structure

A $(2n+1)$-dimensional manifold M is said to be an *almost contact manifold* if its structure group of the linear frame bundle is reducible to $U(n) \times \{1\}$. This is equivalent to the existence of a $(1,1)$-tensor field ϕ, a vector field ξ and a 1-form η satisfying

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1. \tag{1}$$

Then one can find always a compatible Riemann metric, namely which satisfies

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{2}$$

for all vector fields X, Y on M. We call (η, ϕ, ξ, g) an *almost contact metric structure* of M and $M = (M; \eta, \phi, \xi, g)$ an *almost contact metric manifold*. The *fundamental 2-form* Φ is defined by $\Phi(X, Y) = g(\phi X, Y)$. In particular, $d\eta = \Phi$, then M is called an *contact metric manifold*. From (1) and (2) we easily get

$$\phi\xi = 0, \quad \eta \circ \phi = 0, \quad \eta(X) = g(X, \xi). \tag{3}$$

For more details about the general theory of almost contact metric manifolds, we refer to [Blair].

D. E. Blair, Riemannian geometry of contact and symplectic

Almost contact structure and the associated CR-structure

A $(2n+1)$-dimensional manifold M is said to be an *almost contact manifold* if its structure group of the linear frame bundle is reducible to $U(n) \times \{1\}$. This is equivalent to the existence of a $(1,1)$-tensor field ϕ, a vector field ξ and a 1-form η satisfying

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1. \tag{1}$$

Then one can find always a compatible Riemann metric, namely which satisfies

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{2}$$

for all vector fields X, Y on M. We call (η, ϕ, ξ, g) an *almost contact metric structure* of M and $M = (M; \eta, \phi, \xi, g)$ an *almost contact metric manifold.* The *fundamental 2-form* Φ is defined by $\Phi(X, Y) = g(\phi X, Y)$. In particular, $d\eta = \Phi$, then M is called an *contact metric manifold.*

From (1) and (2) we easily get

$$\phi\xi = 0, \quad \eta \circ \phi = 0, \quad \eta(X) = g(X, \xi). \tag{3}$$

For more details about the general theory of almost contact metric manifolds, we refer to [Blair].

D. E. Blair, Riemannian geometry of contact and symplectic

Almost contact structure and the associated CR-structure

A $(2n+1)$-dimensional manifold M is said to be an *almost contact manifold* if its structure group of the linear frame bundle is reducible to $U(n) \times \{1\}$. This is equivalent to the existence of a $(1,1)$-tensor field ϕ, a vector field ξ and a 1-form η satisfying

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1. \tag{1}$$

Then one can find always a compatible Riemann metric, namely which satisfies

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{2}$$

for all vector fields X, Y on M. We call (η, ϕ, ξ, g) an *almost contact metric structure* of M and $M = (M; \eta, \phi, \xi, g)$ an *almost contact metric manifold.* The *fundamental 2-form* Φ is defined by $\Phi(X, Y) = g(\phi X, Y)$. In particular, $d\eta = \Phi$, then M is called an *contact metric manifold.*
From (1) and (2) we easily get

$$\phi\xi = 0, \quad \eta \circ \phi = 0, \quad \eta(X) = g(X, \xi). \tag{3}$$

For more details about the general theory of almost contact metric manifolds, we refer to [Blair].

D. E. Blair, Riemannian geometry of contact and symplectic

Almost contact structure and the associated CR-structure

A $(2n+1)$-dimensional manifold M is said to be an *almost contact manifold* if its structure group of the linear frame bundle is reducible to $U(n) \times \{1\}$. This is equivalent to the existence of a $(1,1)$-tensor field ϕ, a vector field ξ and a 1-form η satisfying

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1. \tag{1}$$

Then one can find always a compatible Riemann metric, namely which satisfies

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{2}$$

for all vector fields X, Y on M. We call (η, ϕ, ξ, g) an *almost contact metric structure* of M and $M = (M; \eta, \phi, \xi, g)$ an *almost contact metric manifold.* The *fundamental 2-form* Φ is defined by $\Phi(X, Y) = g(\phi X, Y)$. In particular, $d\eta = \Phi$, then M is called an *contact metric manifold.* From (1) and (2) we easily get

$$\phi\xi = 0, \quad \eta \circ \phi = 0, \quad \eta(X) = g(X, \xi). \tag{3}$$

For more details about the general theory of almost contact metric manifolds, we refer to [Blair].

D. E. Blair, Riemannian geometry of contact and symplectic

For an almost contact manifold $M = (M; \eta, \phi, \xi)$, the tangent space T_pM of M at each point $p \in M$ is decomposed as $T_pM = D_p \oplus \{\xi\}_p$ (direct sum), where we denote $D_p = \{v \in T_pM | \eta(v) = 0\}$. Then $D : p \to D_p$ defines a distribution and the restriction $J = \phi|_D$ of ϕ to D defines an almost complex structure in D. Define the Levi form L by $L(X, Y) = d\eta(X, JY)$, $X, Y \perp \xi$.

Then we find:

For an almost contact structure, both "non-degeneracy" of the Levi form and "CR-integrability" of the associated almost CR-structure are not guaranteed, in general.

For an almost contact manifold $M = (M; \eta, \phi, \xi)$, the tangent space T_pM of M at each point $p \in M$ is decomposed as $T_pM = D_p \oplus \{\xi\}_p$ (direct sum), where we denote $D_p = \{v \in T_pM | \eta(v) = 0\}$. Then $D : p \to D_p$ defines a distribution and the restriction $J = \phi|_D$ of ϕ to D defines an almost complex structure in D. Define the Levi form L by $L(X, Y) = d\eta(X, JY)$, $X, Y \perp \xi$.

Then we find:
For an almost contact structure, both "non-degeneracy" of the Levi form and "CR-integrability" of the associated almost CR-structure are not guaranteed, in general.

Let M be an oriented real hypersurface of a Kähler manifold $\widetilde{M} = (\widetilde{M}; \widetilde{J}, \widetilde{g})$ and N a global unit normal vector field on M. By $\widetilde{\nabla}$, A we denote the Levi-Civita connection in $\widetilde{M}$ and the shape operator with respect to N, respectively.

Then the Gauss and Weingarten formulas are given respectively by

$$\widetilde{\nabla}_X Y = \nabla_X Y + g(AX, Y)N, \quad \widetilde{\nabla}_X N = -AX$$

for any vector fields X and Y tangent to M, where g denotes the Riemann metric of M induced from $\widetilde{g}$. An eigenvector(resp. eigenvalue) of the shape operator A is called a principal curvature vector(resp. principal curvature).

For any vector field X tangent to M, we put

$$\widetilde{J}X = \phi X + \eta(X)N, \quad \widetilde{J}N = -\xi. \tag{4}$$

We easily see that the structure (η, ϕ, ξ, g) is an almost contact metric structure on M i.e. satisfies (1) and (2).

Real hypersurfaces of Kähler manifolds

Let M be an oriented real hypersurface of a Kähler manifold $\widetilde{M} = (\widetilde{M}; \widetilde{J}, \widetilde{g})$ and N a global unit normal vector field on M. By $\widetilde{\nabla}$, A we denote the Levi-Civita connection in $\widetilde{M}$ and the shape operator with respect to N, respectively.

Then the Gauss and Weingarten formulas are given respectively by

$$\widetilde{\nabla}_X Y = \nabla_X Y + g(AX, Y)N, \quad \widetilde{\nabla}_X N = -AX$$

for any vector fields X and Y tangent to M, where g denotes the Riemann metric of M induced from $\widetilde{g}$. An eigenvector(resp. eigenvalue) of the shape operator A is called a principal curvature vector(resp. principal curvature).

For any vector field X tangent to M, we put

$$\widetilde{J}X = \phi X + \eta(X)N, \quad \widetilde{J}N = -\xi. \tag{4}$$

We easily see that the structure (η, ϕ, ξ, g) is an almost contact metric structure on M i.e. satisfies (1) and (2).

Real hypersurfaces of Kähler manifolds

Let M be an oriented real hypersurface of a Kähler manifold $\widetilde{M} = (\widetilde{M}; \widetilde{J}, \widetilde{g})$ and N a global unit normal vector field on M. By $\widetilde{\nabla}$, A we denote the Levi-Civita connection in $\widetilde{M}$ and the shape operator with respect to N, respectively.

Then the Gauss and Weingarten formulas are given respectively by

$$\widetilde{\nabla}_X Y = \nabla_X Y + g(AX, Y)N, \quad \widetilde{\nabla}_X N = -AX$$

for any vector fields X and Y tangent to M, where g denotes the Riemann metric of M induced from $\widetilde{g}$. An eigenvector(resp. eigenvalue) of the shape operator A is called a principal curvature vector(resp. principal curvature).

For any vector field X tangent to M, we put

$$\widetilde{J}X = \phi X + \eta(X)N, \quad \widetilde{J}N = -\xi. \tag{4}$$

We easily see that the structure (η, ϕ, ξ, g) is an almost contact metric structure on M i.e. satisfies (1) and (2).

Real hypersurfaces of Kähler manifolds

Let M be an oriented real hypersurface of a Kähler manifold $\widetilde{M} = (\widetilde{M}; \widetilde{J}, \widetilde{g})$ and N a global unit normal vector field on M. By $\widetilde{\nabla}$, A we denote the Levi-Civita connection in $\widetilde{M}$ and the shape operator with respect to N, respectively.
Then the Gauss and Weingarten formulas are given respectively by

$$\widetilde{\nabla}_X Y = \nabla_X Y + g(AX, Y)N, \quad \widetilde{\nabla}_X N = -AX$$

for any vector fields X and Y tangent to M, where g denotes the Riemann metric of M induced from $\widetilde{g}$. An eigenvector(resp. eigenvalue) of the shape operator A is called a principal curvature vector(resp. principal curvature).
For any vector field X tangent to M, we put

$$\widetilde{J}X = \phi X + \eta(X)N, \quad \widetilde{J}N = -\xi. \tag{4}$$

We easily see that the structure (η, ϕ, ξ, g) is an almost contact metric structure on M i.e. satisfies (1) and (2).

Real hypersurfaces of Kähler manifolds

Let M be an oriented real hypersurface of a Kähler manifold $\widetilde{M} = (\widetilde{M}; \widetilde{J}, \widetilde{g})$ and N a global unit normal vector field on M. By $\widetilde{\nabla}$, A we denote the Levi-Civita connection in $\widetilde{M}$ and the shape operator with respect to N, respectively.
Then the Gauss and Weingarten formulas are given respectively by

$$\widetilde{\nabla}_X Y = \nabla_X Y + g(AX, Y)N, \quad \widetilde{\nabla}_X N = -AX$$

for any vector fields X and Y tangent to M, where g denotes the Riemann metric of M induced from $\widetilde{g}$. An eigenvector(resp. eigenvalue) of the shape operator A is called a principal curvature vector(resp. principal curvature).
For any vector field X tangent to M, we put

$$\widetilde{J}X = \phi X + \eta(X)N, \quad \widetilde{J}N = -\xi. \tag{4}$$

We easily see that the structure (η, ϕ, ξ, g) is an almost contact metric structure on M i.e. satisfies (1) and (2).

From the condition $\widetilde{\nabla}\widetilde{J} = 0$, the relations (4) and by making use of the Gauss and Weingarten formulas, we have

$$(\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi, \tag{5}$$
$$\nabla_X \xi = \phi AX. \tag{6}$$

By using (5) and (6), we see that a real hypersurface in a Kähler manifold always satisfies the CR-integrability condition.

Proposition 1.

A real hypersurface of a Kähler manifold admits an integrable CR-structure.

But, the non-degeneracy of Levi-form still remains not to be guaranteed.

From the condition $\widetilde{\nabla}\widetilde{J} = 0$, the relations (4) and by making use of the Gauss and Weingarten formulas, we have

$$(\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi, \tag{5}$$
$$\nabla_X \xi = \phi AX. \tag{6}$$

By using (5) and (6), we see that a real hypersurface in a Kähler manifold always satisfies the CR-integrability condition.

Proposition 1.

A real hypersurface of a Kähler manifold admits an integrable CR-structure.

But, the non-degeneracy of Levi-form still remains not to be guaranteed.

From the condition $\widetilde{\nabla}\widetilde{J} = 0$, the relations (4) and by making use of the Gauss and Weingarten formulas, we have

$$(\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi, \tag{5}$$
$$\nabla_X \xi = \phi AX. \tag{6}$$

By using (5) and (6), we see that a real hypersurface in a Kähler manifold always satisfies the CR-integrability condition.

Proposition 1.

A real hypersurface of a Kähler manifold admits an integrable CR-structure.

But, the non-degeneracy of Levi-form still remains not to be guaranteed.

Generalized Tanaka-Webster connections

Modifying Tanno's generalized Tanaka-Webster connection for contact Riemannian manifolds, we define the generalized-Tanaka-Webster (shortly, g.-Tanaka-Webster connection) $\hat{\nabla}^{(k)}$ for real hypersurfaces of Kähler manifolds by

$$\hat{\nabla}^{(k)}_X Y = \nabla_X Y + g(\phi AX, Y)\xi - \eta(Y)\phi AX - k\eta(X)\phi Y \qquad (7)$$

for a non-zero real number k.
We put

$$F_X Y = g(\phi AX, Y)\xi - \eta(Y)\phi AX - k\eta(X)\phi Y. \qquad (8)$$

Then the torsion tensor $\hat{T}$ is given by:

$$\begin{aligned}\hat{T}(X,Y) =& F_X Y - F_Y X \\ =& g((\phi A + A\phi)X, Y)\xi - \eta(Y)\phi AX + \eta(X)\phi AY \\ &- k\big(\eta(X)\phi Y - \eta(Y)\phi X\big)\end{aligned} \qquad (9)$$

Modifying Tanno's generalized Tanaka-Webster connection for contact Riemannian manifolds, we define the generalized-Tanaka-Webster (shortly, g.-Tanaka-Webster connection) $\hat{\nabla}^{(k)}$ for real hypersurfaces of Kähler manifolds by

$$\hat{\nabla}^{(k)}_X Y = \nabla_X Y + g(\phi AX, Y)\xi - \eta(Y)\phi AX - k\eta(X)\phi Y \tag{7}$$

for a non-zero real number k.

We put

$$F_X Y = g(\phi AX, Y)\xi - \eta(Y)\phi AX - k\eta(X)\phi Y. \tag{8}$$

Then the torsion tensor $\hat{T}$ is given by:

$$\begin{aligned}\hat{T}(X,Y) =& F_X Y - F_Y X \\ =& g((\phi A + A\phi)X, Y)\xi - \eta(Y)\phi AX + \eta(X)\phi AY \\ & - k(\eta(X)\phi Y - \eta(Y)\phi X)\end{aligned} \tag{9}$$

Modifying Tanno's generalized Tanaka-Webster connection for contact Riemannian manifolds, we define the generalized-Tanaka-Webster (shortly, g.-Tanaka-Webster connection) $\hat{\nabla}^{(k)}$ for real hypersurfaces of Kähler manifolds by

$$\hat{\nabla}^{(k)}_X Y = \nabla_X Y + g(\phi AX, Y)\xi - \eta(Y)\phi AX - k\eta(X)\phi Y \tag{7}$$

for a non-zero real number k.
We put

$$F_X Y = g(\phi AX, Y)\xi - \eta(Y)\phi AX - k\eta(X)\phi Y. \tag{8}$$

Then the torsion tensor $\hat{T}$ is given by:

$$\begin{aligned}\hat{T}(X, Y) =& F_X Y - F_Y X \\ =& g((\phi A + A\phi)X, Y)\xi - \eta(Y)\phi AX + \eta(X)\phi AY \\ & - k\big(\eta(X)\phi Y - \eta(Y)\phi X\big)\end{aligned} \tag{9}$$

Furthermore, by using (2), (3), (5), (6) and (7) we can see that

$$\hat{\nabla}^{(k)}\eta = 0,\ \hat{\nabla}^{(k)}\xi = 0,\ \hat{\nabla}^{(k)}g = 0,\ \hat{\nabla}^{(k)}\phi = 0, \tag{10}$$

and

$$\hat{T}(X, Y) = 2d\eta(X, Y)\xi,\ X,\ Y \in D.$$

We note that the associated Levi form is

$$L(X, Y) = \frac{1}{2}g((J\bar{A} + \bar{A}J)X, JY),$$

where we denote by $\bar{A}$ the restriction A to D. Then, we have

Proposition 2.([Cho1],[Cho2])

Let $M = (M; \eta, \phi, \xi, g)$ be a real hypersurface of a Kähler manifold. If M satisfies $\phi A + A\phi = 2k\phi$, then the associated CR-structure is strongly pseudo-convex and further the g.-Tanaka-Webster connection $\hat{\nabla}^{(k)}$ coincides with the Tanaka-Webster connection.

Furthermore, by using (2), (3), (5), (6) and (7) we can see that

$$\hat{\nabla}^{(k)}\eta = 0,\ \hat{\nabla}^{(k)}\xi = 0,\ \hat{\nabla}^{(k)}g = 0,\ \hat{\nabla}^{(k)}\phi = 0, \tag{10}$$

and

$$\hat{T}(X, Y) = 2d\eta(X, Y)\xi,\ X,\ Y \in D.$$

We note that the associated Levi form is

$$L(X, Y) = \frac{1}{2}g((J\bar{A} + \bar{A}J)X, JY),$$

where we denote by $\bar{A}$ the restriction A to D. Then, we have

Proposition 2.([Cho1],[Cho2])

Let $M = (M; \eta, \phi, \xi, g)$ be a real hypersurface of a Kähler manifold. If M satisfies $\phi A + A\phi = 2k\phi$, then the associated CR-structure is strongly pseudo-convex and further the g.-Tanaka-Webster connection $\hat{\nabla}^{(k)}$ coincides with the Tanaka-Webster connection.

Furthermore, by using (2), (3), (5), (6) and (7) we can see that

$$\hat{\nabla}^{(k)}\eta = 0,\ \hat{\nabla}^{(k)}\xi = 0,\ \hat{\nabla}^{(k)}g = 0,\ \hat{\nabla}^{(k)}\phi = 0, \tag{10}$$

and

$$\hat{T}(X, Y) = 2d\eta(X, Y)\xi,\ X,\ \ Y \in D.$$

We note that the associated Levi form is

$$L(X, Y) = \frac{1}{2}g((J\bar{A} + \bar{A}J)X, JY),$$

where we denote by $\bar{A}$ the restriction A to D. Then, we have

Proposition 2.([Cho1],[Cho2])

Let $M = (M; \eta, \phi, \xi, g)$ be a real hypersurface of a Kähler manifold. If M satisfies $\phi A + A\phi = 2k\phi$, then the associated CR-structure is strongly pseudo-convex and further the g.-Tanaka-Webster connection $\hat{\nabla}^{(k)}$ coincides with the Tanaka-Webster connection.

J. T. Cho, CR structures on real hypersurfaces of a complex space form, Publ. Math. Debrecen 54(3-4) (1999), 473–487.

J. T. Cho, Levi-parallel hypersurfaces in a complex space form, Tsukuba J. Math. 30(2) (2006), 329–344.

A real hypersurface of a Kähler manifold is called a *contact (type) hypersurface* if M satisfies $\phi A + A\phi = 2k\phi$, $k \neq 0$.

J. T. Cho, CR structures on real hypersurfaces of a complex space form, Publ. Math. Debrecen 54(3-4) (1999), 473–487.

J. T. Cho, Levi-parallel hypersurfaces in a complex space form, Tsukuba J. Math. 30(2) (2006), 329–344.

A real hypersurface of a Kähler manifold is called a *contact (type) hypersurface* if M satisfies $\phi A + A\phi = 2k\phi$, $k \neq 0$.

J. T. Cho, CR structures on real hypersurfaces of a complex space form, Publ. Math. Debrecen 54(3-4) (1999), 473–487.

J. T. Cho, Levi-parallel hypersurfaces in a complex space form, Tsukuba J. Math. 30(2) (2006), 329–344.

A real hypersurface of a Kähler manifold is called a *contact (type) hypersurface* if M satisfies $\phi A + A\phi = 2k\phi$, $k \neq 0$.

Theorem 3.([Okumura], [Kon], [Vernon], [Suh], [Adachi-Kameda-Maeda])

Let M be a real hypersurface of a complex space form $\widetilde{M}_n(c)$. Then M is of contact type if and only if M is locally congruent to one of the following:

(I) in case that $\widetilde{M}_n(c) = P_n\mathbb{C}$ with the Fubini-Study metric,
(A_1) a geodesic hypersphere of radius r, where $0 < r < \frac{\pi}{2}$,
(A_2) a tube of radius r over a totally geodesic $P_\ell\mathbb{C}(1 \leq \ell \leq n-2)$, where $0 < r < \frac{\pi}{2}$,
(B) a tube of radius r over a complex quadric Q_{n-1}, where $0 < r < \frac{\pi}{4}$;

(II) in case that $\widetilde{M}_n(c) = H_n\mathbb{C}$ with the Bergman metric,

(A_0) a horosphere,
(A_1) a geodesic hypersphere or a tube over a complex hyperbolic hyperplane $H_{n-1}\mathbb{C}$,
(A_2) a tube over a totally geodesic $H_\ell\mathbb{C}(1 \leq \ell \leq n-2)$,
(B) a tube over a totally real hyperbolic space $H_n\mathbb{R}$

(III) in case that $\widetilde{M}_n(c) = E_n\mathbb{C}$,

(A) $S^{2n-1}(r)$,
(B) $S^{n-1}(r) \times \mathbb{E}^n$.

Theorem 3.([Okumura], [Kon], [Vernon], [Suh], [Adachi-Kameda-Maeda])

Let M be a real hypersurface of a complex space form $\widetilde{M}_n(c)$. Then M is of contact type if and only if M is locally congruent to one of the following:

(I) in case that $\widetilde{M}_n(c) = P_n\mathbb{C}$ with the Fubini-Study metric,
(A_1) a geodesic hypersphere of radius r, where $0 < r < \frac{\pi}{2}$,
(A_2) a tube of radius r over a totally geodesic $P_\ell\mathbb{C}(1 \leq \ell \leq n-2)$, where $0 < r < \frac{\pi}{2}$,
(B) a tube of radius r over a complex quadric Q_{n-1}, where $0 < r < \frac{\pi}{4}$;

(II) in case that $\widetilde{M}_n(c) = H_n\mathbb{C}$ with the Bergman metric,

(A_0) a horosphere,
(A_1) a geodesic hypersphere or a tube over a complex hyperbolic hyperplane $H_{n-1}\mathbb{C}$,
(A_2) a tube over a totally geodesic $H_\ell\mathbb{C}(1 \leq \ell \leq n-2)$,
(B) a tube over a totally real hyperbolic space $H_n\mathbb{R}$

(III) in case that $\widetilde{M}_n(c) = E_n\mathbb{C}$,

(A) $S^{2n-1}(r)$,
(B) $S^{n-1}(r) \times \mathbb{E}^n$.

Theorem 3.([Okumura], [Kon], [Vernon], [Suh], [Adachi-Kameda-Maeda])

Let M be a real hypersurface of a complex space form $\widetilde{M}_n(c)$. Then M is of contact type if and only if M is locally congruent to one of the following:

(I) in case that $\widetilde{M}_n(c) = P_n\mathbb{C}$ with the Fubini-Study metric,
(A_1) a geodesic hypersphere of radius r, where $0 < r < \frac{\pi}{2}$,
(A_2) a tube of radius r over a totally geodesic $P_\ell\mathbb{C}(1 \leq \ell \leq n-2)$, where $0 < r < \frac{\pi}{2}$,
(B) a tube of radius r over a complex quadric Q_{n-1}, where $0 < r < \frac{\pi}{4}$;

(II) in case that $\widetilde{M}_n(c) = H_n\mathbb{C}$ with the Bergman metric,

(A_0) a horosphere,
(A_1) a geodesic hypersphere or a tube over a complex hyperbolic hyperplane $H_{n-1}\mathbb{C}$,
(A_2) a tube over a totally geodesic $H_\ell\mathbb{C}(1 \leq \ell \leq n-2)$,
(B) a tube over a totally real hyperbolic space $H_n\mathbb{R}$

(III) in case that $\widetilde{M}_n(c) = E_n\mathbb{C}$,

(A) $S^{2n-1}(r)$,
(B) $S^{n-1}(r) \times \mathbb{E}^n$.

Theorem 3.([Okumura], [Kon], [Vernon], [Suh], [Adachi-Kameda-Maeda])

Let M be a real hypersurface of a complex space form $\widetilde{M}_n(c)$. Then M is of contact type if and only if M is locally congruent to one of the following:

(I) in case that $\widetilde{M}_n(c) = P_n\mathbb{C}$ with the Fubini-Study metric,
(A_1) a geodesic hypersphere of radius r, where $0 < r < \frac{\pi}{2}$,
(A_2) a tube of radius r over a totally geodesic $P_\ell\mathbb{C}(1 \leq \ell \leq n-2)$, where $0 < r < \frac{\pi}{2}$,
(B) a tube of radius r over a complex quadric Q_{n-1}, where $0 < r < \frac{\pi}{4}$;

(II) in case that $\widetilde{M}_n(c) = H_n\mathbb{C}$ with the Bergman metric,

(A_0) a horosphere,
(A_1) a geodesic hypersphere or a tube over a complex hyperbolic hyperplane $H_{n-1}\mathbb{C}$,
(A_2) a tube over a totally geodesic $H_\ell\mathbb{C}(1 \leq \ell \leq n-2)$,
(B) a tube over a totally real hyperbolic space $H_n\mathbb{R}$

(III) in case that $\widetilde{M}_n(c) = E_n\mathbb{C}$,

(A) $S^{2n-1}(r)$,
(B) $S^{n-1}(r) \times \mathbb{E}^n$.

M. Okumura, Contact hypersutfaces in certain Kaehlerian manifolds, Tôhoku Math. J. 18 (2) (1966), 74–102.

M. Kon, Pseudo-Einstein real hypersurfaces of complex space forms, J. Diff. Geometry 14 (1979), 339–354.

M. H. Vernon, Contact hypersutfaces in complex hyperbolic space, Tôhoku Math. J. 39 (1987), 215–222.

Y.J. Suh, On real hypersurfaces of a complex space form with η-parallel Ricci tensor, Tsukuba J. Math. 14 (1990), 27–37.

T. Adachi, M. Kameda and S. Maeda, Real hypersurfaces which are contact in a nonflat complex space form, Hokkaido Math. J. 40 (2011), 205–217.

Theorem A. ([Cho1]) *Let M be a real hypersurface of a non-flat complex space form $\widetilde{M}_n(c)$, $c \neq 0$. Then the shape operator A is parallel for the g.-Tanaka-Webster connection if and only if M is locally congruent to a real hypersurfaces of type (A) or type (B) in $P_n\mathbb{C}$ or $H_n\mathbb{C}$*

As the CR-geometric counterpart of local symmetry, we introduce *g.-Tanaka-Webster parallelity in real hypersurfaces of a Kähler manifold*, whose g.-Tanaka-Webster torsion tensor $\hat{T}$ and g.-Tanaka-Webster curvature tensor $\hat{R}$ are parallel with respect to $\hat{\nabla}^{(k)}$:

$$\hat{\nabla}^{(k)}\hat{T} = 0, \quad \hat{\nabla}^{(k)}\hat{R} = 0.$$

Theorem A. ([Cho1]) *Let M be a real hypersurface of a non-flat complex space form $\widetilde{M}_n(c)$, $c \neq 0$. Then the shape operator A is parallel for the g.-Tanaka-Webster connection if and only if M is locally congruent to a real hypersurfaces of type (A) or type (B) in $P_n\mathbb{C}$ or $H_n\mathbb{C}$*

As the CR-geometric counterpart of local symmetry, we introduce *g.-Tanaka-Webster parallelity in real hypersurfaces of a Kähler manifold*, whose g.-Tanaka-Webster torsion tensor $\hat{T}$ and g.-Tanaka-Webster curvature tensor $\hat{R}$ are parallel with respect to $\hat{\nabla}^{(k)}$:

$$\hat{\nabla}^{(k)}\hat{T} = 0, \quad \hat{\nabla}^{(k)}\hat{R} = 0.$$

Theorem A. ([Cho1]) *Let M be a real hypersurface of a non-flat complex space form $\widetilde{M}_n(c)$, $c \neq 0$. Then the shape operator A is parallel for the g.-Tanaka-Webster connection if and only if M is locally congruent to a real hypersurfaces of type (A) or type (B) in $P_n\mathbb{C}$ or $H_n\mathbb{C}$*

As the CR-geometric counterpart of local symmetry, we introduce *g.-Tanaka-Webster parallelity in real hypersurfaces of a Kähler manifold*, whose g.-Tanaka-Webster torsion tensor $\hat{T}$ and g.-Tanaka-Webster curvature tensor $\hat{R}$ are parallel with respect to $\hat{\nabla}^{(k)}$:

$$\hat{\nabla}^{(k)}\hat{T} = 0, \quad \hat{\nabla}^{(k)}\hat{R} = 0.$$

Then we have

Theorem B.([Cho-Kon]) *Let M be a real hypersurface of a non-flat complex space form $\widetilde{M}_n(c)$, $n \geq 3$, $c \neq 0$. Then M is g.-Tanaka-Webster parallel if and only if M is locally congruent to one of real hypersurfaces of type (A) or (B).*

J. T. Cho and M. Kon, The Tanaka-Webster connection and real hypersurfaces in a complex space form, Kodai Math. J. 34(3) (2011), 474–484.

Then we have

Theorem B.([Cho-Kon]) *Let M be a real hypersurface of a non-flat complex space form $\widetilde{M}_n(c)$, $n \geq 3$, $c \neq 0$. Then M is g.-Tanaka-Webster parallel if and only if M is locally congruent to one of real hypersurfaces of type (A) or (B).*

J. T. Cho and M. Kon, The Tanaka-Webster connection and real hypersurfaces in a complex space form, Kodai Math. J. 34(3) (2011), 474–484.

Motivation II.

• Y. T. Siu [Siu] proved the nonexistence of compact smooth Levi-flat hypersurfaces in $P_n\mathbb{C}$ of dimensions ≥ 3.

• When $n = 2$, T. Ohsawa [Ohsawa] proved the nonexistence of compact real analytic Levi-flat hypersurfaces in $P_2\mathbb{C}$.

• It is remarkable that the assumption of compactness has a crucial role. Indeed, there are non-complete examples which are realized as ruled hypersurfaces and Levi-flat in $P_n\mathbb{C}$.

• We also find that there does not exist a Levi-flat Hopf hypersurface in $P_n\mathbb{C}$ or $H_n\mathbb{C}$ (cf. [Cho2]).

• Cho and Kimura give new examples of Levi-flat real hypersurfaces which are not ruled hypersurfaces in $P_2\mathbb{C}$.

T. Ohsawa, Nonexistence of real analytic Levi flat hypersurfaces in $\mathbb{P}^2$, Nagoya Math. J. 158 (2000), 95–98.

Y. T. Siu, Nonexistence of smooth Levi-flat hypersurfaces in complex projective spaces of dimensions ≥ 3, Ann. of Math. 151(2) (2000), 1217–1243.

Motivation II.

• Y. T. Siu [Siu] proved the nonexistence of compact smooth Levi-flat hypersurfaces in $P_n\mathbb{C}$ of dimensions ≥ 3.

• When $n = 2$, T. Ohsawa [Ohsawa] proved the nonexistence of compact real analytic Levi-flat hypersurfaces in $P_2\mathbb{C}$.

• It is remarkable that the assumption of compactness has a crucial role. Indeed, there are non-complete examples which are realized as ruled hypersurfaces and Levi-flat in $P_n\mathbb{C}$.

• We also find that there does not exist a Levi-flat Hopf hypersurface in $P_n\mathbb{C}$ or $H_n\mathbb{C}$ (cf. [Cho2]).

• Cho and Kimura give new examples of Levi-flat real hypersurfaces which are not ruled hypersurfaces in $P_2\mathbb{C}$.

T. Ohsawa, Nonexistence of real analytic Levi flat hypersurfaces in $\mathbb{P}^2$, Nagoya Math. J. 158 (2000), 95–98.

Y. T. Siu, Nonexistence of smooth Levi-flat hypersurfaces in complex projective spaces of dimensions ≥ 3, Ann. of Math. 151(2) (2000), 1217–1243.

Motivation II.

• Y. T. Siu [Siu] proved the nonexistence of compact smooth Levi-flat hypersurfaces in $P_n\mathbb{C}$ of dimensions ≥ 3.

• When $n = 2$, T. Ohsawa [Ohsawa] proved the nonexistence of compact real analytic Levi-flat hypersurfaces in $P_2\mathbb{C}$.

• It is remarkable that the assumption of compactness has a crucial role. Indeed, there are non-complete examples which are realized as ruled hypersurfaces and Levi-flat in $P_n\mathbb{C}$.

• We also find that there does not exist a Levi-flat Hopf hypersurface in $P_n\mathbb{C}$ or $H_n\mathbb{C}$ (cf. [Cho2]).

• Cho and Kimura give new examples of Levi-flat real hypersurfaces which are not ruled hypersurfaces in $P_2\mathbb{C}$.

T. Ohsawa, Nonexistence of real analytic Levi flat hypersurfaces in $\mathbb{P}^2$, Nagoya Math. J. 158 (2000), 95–98.

Y. T. Siu, Nonexistence of smooth Levi-flat hypersurfaces in complex projective spaces of dimensions ≥ 3, Ann. of Math. 151(2) (2000), 1217–1243.

Motivation II.

• Y. T. Siu [Siu] proved the nonexistence of compact smooth Levi-flat hypersurfaces in $P_n\mathbb{C}$ of dimensions ≥ 3.

• When $n = 2$, T. Ohsawa [Ohsawa] proved the nonexistence of compact real analytic Levi-flat hypersurfaces in $P_2\mathbb{C}$.

• It is remarkable that the assumption of compactness has a crucial role. Indeed, there are non-complete examples which are realized as ruled hypersurfaces and Levi-flat in $P_n\mathbb{C}$.

• We also find that there does not exist a Levi-flat Hopf hypersurface in $P_n\mathbb{C}$ or $H_n\mathbb{C}$ (cf. [Cho2]).

• Cho and Kimura give new examples of Levi-flat real hypersurfaces which are not ruled hypersurfaces in $P_2\mathbb{C}$.

T. Ohsawa, Nonexistence of real analytic Levi flat hypersurfaces in $\mathbb{P}^2$, Nagoya Math. J. 158 (2000), 95–98.

Y. T. Siu, Nonexistence of smooth Levi-flat hypersurfaces in complex projective spaces of dimensions ≥ 3, Ann. of Math. 151(2) (2000), 1217–1243.

Motivation II.

• Y. T. Siu [Siu] proved the nonexistence of compact smooth Levi-flat hypersurfaces in $P_n\mathbb{C}$ of dimensions ≥ 3.

• When $n = 2$, T. Ohsawa [Ohsawa] proved the nonexistence of compact real analytic Levi-flat hypersurfaces in $P_2\mathbb{C}$.

• It is remarkable that the assumption of compactness has a crucial role. Indeed, there are non-complete examples which are realized as ruled hypersurfaces and Levi-flat in $P_n\mathbb{C}$.

• We also find that there does not exist a Levi-flat Hopf hypersurface in $P_n\mathbb{C}$ or $H_n\mathbb{C}$ (cf. [Cho2]).

• Cho and Kimura give new examples of Levi-flat real hypersurfaces which are not ruled hypersurfaces in $P_2\mathbb{C}$.

T. Ohsawa, Nonexistence of real analytic Levi flat hypersurfaces in $\mathbb{P}^2$, Nagoya Math. J. 158 (2000), 95–98.

Y. T. Siu, Nonexistence of smooth Levi-flat hypersurfaces in complex projective spaces of dimensions ≥ 3, Ann. of Math. 151(2) (2000), 1217–1243.

• Y. T. Siu [Siu] proved the nonexistence of compact smooth Levi-flat hypersurfaces in $P_n\mathbb{C}$ of dimensions ≥ 3.

• When $n = 2$, T. Ohsawa [Ohsawa] proved the nonexistence of compact real analytic Levi-flat hypersurfaces in $P_2\mathbb{C}$.

• It is remarkable that the assumption of compactness has a crucial role. Indeed, there are non-complete examples which are realized as ruled hypersurfaces and Levi-flat in $P_n\mathbb{C}$.

• We also find that there does not exist a Levi-flat Hopf hypersurface in $P_n\mathbb{C}$ or $H_n\mathbb{C}$ (cf. [Cho2]).

• Cho and Kimura give new examples of Levi-flat real hypersurfaces which are not ruled hypersurfaces in $P_2\mathbb{C}$.

T. Ohsawa, Nonexistence of real analytic Levi flat hypersurfaces in $\mathbb{P}^2$, Nagoya Math. J. 158 (2000), 95–98.

Y. T. Siu, Nonexistence of smooth Levi-flat hypersurfaces in complex projective spaces of dimensions ≥ 3, Ann. of Math. 151(2) (2000), 1217–1243.

Motivation II.

• Y. T. Siu [Siu] proved the nonexistence of compact smooth Levi-flat hypersurfaces in $P_n\mathbb{C}$ of dimensions ≥ 3.

• When $n = 2$, T. Ohsawa [Ohsawa] proved the nonexistence of compact real analytic Levi-flat hypersurfaces in $P_2\mathbb{C}$.

• It is remarkable that the assumption of compactness has a crucial role. Indeed, there are non-complete examples which are realized as ruled hypersurfaces and Levi-flat in $P_n\mathbb{C}$.

• We also find that there does not exist a Levi-flat Hopf hypersurface in $P_n\mathbb{C}$ or $H_n\mathbb{C}$ (cf. [Cho2]).

• Cho and Kimura give new examples of Levi-flat real hypersurfaces which are not ruled hypersurfaces in $P_2\mathbb{C}$.

T. Ohsawa, Nonexistence of real analytic Levi flat hypersurfaces in $\mathbb{P}^2$, Nagoya Math. J. 158 (2000), 95–98.

Y. T. Siu, Nonexistence of smooth Levi-flat hypersurfaces in complex projective spaces of dimensions ≥ 3, Ann. of Math. 151(2) (2000), 1217–1243.

- Ruled real hypersurfaces in $P_n\mathbb{C}$ or $H_n\mathbb{C}$:

Such a space is *a foliated real hypersurface whose leaves are complex hyperplanes $P_{n-1}\mathbb{C}$ or $H_{n-1}\mathbb{C}$, respectively in $P_n\mathbb{C}$ or $H_n\mathbb{C}$*. That is, let $\gamma : I \to \widetilde{M}_n(c)$ be a regular curve in $\widetilde{M}_n(c)$ ($P_n\mathbb{C}$ or $H_n\mathbb{C}$). Then for each $t \in I$, let $M^{(t)}_{n-1}(c)$ be a totally geodesic complex hypersurfaces which is orthogonal to holomorphic plane $\mathrm{Span}\{\dot\gamma, J\dot\gamma\}$. We have a ruled real hypersurface $M = \bigcup_{t\in I} M^{(t)}_{n-1}(c)$. Note that a ruled real hypersurface is non-Hopf and particularly it is non-complete real hypersurface in $P_n\mathbb{C}$.

The shape operator A is written by the following form:

$$
\begin{aligned}
&A\xi = \mu\xi + \nu V \ (\nu \neq 0),\\
&AV = \nu\xi,\\
&AX = 0 \text{ for any } X \perp \xi, V,
\end{aligned} \tag{11}
$$

where V is a unit vector orthogonal to ξ, and μ, ν are differentiable functions on M. Then, we easily see that *ruled real hypersurfaces in $P_n\mathbb{C}$ or in $H_n\mathbb{C}$ are Levi-flat.*

- Ruled real hypersurfaces in $P_n\mathbb{C}$ or $H_n\mathbb{C}$:

Such a space is *a foliated real hypersurface whose leaves are complex hyperplanes* $P_{n-1}\mathbb{C}$ *or* $H_{n-1}\mathbb{C}$, *respectively in* $P_n\mathbb{C}$ *or* $H_n\mathbb{C}$. That is, let $\gamma : I \to \widetilde{M}_n(c)$ be a regular curve in $\widetilde{M}_n(c)$ ($P_n\mathbb{C}$ or $H_n\mathbb{C}$). Then for each $t \in I$, let $M^{(t)}_{n-1}(c)$ be a totally geodesic complex hypersurfaces which is orthogonal to holomorphic plane Span$\{\dot{\gamma}, J\dot{\gamma}\}$. We have a ruled real hypersurface $M = \bigcup_{t\in I} M^{(t)}_{n-1}(c)$. Note that a ruled real hypersurface is non-Hopf and particularly it is non-complete real hypersurface in $P_n\mathbb{C}$.

The shape operator A is written by the following form:

$$\begin{aligned} &A\xi = \mu\xi + \nu V \ (\nu \neq 0), \\ &AV = \nu\xi, \\ &AX = 0 \text{ for any } X \perp \xi, V, \end{aligned} \tag{11}$$

where V is a unit vector orthogonal to ξ, and μ, ν are differentiable functions on M. Then, we easily see that *ruled real hypersurfaces in* $P_n\mathbb{C}$ *or in* $H_n\mathbb{C}$ *are Levi-flat.*

Ruled real hypersurfaces

- Ruled real hypersurfaces in $P_n\mathbb{C}$ or $H_n\mathbb{C}$:

Such a space is *a foliated real hypersurface whose leaves are complex hyperplanes $P_{n-1}\mathbb{C}$ or $H_{n-1}\mathbb{C}$, respectively in $P_n\mathbb{C}$ or $H_n\mathbb{C}$*. That is, let $\gamma : I \to \widetilde{M}_n(c)$ be a regular curve in $\widetilde{M}_n(c)$ ($P_n\mathbb{C}$ or $H_n\mathbb{C}$). Then for each $t \in I$, let $M^{(t)}_{n-1}(c)$ be a totally geodesic complex hypersurfaces which is orthogonal to holomorphic plane $\mathrm{Span}\{\dot\gamma, J\dot\gamma\}$. We have a ruled real hypersurface $M = \bigcup_{t\in I} M^{(t)}_{n-1}(c)$. Note that a ruled real hypersurface is non-Hopf and particularly it is non-complete real hypersurface in $P_n\mathbb{C}$.

The shape operator A is written by the following form:

$$\begin{aligned} A\xi &= \mu\xi + \nu V \ (\nu \neq 0),\\ AV &= \nu\xi, \\ AX &= 0 \text{ for any } X \perp \xi, V, \end{aligned} \tag{11}$$

where V is a unit vector orthogonal to ξ, and μ, ν are differentiable functions on M. Then, we easily see that *ruled real hypersurfaces in $P_n\mathbb{C}$ or in $H_n\mathbb{C}$ are Levi-flat.*

- Ruled real hypersurfaces in $P_n\mathbb{C}$ or $H_n\mathbb{C}$:

Such a space is *a foliated real hypersurface whose leaves are complex hyperplanes $P_{n-1}\mathbb{C}$ or $H_{n-1}\mathbb{C}$, respectively in $P_n\mathbb{C}$ or $H_n\mathbb{C}$*. That is, let $\gamma : I \to \widetilde{M}_n(c)$ be a regular curve in $\widetilde{M}_n(c)$ ($P_n\mathbb{C}$ or $H_n\mathbb{C}$). Then for each $t \in I$, let $M^{(t)}_{n-1}(c)$ be a totally geodesic complex hypersurfaces which is orthogonal to holomorphic plane $\mathrm{Span}\{\dot\gamma, J\dot\gamma\}$. We have a ruled real hypersurface $M = \bigcup_{t\in I} M^{(t)}_{n-1}(c)$. Note that a ruled real hypersurface is non-Hopf and particularly it is non-complete real hypersurface in $P_n\mathbb{C}$.

The shape operator A is written by the following form:

$$\begin{aligned} A\xi &= \mu\xi + \nu V \ (\nu \neq 0), \\ AV &= \nu\xi, \\ AX &= 0 \text{ for any } X \perp \xi, V, \end{aligned} \tag{11}$$

where V is a unit vector orthogonal to ξ, and μ, ν are differentiable functions on M. Then, we easily see that *ruled real hypersurfaces in $P_n\mathbb{C}$ or in $H_n\mathbb{C}$ are Levi-flat.*

M. Kimura and S. Maeda, On real hypersurfaces of a complex projective space, Math. Z. 202 (1989), 299–311.

J. Berndt and H. Tamaru [Berndt-Tamaru] classified all homogeneous real hypersurfaces in $H_n\mathbb{C}$. In their classification homogeneous ruled minimal hypersurfaces, called type (S), appeared.

J. Berndt and H. Tamaru, Cohomogeneity one actions on noncompact symmetric spaces of rank one, Trans. Amer. Math. Soc. 359 (2007), no. 7, 3425–3438.

M. Kimura and S. Maeda, On real hypersurfaces of a complex projective space, Math. Z. 202 (1989), 299–311.

J. Berndt and H. Tamaru [Berndt-Tamaru] classified all homogeneous real hypersurfaces in $H_n\mathbb{C}$. In their classification homogeneous ruled minimal hypersurfaces, called type (S), appeared.

J. Berndt and H. Tamaru, Cohomogeneity one actions on noncompact symmetric spaces of rank one, Trans. Amer. Math. Soc. 359 (2007), no. 7, 3425–3438.

Levi-umbilicity

If the Levi-form L is proportional to the induced metric g by a non-zero constant, then M is said to be *Levi-umbilical*. M is said to be *Levi-parallel* (for the generalized Tanaka-Webster connection) if M satisfies $\hat{\nabla} L = 0$. (well-defined!!)

Then we find at once that

Fact:

• Levi-flat and Levi-umbilical real hypersurfaces in Kählerian manifolds $\Rightarrow$ Levi-parallel.

Recently, Cho and Kimura classified Levi-umbilical real hypersurfaces in a complex space form $\widetilde{M}_n(c)$, $n \geq 3$.

J. T. Cho and M. Kimura, Levi-umbilical real hypersurfaces in a complex space form, preprint

If the Levi-form L is proportional to the induced metric g by a non-zero constant, then M is said to be *Levi-umbilical*. M is said to be *Levi-parallel* (for the generalized Tanaka-Webster connection) if M satisfies $\hat{\nabla} L = 0$. (well-defined!!)

Then we find at once that

Fact:

• Levi-flat and Levi-umbilical real hypersurfaces in Kählerian manifolds $\Rightarrow$ Levi-parallel.

Recently, Cho and Kimura classified Levi-umbilical real hypersurfaces in a complex space form $\widetilde{M}_n(c)$, $n \geq 3$.

J. T. Cho and M. Kimura, Levi-umbilical real hypersurfaces in a complex space form, preprint

Levi-umbilicity

If the Levi-form L is proportional to the induced metric g by a non-zero constant, then M is said to be *Levi-umbilical*. M is said to be *Levi-parallel* (for the generalized Tanaka-Webster connection) if M satisfies $\hat{\nabla} L = 0$. (well-defined!!)

Then we find at once that

Fact:

• Levi-flat and Levi-umbilical real hypersurfaces in Kählerian manifolds $\Rightarrow$ Levi-parallel.

Recently, Cho and Kimura classified Levi-umbilical real hypersurfaces in a complex space form $\widetilde{M}_n(c)$, $n \geq 3$.

J. T. Cho and M. Kimura, Levi-umbilical real hypersurfaces in a complex space form, preprint

Theorem 4.

If a real hypersurface M of a complex space form $\widetilde{M}_n(c)$ is Levi-umbilical, then $n = 2$ or M is a Hopf and further a contact-type hypersurface.

• We give a construction of Levi-umbilical non-Hopf hypersurfaces in $P_2\mathbb{C}$.

M. Kimura, Some non-homogeneous real hypersurfaces in a complex projective space I (Construction), Bull. Fac. Educ. Ibaraki Univ. 44 (1995), 1–16.

Studies in progress

Problem 1.

Study on Levi-parallel real hypersurfaces in a complex space form.

Problem 2.

Classify Levi-umbilical real hypersurfaces in a Hermitian symmetric space.

Problem 1.

Study on Levi-parallel real hypersurfaces in a complex space form.

Problem 2.

Classify Levi-umbilical real hypersurfaces in a Hermitian symmetric space.

Studies in progress

Problem 1.

Study on Levi-parallel real hypersurfaces in a complex space form.

Problem 2.

Classify Levi-umbilical real hypersurfaces in a Hermitian symmetric space.

Denote by $G_2(\mathbb{C}^{m+2})$ the set of all complex two-dimensional linear subspaces in $\mathbb{C}^{m+2}$. This Riemannian symmetric space $G_2(\mathbb{C}^{m+2})$ has a remarkable geometric structure. It is well-known that $G_2(\mathbb{C}^{m+2})$ is the unique compact, irreducible, Kahler, quaternionic Kahler manifold which is not a hyper-Kahler manifold. Then, we find that not only an almost contact metric structure (η, ϕ, ξ, g) but also an almost contact metric 3-structure $(\eta_\nu, \phi_\nu, \xi_\nu, g)$ $(\nu = 1, 2, 3)$ enjoys on M. Here, the almost contact 3-structure vector fields $\{\xi_1, \xi_2, \xi_3\}$ for the 3-dimensional distribution $D^\perp$ of M in $G_2(\mathbb{C}^{m+2})$ are defined by $\xi_\nu = -J_\nu N$ $(\nu = 1, 2, 3)$, where J_ν denotes a canonical local basis of a quaternionic Käler structure $\mathfrak{J}$, such that $T_x M = D \oplus D^\perp$, $x \in M$.

Some recent results

Theorem 5. (I. Jeong, H. Lee and Y.J. Suh)

There does not exist any Hopf hypersurface, $\alpha \neq 2k$, in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, with parallel shape operator in the generalized Tanaka-Webster connection.

I. Jeong, H. Lee and Y.J. Suh, Real hypersurfaces in a complex two-plane Grassmannians with generalized Tanaka-Webster parallel shape operator, Kodai Math. J. 34 (2011), 352–366.

Theorem 6. (Y.J. Suh)

There do not exist Hopf real hypersurfaces in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, whose Ricci tensor is parallel (with respect to Levi-Civita connection).

Y.J. Suh, Real hypersurfaces in comploex two-plane Grassmannians with parallel Ricci tensor, Proc. Royal Soc. Edinb., 142A(2012), 1309–1324.

Theorem 7. (J. D. Pérez and Y.J. Suh)

There do not exist Hopf real hypersurfaces in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, whose Ricci tensor is parallel with respect to the g.-Tanaka- Webster connection.

J. D. Pérez and Y.J. Suh, Real hypersurfaces in complex two-plane Grassmannians with parallel Ricci tensor in generalized Tanaka-Webster connection, preprint

Thus, we have

Corollary 8.

• There do not exist Hopf and locally symmetric ($\nabla R = 0$) real hypersurfaces in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$.
• There do not exist Hopf and g.-Tanaka-Webster parallel ($\hat{\nabla}\hat{R} = 0$ and $\hat{\nabla}\hat{T} = 0$) real hypersurfaces in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$.

Question: Could we find a suitable parallelism in real hypersurfaces in $G_2(\mathbb{C}^{m+2})$?

Levi-umbilical hypersurfaces of $G_2(\mathbb{C}^{m+2})$

Very recently, Cho and Kimura proved

Theorem 9. (J.T. Cho and M. Kimura)

Let M be a Levi-umbilical real hypersurface with constant mean curvature in $G_2(\mathbb{C}^{m+2})$ or $G_2^*(\mathbb{C}^{m+2})$, $m \geq 3$.

(I) If $\widetilde{M}_n(c) = G_2(\mathbb{C}^{m+2})$, then m is even, say $m = 2l$, and M is locally congruent to a tube around the totally geodesic $\mathbb{H}P^l \subset G_2(\mathbb{C}^{2l+2})$.

(II) *If* $\widetilde{M}_n(c) = G_2^*(\mathbb{C}^{m+2})$, *then* M *is congruent to one of the following*:

(1) *a horosphere whose center at infinity is a singular point of type* $JX \perp \mathfrak{J}X$,
(2) m *is even, say* $m = 2l$, *and* M *is locally congruent to a tube around the totally geodesic* $\mathbb{H}P^l \subset G_2^*(\mathbb{C}^{2l+2})$.

J. T. Cho and M. Kimura, Levi-umbilical real hypersurfaces in a complex complex two plane Grassmannians and its non-compact dual, preprint

The Grassmannian $G_2^*(\mathbb{C}^{m+2})$ has two types of singular tangent vectors X, namely of type $JX \in \mathfrak{J}X$ and $JX \perp \mathfrak{J}X$. All other tangent vectors are regular. This gives a corresponding concept of singular and regular points at infinity. For more details, we refer to [Berndt-Suh].

J. Berndt and Y.J. Suh, Hypersurfaces in noncompact complex Grassmannians of rank two, Internat. J. Math. 23(10) (2012) 1250103 (35 pages).

Problem II

Problem II.

Study and classify Levi-umbilical real hypersurfaces in Hermitian symmetric spaces.

Thank you for your attention !