Integrable Curves and Surfaces

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Metin Gürses

Bilkent University, Ankara, Turkey

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Surface theory in $\mathbb{R}^3$ plays a crucial role in differential geometry, partial differential equations (PDEs), string theory, general theory of relativity, and biology [Parthasarthy and Viswanathan, 2001] - [Ou-Yang et. al., 1999].
There are some special subclasses of 2-surfaces which arise in the branches of science.

- Minimal surfaces: $H = 0$,
- Surfaces with constant mean curvature : $H = constant$,
- Surfaces with constant positive Gaussian curvature: $K = constant > 0$,
- Surfaces with constant negative Gaussian curvature: $K = constant < 0$,
- Surfaces with harmonic inverse mean curvature: $\nabla^2(1/H) = 0$,

- Bianchi surfaces: $\nabla^2(1/\sqrt{K}) = 0$ and $\nabla^2(1/\sqrt{-K}) = 0$, for positive Gaussian curvature and negative Gaussian curvature, respectively,
- Weingarten surfaces: $f(H, K) = 0$. For example; linear Weingarten surfaces, $c_1\, H + c_2\, K = c_3$, and quadratic Weingarten surfaces, $c_4\, H^2 + c_5\, H\, K + c_6\, K^2 + c_7\, H + c_8\, K = c_9$, where c_j are constants, $j = 1, 2, ..., 9$,
- Willmore surfaces: $\nabla^2 H + 2\, H(H^2 - K) = 0$,
- Surfaces that solve the shape equation of lipid membrane: $p - 2\,\omega H + k_c \nabla^2(2H) + k_c(2H + c_0)(2H^2 - c_0 H - 2K) = 0$, where p, ω, k_c, and c_0 are constants.

Soliton equations play a very practical role for the construction of surfaces.

The theory of nonlinear soliton equations was developed in 1960s.

Lax representation of nonlinear PDEs consists of two linear equations which are called Lax equations

$$\Phi_x = U\,\Phi, \quad \Phi_t = V\,\Phi, \tag{1}$$

and their compatibility condition

$$U_t - V_x + [U, V] = 0, \tag{2}$$

where x and t are independent variables. Here U and V are called Lax pairs.

The relation of 2-surfaces and integrable equations is established by the use of Lie groups and Lie algebras.

Using this relation, soliton surface theory was first developed by Sym [Sym, 1982]-[Sym, 1985]. He obtained the immersion function by using the deformation of Lax equations for integrable equations.

$$F = \Phi^{-1} \frac{\partial \Phi}{\partial \lambda}, \tag{3}$$

Fokas and Gel'fand [Fokas and Gelfand, 1996] generalized Sym's result and find more general immersion function.

$$\begin{aligned} F = {} & \alpha_1 \Phi^{-1} U \, \Phi + \alpha_2 \Phi^{-1} V \, \Phi + \alpha_3 \Phi^{-1} \frac{\partial \Phi}{\partial \lambda} + \alpha_4 \, x \, \Phi^{-1} U \, \Phi \\ & + \alpha_5 \, t \, \Phi^{-1} V \, \Phi + \Phi^{-1} M \, \Phi, \end{aligned} \tag{4}$$

where α_i, $i = 1, 2, 3, 4, 5$ and $M \in \mathfrak{g}$ are constants.

On the other hand, there are some surfaces that arise from a variational principle for a given Lagrange function. Examples of this type are

- Minimal surfaces
- Constant mean curvature surfaces
- Linear Weingarten surfaces
- Willmore surfaces
- Surfaces solving the shape equation

Taking more general Lagrange function of the mean and Gaussian curvatures of the surface, we may find more general surfaces that solve the generalized shape equation (see [Tu and Ou-Yang, 2004]-[Tu, 2001]) . Examples for this type of surfaces can be found in [Gürses, 2002] - [Gürses and Tek, 2014].

Examples of some of these surfaces such as Bianchi surfaces [Bobenko, 1990] and Willmore surfaces [Willmore, 1982], [Willmore, 2000] are very rare.

The main reason is the difficulty of solving corresponding differential equations.

For this purpose, some indirect methods [Sym, 1982]-[Cieslinski, 1997] have been developed for the construction of two surfaces in $\mathbb{R}^3$ and in M_3.

Among all different methods, *soliton surface technique* is a very effective method.
In this method, one mainly uses the deformations of the Lax equations of the integrable equations
[Sym, 1982],[Fokas and Gelfand, 1996],[Ceyhan et. al., 2000],[Fokas et. al.

- Sine Gordon (SG) equation
- Korteweg de Vries (KdV) equation
- Modified Korteweg de Vries (mKdV) equation
- Nonlinear Schrödinger (NLS) equation

This talk contains a collection of the authors' works on surfaces and curves, in particular on soliton surfaces [Gürses, 1998],[Gürses, 1984],[Ceyhan et. al., 2000],[Gürses, 2002], [Tek, 2007], [Tek, 2007],[Tek, 2009],[Tek, 2015],[Gürses and Tek, 2014] [Gürses and Nutku, 1981].

Let $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ defines a triad at every point of curve and forms a base at that point. Here $\mathbf{t}, \mathbf{n}$, and $\mathbf{b}$ denote tangent, normal, and binormal vectors, respectively (Frenet Frame).

Let $\langle , \rangle$ be the standard inner product in $\mathbb{R}^3$. The orthonormal base $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ satisfy the following orthogonal conditions

$$\langle \mathbf{t}, \mathbf{t} \rangle = \langle \mathbf{n}, \mathbf{n} \rangle = \langle \mathbf{b}, \mathbf{b} \rangle = 1. \tag{5}$$

This triad is called *Serret-Frenet* (SF) triad and it's change with respect to t is defined by the following SF equations

$$\dot{\mathbf{t}} = k\mathbf{n}, \tag{6}$$

$$\dot{\mathbf{n}} = -k\mathbf{t} - \tau\mathbf{b}, \tag{7}$$

$$\dot{\mathbf{b}} = \tau\mathbf{n}. \tag{8}$$

Let E and Ω be defined, respectively, as

$$E = (\mathbf{t}, \mathbf{n}, \mathbf{b})^T, \tag{9}$$

$$\Omega = \begin{pmatrix} 0 & k & 0 \\ -k & 0 & -\tau \\ 0 & \tau & 0 \end{pmatrix}. \tag{10}$$

Here Ω is an antisymmetric and traceless matrix.

We can write the SF equations in terms of E and Ω as follows

$$\frac{dE}{dt} = \Omega E. \tag{11}$$

Instead of $\mathbb{R}^3$, if we take three dimensional Minkowski space ($\mathbb{M}_3$), SF equations will be different.

Let $\langle , \rangle$ be inner product in $\mathbb{M}_3$. The orthonormal base $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ satisfy the following orthogonal conditions

$$\langle \mathbf{t}, \mathbf{t} \rangle = 1, \; \langle \mathbf{n}, \mathbf{n} \rangle = -1, \; \langle \mathbf{b}, \mathbf{b} \rangle = -1. \tag{12}$$

With this orthogonal conditions SF equations take the following form

$$\dot{\mathbf{t}} = k\mathbf{n}, \tag{13}$$

$$\dot{\mathbf{n}} = k\mathbf{t} - \tau\mathbf{b}, \tag{14}$$

$$\dot{\mathbf{b}} = \tau\mathbf{n}. \tag{15}$$

For the relation between soliton equations and SF equations in different three dimensional geometries ($\mathbb{R}^3$ or $\mathbb{M}_3$) with different signature see [Gürses, 1998].

Let's define a surface in $\mathbb{R}^3$ as a map $\mathbf{Y} : \mathcal{O} \to \mathbb{R}^3$, where $\mathcal{O}$ is an open set in $\mathbb{R}^2$.

Position vector of the surface at every point is defined as $\mathbf{Y}(x,t) = (y^1(x,t), y^2(x,t), y^3(x,t))$, where $(x,t) \in \mathcal{O}$.

Define a triad $\{\mathbf{Y}_x, \mathbf{Y}_t, \mathbf{N}\}$ at every point of surface and forms a basis for $\mathbb{R}^3$ at these points.

Here $\mathbf{Y}_{,x}$ and $\mathbf{Y}_{,t}$ are the tangent vectors of the surface, $\mathbf{N}$ is a unit normal vector. For the smooth surfaces $\mathbf{N}$ is given as

$$\mathbf{N} = \frac{\mathbf{Y}_{,x} \times \mathbf{Y}_{,t}}{|\mathbf{Y}_{,x} \times \mathbf{Y}_{,t}|}. \tag{16}$$

The equations which gives the change of this triad is called Gauss-Wengarten (GW) equations and they are given as

$$\mathbf{Y}_{,ij} = \Gamma^k_{ij}\,\mathbf{Y}_{,k} + h_{ij}\,\mathbf{N}, \tag{17}$$

$$\mathbf{N}_{,i} = -g^{kl}h_{li}\,\mathbf{Y}_{,k}, \tag{18}$$

g_{ij} and h_{ij} denote the coefficients of the first and second fundamental forms, respectively.

We can find the fundamental forms as

$$g_{ij} = \langle \mathbf{Y}_{,i}, \mathbf{Y}_{,j} \rangle, \tag{19}$$

$$h_{ij} = \langle \mathbf{N}, \mathbf{Y}_{,ij} \rangle = -\langle \mathbf{N}_{,i}, \mathbf{Y}_{,j} \rangle \tag{20}$$

The Christoffel symbol Γ^i_{jk} is defined as

$$\Gamma^i_{jk} = \frac{1}{2}g^{il}[g_{lj,k} + g_{lk,j} - g_{jk,l}]. \tag{21}$$

Compatibility conditions of GW equations given in Eqs. (17) and (18) yields Gauss-Codazzi (GC) equations

$$R^i_{jkl} = g^{im}(h_{mk}h_{jl} - h_{ml}h_{jl}), \tag{22}$$

$$h_{ij,k} - \Gamma^m_{ik}h_{mj} = h_{ik,j} - \Gamma^m_{ij}h_{mk}, \tag{23}$$

The Gaussian (K) and mean (H) curvatures of a surface in $\mathbb{R}^3$ are given as

$$K = \det(g^{-1}h), \;\; H = \frac{1}{2}\mathrm{trace}(g^{-1}h) \tag{24}$$

Now we give the following proposition locally.

Proposition

Let $\mathbf{Y}_{,x}(x,t)$ and $\mathbf{Y}_{,t}(x,t)$ be independent differentiable vectors in $\mathbb{R}^3$. If $\mathbf{Y}_{,xt} = \mathbf{Y}_{,tx}$, then there exist a unique surface that accept these vectors as its tangent vectors at every point of it. (It is unique except isometric ones.)

Example

If we consider the first fundamental form as

$$ds_I^2 = \sin^2\theta dx^2 + \cos^2\theta dt^2, \tag{25}$$

Gaussian curvature satisfy the following equation

$$\theta_{xx} - \theta_{tt} = \frac{1}{2} K \sin(2\theta). \tag{26}$$

If we take the first fundamental form as

$$ds_I^2 = du^2 + dv^2 - 2\cos\theta du\, dv, \tag{27}$$

Gaussian curvature satisfy the following equation

$$\theta_{uv} = K \sin\theta. \tag{28}$$

We shall be interested in curves where their curvature k and torsion τ satisfies certain coupled nonlinear PDEs.

From the fundamental theorem of the local theory of curves there exists up to isometries, a unique curve for given the functions k and τ.

Hence every distinct solution of PDE satisfied by k and τ define a unique, up to isometries, a unique curve in $\mathbb{R}^3$ or in M_3.

For the plane curves, given the curvature function k , up to isometries, we can determine the corresponding curve uniquely. As an example let

$$k = 1/\cosh^2 s$$

then the corresponding curve is a catenary

$$\alpha = (s, \cosh s)$$

.

SF equations defines how the SF triad $E = (\mathbf{t}, \mathbf{n}, \mathbf{b})^T$ defined at every point of the curve moves along the curve.

If the curve moves on a surface (S) at the same time, we should be able to write how it changes in the direction of the movement.

Let the surface be parameterized as $(s, t) \in \mathcal{O} \to S$, such that s is arc length parameter and t is the second parameter of the surface S.

The motion of curve is defined as the derivative of the SF triad with respect to variable t as

$$\frac{dE}{dt} = \Gamma\, E, \tag{29}$$

Here Γ is a traceless 3×3 matrix. The entries of this matrix is not free. As we mentioned earlier, SF equations can be written in the following form

$$\frac{dE}{ds} = \Omega E \tag{30}$$

where Ω is given by Eq. (10).

The compatibility condition of SF equations and the equation [(29)] defines the t change gives the following equation

$$\Omega_t - \Gamma_s + \Omega\,\Gamma - \Gamma\,\Omega = 0. \tag{31}$$

Using this equation we can find the entries of Γ in terms of the curvature k and the torsion τ.

For the plane curves, the change of the position vector α with respect to s and $t$are given by

$$\frac{d\alpha}{ds} = \mathbf{t}, \tag{32}$$

$$\frac{d\alpha}{dt} = p\mathbf{n} + w\mathbf{t}, \tag{33}$$

where p and w are some functions of s and t.

The compatibility conditions of the equations [Eqs. (32) and (33)] results the following equations

$$w_s = k\,p, \tag{34}$$

$$\mathbf{t}_t = (p_s + kw)\mathbf{n}. \tag{35}$$

Eq. (35) gives the t change of the tangent vector.

If we look the compatibility conditions of this equation with SF equation $d\mathbf{t}/ds = k\,\mathbf{n}$, we obtain the following equations

$$k_t = (p_s + w\,k)_s, \tag{36}$$

$$\mathbf{n}_t = -(p_s + kw)\mathbf{t}. \tag{37}$$

s derivative of the normal vector from SF equation is given as

$$d\mathbf{n}/ds = -k\mathbf{t}. \tag{38}$$

Hence the entries of the Γ matrix are found.

If we substitute w given by Eq. (34) into the equation [Eq.(36)], we obtain the following equation

$$k_t = D^2 p + k^2 p + k_s \int kp\, ds, \tag{39}$$

where $D = \partial / \partial s$.

This equation reminds us the recursion operator $\mathcal{R}$ of the mKdV equation.

Eq. (39) can take the appropriate form with a simple calculation.

$$k_t = \mathcal{R}\, p, \;\; \mathcal{R} = D^2 + k^2 + k_s D^{-1} k, \tag{40}$$

where $\mathcal{R}$ is the recursion operator of mKdV equation and D^{-1} is the integral operator

For example, if we take $p = k_s$, then k satisfies the mKdV equation

$$k_t = k_{sss} + \frac{3}{2} k^2 k_s. \tag{41}$$

In general, $p(s,t)$ is a free function. When we take $p = \mathcal{R}^n k_s$, Eq. (40) gives the mKdV hierarchy, where n is positive integer.

Every solution of the mKdV equation, especially soliton solutions, give different curves in the plane.

As far as we know, this side of the problem hasn't been worked that much.

Another point that we should mention is the arbitrary choice of the function p results different curves whereas the equation satisfied by k does not have to be integrable.

For example, if we choose $p = k k_s$, the equation we obtain from Eq. (40) is not integrable.

In $\mathbb{R}^3$, there will be a separate equation for the torsion function τ.

Following the similar approach, a coupled nonlinear PDEs are obtained for k and τ. For more details see [Gürses, 1998] .

Minkowski plane curves are also studied in that article.

In this way, some new equations are obtained those couldn't be obtained from the plane curves in $\mathbb{R}^3$.

As an example, let's consider a three dimensional general space with the signature $1+2\epsilon$ (ϵ is 1 and -1 for $\mathbb{R}^3$ and $\mathbb{M}_3$, respectively).

We can find mKdV equation with both signature on plane curves as $k_t = k_{sss} + (3/2)\epsilon k^2\, k_s$ using the method described above.

Additionally for obtaining the NLS equation one can look [Hasimoto, 1972] and [Lamb, 1977].

Hasimoto is trying to find relationship between tornado in nature and solutions of the NLS equation.

If a given PDE satisfy one of the following is called *integrable*;

i) It has a Lax representation,
ii) It has Painleve property,
iii) It has zero curvature representation,
iv) It has Bäcklund transformation,
v) There exist infinitely many conserved quantities,
vi) It has a recursion operator

Lax equations have different formulations depending on the algebra of the Lax operator belongs to.

For example, Lax operator can be in pseudo-differential operator algebra, polynomial algebra, or matrix algebra.

We will consider the Lax representation in the matrix algebra. Since we will work with single PDE, 2×2 matrices will be enough.

Definition

(Lax Equations) Let $\Phi(x,t,\lambda)$ be $SU(2)$ valued function such that $(x,t) \in \mathcal{O} \subset \mathbb{R}^2$, and $\lambda \in \mathbb{C}$ is spectral parameter. Lax equations are defined as

$$\Phi_x = U\Phi, \;\; \Phi_t = V\Phi, \tag{42}$$

where $U(x,t,\lambda)$ and $V(x,t,\lambda)$ are $\mathfrak{su}(2)$ valued functions and they satisfy the following equation

$$U_t - V_x + [U,V] = 0. \tag{43}$$

Eq. (43) is the compatibility condition of the Eq. (42). U and V are called as Lax pairs.

Example

(Sine-Gordon Equation) If we consider the following Lax pairs U and V

$$U = \frac{\mathrm{i}}{2}(-u_x\sigma_1 + \lambda\sigma_3),\; V = \frac{\mathrm{i}}{2\lambda}(\sin(u)\sigma_2 - \cos(u)\sigma_3), \tag{44}$$

then the function $u(x,t)$ satisfy the sine-Gordon equation

$$u_{xt} = \sin(u), \tag{45}$$

where λ is the spectral parameter and σ_k denote the standard Pauli sigma matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\; \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},\; \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{46}$$

Example

(mKdV Equation) The Lax pairs for mKdV equation is given as

$$U = \frac{\mathrm{i}}{2}(\lambda\sigma_3 - u\sigma_1), \; V = -\frac{\mathrm{i}}{2}([\lambda^3 - \frac{1}{2}\lambda u^2]\sigma_3 + v_1\sigma_1 + v_2\sigma_2), \qquad (47)$$

where $v_1 = u_{xx} + u^3/2 - \lambda^2 u, \;\; v_2 = -\lambda u_x$. Here the function $u(x,t)$ satisfy the mKdV equation

$$u_t = u_{xxx} + \frac{3}{2}u^2 u_x. \qquad (48)$$

Soliton surface technique is a method to construct 2-surfaces in $\mathbb{R}^3$ and in M_3.

In the literature, there are certain surfaces corresponding to certain integrable equations such as SG, sinh-Gordon, KdV, mKdV, and NLS equations [Bobenko, 1994], [Bobenko, 1990], [Melko and Sterling, 1994], [Sym, 1982],[Fokas and Gelfand, 1996],[Ceyhan et. al., 2000],[Fokas et. al. ,[Gürses and Tek, 2014].

Symmetries of the integrable equations for given Lax pairs play an essential role in this method which was first started by Sym [Sym, 1982]-[Sym, 1985] and then it was generalized by Fokas and Gel'fand [Fokas and Gelfand, 1996], Fokas *at al.* [Fokas et. al., 2000], [Ceyhan et. al., 2000] and Cieśliński [Cieslinski, 1997].

Now by considering surfaces in a Lie group and in the corresponding Lie algebra, we give the general theory.

Let G be a Lie group and $\mathfrak{g}$ be the corresponding Lie algebra.

We give the theory for dim $\mathfrak{g} = 3$, it is possible to generalize it for finite dimension n.

Assume that there exists an inner product $\langle , \rangle$ on $\mathfrak{g}$ such that for g_1, $g_2 \in \mathfrak{g}$ as $\langle g_1, g_2 \rangle$.

Let $\{\mathrm{e}_1, \mathrm{e}_2, \mathrm{e}_3\}$ be the orthonormal basis in $\mathfrak{g}$ such that $\langle \mathrm{e}_i, \mathrm{e}_j \rangle = \delta_{ij}$ $(i, j = 1, 2, 3)$, where δ_{ij} is the Kronecker delta.

Theorem

Let U, V, A, and B be $\mathfrak{g}$ valued differentiable functions of x, t, and λ for every $(x,t) \in \mathcal{O} \subset \mathbb{R}^2$ and $\lambda \in \mathbb{R}$. Assume that U, V, A, and B satisfy the following equations

$$U_t - V_x + [U,V] = 0, \tag{49}$$

and

$$A_t - B_x + [A,V] + [U,B] = 0. \tag{50}$$

Then the following equations

$$\Phi_x = U\,\Phi, \quad \Phi_t = V\,\Phi, \tag{51}$$

and

$$F_x = \Phi^{-1}\,A\,\Phi, \quad F_t = \Phi^{-1}\,B\,\Phi, \tag{52}$$

define surfaces $\Phi \in G$ and $F \in \mathfrak{g}$, respectively.

Then we have the following theorem

Theorem

The first and second fundamental forms are

$$(ds_I)^2 \equiv g_{ij} dx^i\, dx^j, \quad (ds_{II})^2 \equiv h_{ij} dx^i\, dx^j \tag{53}$$

where $i, j = 1, 2$, $x^1 = x$, $x^2 = t$, g_{ij} *and* h_{ij} *are given as*

$$g_{11} = \langle A, A \rangle,\ g_{12} = g_{21} = \langle A, B \rangle,\ g_{22} = \langle B, B \rangle, \tag{54}$$

$$h_{11} = \langle A_x + [A, U], C \rangle,\ h_{12} = h_{21} = \langle A_t + [A, V], C \rangle, \tag{55}$$

$$h_{22} = \langle B_t + [B, V], C \rangle,$$

$$C = [A, B]/\|[A, B]\|,\ \|A\| = \sqrt{|\langle A, A \rangle|}. \tag{56}$$

The Gaussian and mean curvatures of the surface are

$$K = det(g^{-1})h, \quad H = \frac{1}{2} trace(g^{-1}h). \tag{57}$$

The matrices A and B relates the differential equations and surfaces by equations in Eq. (52).

In general, solving Eq. (50) to find A and B and expressing the position vector are difficult.

In order to overcome this difficulty we define an operator operator δ.

Definition

Let δ be an operator acts on differentiable functions and satisfy the following conditions

$$\delta\partial_x = \partial_x\delta,\ \ \delta\partial_t = \partial_t\delta, \tag{58}$$

$$\delta(f\,g) = g\delta(f) + f\delta(g) \tag{59}$$

$$\delta(af + bg) = a\delta(f) + b\delta(g) \tag{60}$$

Here f and g are differentiable functions, a and b are constant. We call such operators as deformation operators.

The following proposition gives a solution for finding A and B matrices.

Proposition

Let Φ, U, and V are the matrices satisfying equations in Eqs. (49) and (51), A and B defines as $A = \delta U$ and $B = \delta V$, respectively. Equation for A and B in Eq. (50) is automatically satisfied and we have the following equations

$$(\Phi^{-1}\delta\Phi)_x = \Phi^{-1}A\Phi, \tag{61}$$

$$(\Phi^{-1}\delta\Phi)_t = \Phi^{-1}B\Phi. \tag{62}$$

In the following proposition, we give the relation that directly connects deformation operators and surfaces.

Proposition

Let F be $\mathfrak{g}$ valued position vector. The position vector F and its partial derivatives are given as

$$F = \Phi^{-1}\delta\Phi, \tag{63}$$

$$F_x = \Phi^{-1}A\Phi, \tag{64}$$

$$F_t = \Phi^{-1}B\Phi, \tag{65}$$

Now finding deformation operators in soliton theory and hence determining the matrices A and B becomes an important step.

The following proposition answers the question how to find A and B without solving the equation in constructing the surfaces.

Proposition

The followings are the deformation operators of soliton equations:

a) *Nonlinear integrable equations are invariant under spectral parameter deformation. In this case, the deformation operator is* $\delta = \partial/\partial\lambda$*. Hence* A *and* B *matrices are given as*

$$A = \frac{\partial U}{\partial \lambda}, \ B = \frac{\partial V}{\partial \lambda}, \tag{66}$$

and position vector of the surface and its derivatives take the following forms

$$F = \Phi^{-1} \frac{\partial \Phi}{\partial \lambda}, F_x = \Phi^{-1} \frac{\partial U}{\partial \lambda} \Phi, F_t = \Phi^{-1} \frac{\partial V}{\partial \lambda} \Phi. \tag{67}$$

This type of relation first studied by Sym [Sym, 1982]-[Sym, 1985].

Proposition

b) *Under Gauge transformation Φ, U, and V change as*

$$\Phi' = S\Phi, U' = SUS^{-1} + S_x S^{-1}, V' = SVS^{-1} + S_t S^{-1}, \tag{68}$$

These Gauge transformations define a new δ operator. If we let $S = I + \epsilon M$ such that $\epsilon^2 = 0$, then we get $\delta\Phi = M\Phi$. M is any traceless 2×2 matrix. The matrices A and B are obtained by

$$A = \delta U = \frac{\partial M}{\partial x} + [M, U], B = \delta V = \frac{\partial M}{\partial t} + [M, U], \tag{69}$$

and the position vector of the surface is given as

$$F = \Phi^{-1} M \Phi. \tag{70}$$

For more information can be found in [Fokas and Gelfand, 1996] - [Ceyhan et. al., 2000], [Cieslinski, 1997].

Proposition

c) *Symmetries of the nonlinear integrable equations are another type of deformation. These are two types. First one is classical Lie symmetries which preserve the differential equation. The second is the generalized symmetries of nonlinear integrable equations. The latter transformation maps solutions to solutions. Deformation operator for these symmetries is taken as Freche't derivative (see [Fokas and Gelfand, 1996] and [Ceyhan et. al., 2000]). In other words, for a differentiable function* F*,* $\delta F(x)$ *is defines as*

$$\delta F(x) = \lim_{\epsilon \to 0} \frac{df(x+\epsilon t)}{d\epsilon}. \tag{71}$$

For this deformation, the matrices A *and* B*, and the position vector of the surface take the following form*

$$A = \delta U, B = \delta V, F = \Phi^{-1}\delta\Phi. \tag{72}$$

Proposition

d) *The deformation of parameters for solution of integrable equation is the fourth deformation. This is introduced by [Gürses and Tek, 2015]. In this case, A, B, and F are obtained as*

$$A_i = \frac{\partial U}{\partial \xi_i}, \tag{73}$$

$$B_i = \frac{\partial V}{\partial \xi_i}, \tag{74}$$

$$F_i = \Phi^{-1}\frac{\partial \Phi}{\partial \xi_i}, \quad i = 1, 2, \ldots, N \tag{75}$$

Here ξ_i are parameters of solution $u(x, t, \xi_i)$ of integrable nonlinear equations, where $i = 1, 2, \ldots, N$. Here N is the number of parameters.

Now we give two proposition about the surface of sphere.

Proposition

For any differential equation, if the determinant of the matrix M, given in Proposition 7.6 **b)**, *is constant, i.e.* $\det M = R^2 =$ *constant, the corresponding surface is a sphere with radius R.*

Some of the differential equations have transitional symmetry in either x direction or t direction (or in both direction).

In this case the deformation operator can be considered as $\delta = \partial_x$ or $\delta = \partial_t$.

Here we consider the transition in both directions such as $\delta = a\partial_x + b\partial_t$, where a and b are arbitrary constants.

Proposition

If $\delta = a\partial_x + b\partial_t$ is the symmetry operator such that a and b are free parameters and $\det(aU + bV) = constant = R^2$, the corresponding surface is a surface of sphere with radius R.

If $\delta = a\partial_x + b\partial_t$, using Eq. (72) in Proposition 7.6 and Eq. (51), we obtain F as

$$F = \Phi^{-1}(aU + bV)\Phi, \tag{76}$$

which yields

$$\det F = \det(aU + bV) = R^2. \tag{77}$$

In 1833, Poisson considered the free energy of a solid shell as

$$\mathcal{F} = \oint\!\!\!\oint_S H^2 dA. \tag{78}$$

Here S is a smooth closed surface, A and H denote the surface area and mean curvature of the surface S.

Proposition

Let S be a smooth closed surface, and K and H be Gaussian and mean curvatures of the surface, respectively. Variation of functional $\mathcal{F}$ in Eq. (78) gives the following Euler-Lagrange equation [Willmore, 1982]

$$\nabla^2 H + 2H(H^2 - K) = 0. \tag{79}$$

Here ∇^2 is the Laplace-Beltrami operator defined as

$$\nabla^2 = \frac{1}{\sqrt{\tilde{g}}} \frac{\partial}{\partial x^i} \left(\sqrt{\tilde{g}} g^{ij} \frac{\partial}{\partial x^j} \right), \tag{80}$$

where $\tilde{g} = \det(g_{ij})$, g^{ij} is the inverse components of the first fundamental form, and $i, j = 1, 2$, where $x^1 = x$, $x^2 = t$. Solutions of Eq. (79) are called Willmore surfaces.

Helfrich [Helfrich, 1973] obtained the curvature energy per unit area of the bilayer as

$$\mathcal{E}_{lb} = (k_c/2)\,(2H + c_0)^2 + \bar{k}K, \tag{81}$$

where k_c and $\bar{k}$ are elastic constants, and c_0 is spontaneous curvature of the lipid bilayer. Using the Helfrich curvature energy Eq. (81), the free energy functional of the lipid vesicle is written as

$$\mathcal{F} = \oiint_S (\mathcal{E}_{lb} + \omega) dA + p \iiint_V dV. \tag{82}$$

Ou-Yang and Helfrich [Ou-Yang and Helfrich, 1987] obtained shape equation of the bilayer by taking the first variation of free energy $\mathcal{F}$ in Eq. (82). We give this result in the following proposition.

Proposition

Let S be a smooth surface of lipid vesicle, V be the volume enclosed by the surface, p and ω be osmotic pressure and surface tension of the vesicle, respectively. First order variation of the functional in Eq. (82) *yields the following Euler-Lagrange equation [Ou-Yang and Helfrich, 1987]*

$$p - 2\omega H + k_c \nabla^2 (2H) + k_c (2H + c_0)(2H^2 - c_0 H - 2K) = 0. \qquad (83)$$

Later Ou-Yang *et al.* considered the more general energy functional

$$\mathcal{F} = \oiint_S \mathcal{E}(H, K) dA + p \iiint_V dV. \qquad (84)$$

[Ou-Yang et. al., 1999], [Tu and Ou-Yang, 2004]-[Tu, 2001]. Here $\mathcal{E}$ is function of mean and Gaussian curvature H and K, respectively, p is a constant, and V is the volume enclosed within the surface S.

Proposition

Let S be a closed smooth surface. The first variation of $\mathcal{F}$ given in Eq. (84) *results a highly nonlinear Euler-Lagrange equation as [Ou-Yang et. al., 1999], [Tu and Ou-Yang, 2004]-[Tu and Ou-Yang, 2005]*

$$(\nabla^2 + 4H^2 - 2K)\frac{\partial \mathcal{E}}{\partial H} + 2(\nabla \cdot \bar{\nabla} + 2KH)\frac{\partial \mathcal{E}}{\partial K} - 4H\mathcal{E} + 2p = 0, \quad (85)$$

where ∇^2 *is the Laplace-Beltrami operator given in Eq.* (80) *and* $\nabla \cdot \bar{\nabla}$ *is defined as*

$$\nabla \cdot \bar{\nabla} = \frac{1}{\sqrt{\tilde{g}}}\frac{\partial}{\partial x^i}\left(\sqrt{\tilde{g}}Kh^{ij}\frac{\partial}{\partial x^j}\right). \quad (86)$$

For open surfaces, we let $p = 0$*.*

Some of the surfaces can be obtained from a variational principle for a suitable choice of $\mathcal{E}$ are given as:

(a) Minimal surfaces: $\mathcal{E} = 1, \quad p = 0$;

(b) Surfaces with constant mean curvature: $\mathcal{E} = 1$;

(c) Linear Weingarten surfaces: $\mathcal{E} = aH + b$, where a and b are some constants, $aK + 2bH - p = 0$;

(d) Willmore surfaces: $\mathcal{E} = H^2$ [Willmore, 1982], [Willmore, 2000];

(e) Surfaces that solve the shape equation of lipid membrane: $\mathcal{E} = (H - c)^2$, where c is a constant [Ou-Yang et. al., 1999], [Tu and Ou-Yang, 2004]-[Mladenov, 2002];

(f) Shape equation of closed lipid bilayer: $\mathcal{E} = (k_c/2)\,(2H + c_0)^2 + \bar{k}K$, where k_c and $\bar{k}$ are elastic constants, and c_0 is the spontaneous curvature of the lipid bilayer [Ou-Yang and Helfrich, 1987].

Definition

Surfaces that solve the following equation

$$\nabla^2 H + aH^3 + bH\,K = 0, \tag{87}$$

are called Willmore-like surfaces, where a *and* b *are arbitrary constants.*

Remark

When $a = 2$ *and* $b = -2$*, the surface becomes Willmore surface which arise from a variational problem.*

In this section, we obtain surfaces in $\mathbb{R}^3$ using soliton surface technique and variational principle.

Consider the immersion F of $\mathcal{U} \in \mathbb{R}^2$ into $\mathbb{R}^3$.

Let's denote the tangent space by $T_{(x,t)}S$ of the surface S. A basis for the $T_{(x,t)}S$ can be defined as $\{F_x, F_t, N\}$.

Here S is a surface parameterized by $F(x,t)$.

Let's denote the first and second fundamental forms, respectively, as

$$ds_I^2 \equiv g_{ij} dx^i dx^j, \text{ and } ds_{II}^2 \equiv h_{ij} dx^i dx^j \tag{88}$$

where $i, j = 1, 2$, and $x^1 = x$, $x^2 = t$.

We use Lie group and its Lie algebra to develop surfaces using integrable equations.

To study the immersions in $\mathbb{R}^3$, we use SU(2) as a Lie group and $\mathfrak{su}(2)$ as its corresponding Lie algebra.

Consider $e_k = -\mathrm{i}\sigma_k$, $k = 1, 2, 3$ as a basis for the Lie algebra $\mathfrak{su}(2)$. Here σ_k denotes the standard Pauli sigma matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{89}$$

Consider the following inner product defined on Lie algebra $\mathfrak{su}(2)$

$$\langle X, Y \rangle = -\frac{1}{2}\,\mathrm{trace}(XY), \tag{90}$$

where X, $Y \in \mathfrak{su}(2)$ and $[.,.]$ denotes the usual commutator.

We follow Fokas and Gelfand's approach introduced in Section 7 to construct surfaces using integrable equations such as mKdV, SG, and NLS equations.

- We start with $\mathfrak{su}(2)$ valued Lax pairs U and V of these integrable equations.
- We use the deformations that we introduced in Section 7 to find the matrices $A = \delta U$ and $B = \delta V$ that satisfy Eq. (50).
- Using the matrices U, V, A and B we find the first and second fundamental forms g and h of the surfaces corresponding to mKdV, SG, and NLS equations.
- We also find Gaussian (K) and mean (H) curvatures of these surfaces using first and second fundamental forms.

$$K = det(g^{-1})h, \quad H = \frac{1}{2} trace(g^{-1}h)$$

- Finding K and H allows us to classify some of these surfaces (Weingarten, Willmore , etc.,)

- Furthermore, in order to find the position vector $F = \Phi^{-1}\,\delta\,\Phi$ explicitly, we solve the Lax equations of the integrable equation using the Lax pairs U and V, and a solution (in particular soliton solutions) of the integrable equation that we consider.
- Considering some special values of the parameters in the position vectors, we plot some of these surfaces that we obtained using integrable equations. We also obtain some Willmore-like surfaces and surfaces that satisfy generalized shape equation.

In this section we use spectral parameter deformation of the Lax pairs of mKdV equation. In this section we closely follow the references [Ceyhan et. al., 2000] and [Tek, 2007].

Let u satisfy the mKdV equation

$$u_t = u_{xxx} + \frac{3}{2}u^2 u_x. \tag{91}$$

When we use the travelling wave ansatz $u_t - \alpha u_x = 0$ in mKdV equation [Eq. (91)], we obtain

$$u_{xx} = \alpha u - \frac{u^3}{2}. \tag{92}$$

The Lax pairs for the mKdV equation in Eq. (92) are given as

$$U = \frac{i}{2}\begin{pmatrix} \lambda & -u \\ -u & -\lambda \end{pmatrix}, \tag{93}$$

$$V = -\frac{i}{2}\begin{pmatrix} \frac{1}{2}u^2-(\alpha+\alpha\lambda+\lambda^2) & (\alpha+\lambda)u-iu_x \\ (\alpha+\lambda)u+iu_x & -\frac{1}{2}u^2+(\alpha+\alpha\lambda+\lambda^2) \end{pmatrix}, \tag{94}$$

and λ is a spectral parameter.

In the following proposition, using the Lax pairs of mKdV equation and their spectral parameter deformation we obtain the surfaces for mKdV equation.

Proposition

Let u be a travelling wave solution of the mKdV equation given in Eq. (92) and $\mathfrak{su}(2)$ valued Lax pairs U and V are defined by Eqs. (93) and (94), respectively. The matrices A and B are defined as spectral parameter deformations of the Lax pairs U and V, respectively, as

$$A = \mu \frac{\partial U}{\partial \lambda} = \frac{i}{2} \begin{pmatrix} \mu & 0 \\ 0 & -\mu \end{pmatrix}, \tag{95}$$

$$B = \mu \frac{\partial V}{\partial \lambda} = -\frac{i}{2} \begin{pmatrix} -(\alpha\,\mu + 2\,\mu\,\lambda) & \mu\,u \\ \mu\,u & \alpha\,\mu + 2\,\mu\,\lambda \end{pmatrix}, \tag{96}$$

where μ is a constant and λ is the spectral parameter.

Proposition

First and second fundamental forms of the surface S are given as

$$(ds_I)^2 \equiv g_{jk}\, dx^j\, dx^k = \frac{\mu^2}{4}\Big([dx + (\alpha + 2\,\lambda)dt]^2 + u^2\, dt^2\Big), \quad (97)$$

$$(ds_{II})^2 \equiv h_{jk}\, dx^j\, dx^k = \frac{\mu\, u}{2}\big(dx + (\alpha + \lambda)dt\big)^2 + \frac{\mu\, u}{4}(u^2 - 2\,\alpha)dt^2, \quad (98)$$

and the other two important geometric invariants of the surface, namely Gaussian and mean curvatures are given as

$$K = \frac{2}{\mu^2}\big(u^2 - 2\,\alpha\big), \quad H = \frac{1}{2\mu\, u}\big(3\,u^2 + 2\,(\lambda^2 - \alpha)\big), \quad (99)$$

where $x^1 = x$, $x^2 = t$.

The following proposition gives surfaces belongs to Weingarten surfaces [Ceyhan et. al., 2000], [Tek, 2007].

Proposition

Let u be a travelling wave solution of the mKdV equation given in Eq. (92) and S be the surface obtained in Proposition 9.1. Then the surface S is a Weingarten surface that has the following algebraic relation between K and H

$$8\mu^2 H^2(4\alpha + \mu^2 K) = (8\alpha + 4\lambda^2 + 3\mu^2 K)^2. \tag{100}$$

When $\alpha = \lambda^2$ in Proposition 9.1, the surface reduces to a quadratic Weingarten surface which has the following relation

$$16\mu^2 H^2 = 18(\mu^2 K + 4\lambda^2). \tag{101}$$

Integrating the equation given in Eq. (92) and taking the integration constant zero, we obtain the following equation

$$u_x^2 = \alpha\, u^2 - \frac{u^4}{4}. \tag{102}$$

The following proposition gives another class of mKdV surfaces, namely Willmore-like surfaces [Tek, 2007].

Proposition

Let u satisfy Eq. (102) *and S be the surface obtained using spectral parameter deformation in Proposition 9.1. Then the surface S is called a Willmore-like surface. This means that the Gaussian and mean curvatures of the surface S satisfy the following equation*

$$\nabla^2 H + aH^3 + bH\,K = 0, \tag{103}$$

where

$$a = \frac{4}{9}, \quad b = 1, \quad \alpha = \lambda^2, \tag{104}$$

and λ is an arbitrary constant.

In the following proposition we investigate mKdV surfaces which arise from a variational principle.

Proposition

Let u satisfy Eq. (102) *and S be the surface obtained using spectral parameter deformation in Proposition 9.1. Then there are mKdV surfaces satisfying the generalized shape equation*

$$(\nabla^2 + 4H^2 - 2K)\frac{\partial \mathcal{E}}{\partial H} + 2(\nabla \cdot \bar{\nabla} + 2KH)\frac{\partial \mathcal{E}}{\partial K} - 4H\mathcal{E} + 2p = 0, \quad (105)$$

where Lagrange functions are polynomials of Gaussian and mean curvatures of the surface S.

Now we give some examples of polynomial Lagrange functions of H and K that solve the equation given in [Eq. (105)] and provide the constraints [Tek, 2007].

Example

i) for $N=3$:
$\mathcal{E} = a_1\, H^3 + a_2\, H^2 + a_3\, H + a_4 + a_5\, K + a_6\, K\, H,$

- $\alpha = \lambda^2, \;\; a_1 = -\dfrac{p\,\mu^4}{72\,\lambda^4}, \;\; a_2 = a_3 = a_4 = 0, \;\; a_6 = \dfrac{p\,\mu^4}{32\,\lambda^4},$
where $\lambda \neq 0$, and μ, p, and a_5 are arbitrary constants;

ii) for $N=4$:
$\mathcal{E} =$
$a_1\, H^4 + a_2\, H^3 + a_3\, H^2 + a_4\, H + a_5 + a_6\, K + a_7\, K\, H + a_8\, K^2 + a_9\, K\, H^2,$

- $\alpha = \lambda^2, \;\; a_2 = -\dfrac{p\,\mu^4}{72\,\lambda^4}, \;\; a_3 = -\dfrac{8\,\lambda^2}{15\,\mu^2}\,(27\,a_1 - 8\,a_8), \;\; a_4 = 0,$
- $a_5 = \dfrac{\lambda^4}{5\,\mu^4}\,(81\,a_1 + 16\,a_8), \;\; a_7 = \dfrac{p\,\mu^4}{32\,\lambda^4}, \;\; a_9 = -\dfrac{1}{120}\,(189\,a_1 + 64\,a_8),$
where $\lambda \neq 0$, $\mu \neq 0$, and p, a_1, a_6, and a_8 are arbitrary constants;

Example

iii) for $N = 5$:

$$\mathcal{E} = a_1\, H^5 + a_2\, H^4 + a_3\, H^3 + a_4\, H^2 + a_5\, H + a_6 + a_7\, K + a_8\, K\, H + a_9\, K^2 + a_{10}\, K\, H^2 + a_{11}\, K^2\, H + a_{12}\, K\, H^3,$$

iv) for $N = 6$:

$$\mathcal{E} = a_1\, H^6 + a_2\, H^5 + a_3\, H^4 + a_4\, H^3 + a_5\, H^2 + a_6\, H + a_7 + a_8\, K + a_9\, K\, H + a_{10}\, K^2 + a_{11}\, K\, H^2 + a_{12}\, K^2\, H + a_{13}\, K\, H^3 + a_{14}\, K^3 + a_{15}\, K^2\, H^2 + a_{16}\, K\, H^4,$$

For general $N \geq 3$, from the above examples, the polynomial function $\mathcal{E}$ takes the following form

$$\mathcal{E} = \sum_{n=0}^{N} H^n \sum_{l=0}^{\lfloor \frac{(N-n)}{2} \rfloor} a_{nl} K^l,$$

where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x, and a_{nl} are constants.

In the previous section, we obtained local invariants of the mKdV surfaces.

We also classified some of these surfaces such as Weingarten surfaces, Willmore-like surfaces and surfaces that solves generalized shape equation.

It is also important to determine the position vector of the mKdV surfaces.

We start with one soliton solution of mKdV equation given in Eq. (92).

Consider the following one soliton solution

$$u = k_1 \,\mathrm{sech}\xi, \tag{106}$$

where $\alpha = k_1^2/4$ in Eq. (92) and $\xi = k_1 \, \left(k_1^2 \, t + 4\, x\right)/8$.

Using this one soliton solution and corresponding matrix Lax pairs U and V given by Eqs. (93) and (94) of mKdV equation, we solve the Lax equations given in Eq. (51).

The solution of Lax equation is a 2×2 matrix Φ

$$\Phi = \begin{pmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{pmatrix}. \tag{107}$$

We find these components as

$$\Phi_{11} = -\frac{i}{k_1} A_1 \, \mathrm{e}^{i\left(k_1^2+4\lambda^2\right)t/8} \tag{108}$$
$$\cdot \left(2\,\lambda + i\,k_1\,\tanh\xi\right)\left(\tanh\xi+1\right)^{i\lambda/2k_1}\left(\tanh\xi-1\right)^{-i\lambda/2k_1}$$
$$+\, i\,k_1\,B_1\,\mathrm{e}^{-i\left(k_1^2+4\lambda^2\right)t/8}\left(\tanh\xi-1\right)^{i\lambda/2k_1}\left(\tanh\xi+1\right)^{-i\lambda/2k_1}\operatorname{sech}\xi,$$

$$\Phi_{12} = -\frac{i}{k_1} A_2 \, \mathrm{e}^{i\left(k_1^2+4\lambda^2\right)t/8} \tag{109}$$
$$\cdot \left(2\,\lambda + i\,k_1\,\tanh\xi\right)\left(\tanh\xi+1\right)^{i\lambda/2k_1}\left(\tanh\xi-1\right)^{-i\lambda/2k_1}$$
$$+\, i\,k_1\,B_2\,\mathrm{e}^{-i\left(k_1^2+4\lambda^2\right)t/8}\left(\tanh\xi-1\right)^{i\lambda/2k_1}\left(\tanh\xi+1\right)^{-i\lambda/2k_1}\operatorname{sech}\xi,$$

$$\Phi_{21} = i\,A_1\,\mathrm{e}^{i\left(k_1^2+4\lambda^2\right)t/8}\,(\tanh\xi+1)^{i\lambda/2k_1}\,(\tanh\xi-1)^{-i\lambda/2k_1}\,\mathrm{sech}\;\xi$$
$$+\,B_1\,\mathrm{e}^{-i\left(k_1^2+4\lambda^2\right)t/8}\,(k_1\tanh\xi+2i\lambda)\,(\tanh\xi-1)^{i\lambda/2k_1}\,(\tanh\xi+1)^{-i\lambda/2} \tag{110}$$

$$\Phi_{22} = i\,A_2\,\mathrm{e}^{i\left(k_1^2+4\lambda^2\right)t/8}\,(\tanh\xi+1)^{i\lambda/2k_1}\,(\tanh\xi-1)^{-i\lambda/2k_1}\,\mathrm{sech}\;\xi$$
$$+\,B_2\,\mathrm{e}^{-i\left(k_1^2+4\lambda^2\right)t/8}\,(k_1\tanh\xi+2i\lambda)\,(\tanh\xi-1)^{i\lambda/2k_1}\,(\tanh\xi+1)^{-i\lambda/2} \tag{111}$$

The determinant of the matrix Φ is given as

$$\det(\Phi) = [(k_1^2+4\lambda^2)/k_1]\,(A_1B_2-A_2B_1)\neq 0. \tag{112}$$

In order to find the immersion function F explicitly, we first find F_x and F_t given in the following form

$$F_x = \Phi^{-1} A \Phi, \; F_t = \Phi^{-1} B \Phi. \tag{113}$$

We solve the resultant equation by letting $A_1 = A_2$,
$B_1 = (A_1/k_1)\mathrm{e}^{\pi\lambda/k_1}$, $B_2 = -B_1$ and obtain the function F as

$$F = \mathrm{e}_1 y_1 + \mathrm{e}_2 y_2 + \mathrm{e}_3 y_3 \tag{114}$$

where y_1, y_2, and y_3 are given as

$$y_1 = \frac{1}{4\,k_1\,(e^{2\xi}+1)} W_1 \left(\Omega_1\,(e^{2\xi}+1) + 32 k_1\right), \tag{115}$$

$$y_2 = -4\,W_1\,\cos\Omega_2\,\mathrm{sech}\,\xi, \tag{116}$$

$$y_3 = 4\,W_1\,\sin\Omega_2\,\mathrm{sech}\,\xi. \tag{117}$$

Here e_1, e_2, e_3 form a basis for $\mathfrak{su}(2)$, Ω_1, Ω_2, and W_1 are given as

$$W_1 = -\frac{\mu\,k_1}{2\,(k_1^2 + 4\,\lambda^2)}, \tag{118}$$

$$\Omega_1 = \left(t\,[8\,\lambda + k_1^2] + 4\,x\right)\left(k_1^2 + 4\,\lambda^2\right), \tag{119}$$

$$\Omega_2 = t\left(\lambda^2 + \frac{1}{4}\,k_1^2[1+\lambda]\right) + x\,\lambda, \tag{120}$$

$$\xi = \frac{k_1^3}{8}(t + \frac{4x}{k_1^2}). \tag{121}$$

We obtained the position vector $\mathbf{Y} = (y_1, y_2, y_3)$ of the mKdV surfaces corresponds to the spectral parameter deformation in Eqs. (115) - (117).

We plot some of these mKdV surfaces for some special values of the constants μ, λ, and k_1.

Example: Taking $\mu = 5$, $k_1 = 1.5$, and changing λ as $a)\ \lambda = 1$, $b)\ \lambda = 1.3$, $c)\ \lambda = 1.6$, $d)\ \lambda = 2$, in Eqs. (115) - (117), we get

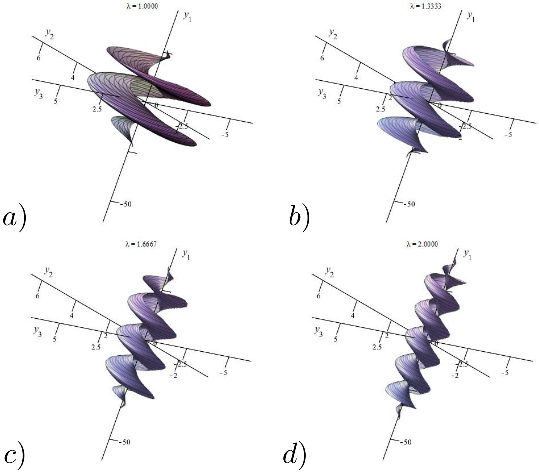


Figure: (a)-(d) $(x,t) \in [-3,3] \times [-3,3]$

Example: Taking $\mu = 2$, $\lambda = 0$, and $k_1 = 1.25$, in Eqs. (115) - (117), we get the surface given in Figure 2.

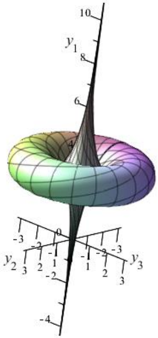


Figure: $(x, t) \in [-8, 8] \times [-8, 8]$

Example: Taking $\mu = 3$, $k_1 = -2$, and changing λ as $a)\ \lambda = 0.08$, $b)\ \lambda = 0.2$, $c)\ \lambda = 0.5$, $d)\ \lambda = 0.8$, in Eqs. (115) - (117), we get

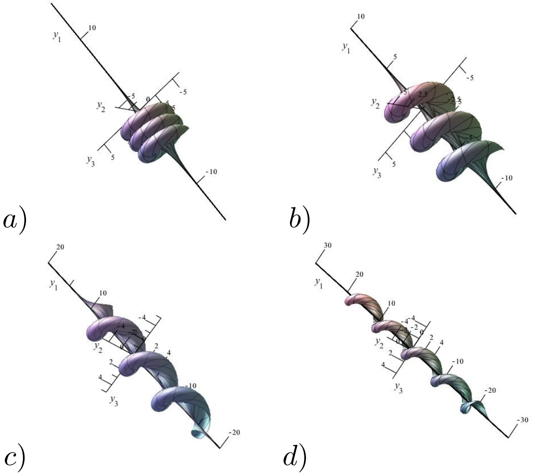


Figure: (a)-(d) $(x, t) \in [-8, 8] \times [-8, 8]$

When we consider a combination of the spectral parameter and gauge deformations of Lax pairs U and V, the matrices A and B are obtained as

$$A = \mu \frac{\partial U}{\partial \lambda} + \nu[\sigma_2, U], \;\; B = \mu \frac{\partial V}{\partial \lambda} + \nu[\sigma_2, U]. \tag{122}$$

Here we just give the Gaussian and mean curvatures of the surfaces of the surface as

$$K = \frac{2\,u\,(u^2 - 2\,\alpha)}{\nu\Big(2\,\nu u[u^2 - 2\,\alpha] - 3\,\mu u^2 - 2\mu(\lambda^2 - \alpha)\Big) + \mu^2 u}, \tag{123}$$

$$H = \frac{\mu(3\,u^2 + 2(\lambda^2 - \alpha)) - 4\,u\,\nu(u^2 - 2\,\alpha)}{2\,\nu\Big(2\,\nu\,u[u^2 - 2\,\alpha] - 3\,\mu\,u^2 - 2\,\mu(\lambda^2 - \alpha)\Big) + 2\,\mu^2\,u}. \tag{124}$$

mKdV surfaces obtained from spectral-gauge deformation do not belong to Willmore-like surfaces and surfaces that solve the generalized shape equation.

The components of the position vector $\mathbf{Y} = (y_1, y_2, y_3)$ for mKdV surfaces correspond to spectral-gauge deformation are given as

$$y_1 = -W_2 \, \frac{e^{2\xi} - 1}{(e^{2\xi} + 1)} \operatorname{sech} \xi - W_3 \, \Omega_3 - W_4 \frac{1}{e^{2\xi} + 1}, \tag{125}$$

$$y_2 = \left[\frac{1}{2} \, W_4 \operatorname{sech} \xi + W_5 \, \frac{\left(e^{4\xi} + 1\right)}{(e^{2\xi} + 1)^2} - W_6 \operatorname{sech}^2 \xi \right] \cos \Omega_2 \tag{126}$$

$$+W_7 \, \frac{\left(e^{2\xi} - 1\right)}{\left(e^{2\xi} + 1\right)} \sin \Omega_2,$$

$$y_3 = \left[\frac{1}{2} \, W_4 \operatorname{sech} \xi + W_5 \, \frac{\left(e^{4\xi} + 1\right)}{(e^{2\xi} + 1)^2} - W_6 \operatorname{sech}^2 \xi \right] \sin \Omega_2 \tag{127}$$

$$-W_7 \, \frac{\left(e^{2\xi} - 1\right)}{\left(e^{2\xi} + 1\right)} \cos \Omega_2,$$

where

$$W_2 = \frac{2\, k_1^2\, \nu}{k_1^2 + 4\lambda^2}, \quad W_3 = \frac{\mu}{8}, \tag{128}$$

$$W_4 = \frac{4\, \mu\, k_1^2}{k_1^2 + 4\lambda^2}, \quad W_5 = \frac{\nu\, (k_1^2 - 4\lambda^2)}{k_1^2 + 4\lambda^2}, \tag{129}$$

$$W_6 = \frac{\nu\, (4\, \lambda^2 + 3\, k_1^2)}{2(k_1^2 + 4\lambda^2)}, \quad W_7 = \frac{4\, \lambda\, k_1^2\, \nu}{k_1^2 + 4\lambda^2}, \tag{130}$$

$$\Omega_2 = t\left(\lambda^2 + k_1^2\, [1+\lambda]/4\right) + x\lambda, \tag{131}$$

$$\Omega_3 = \left(t\, [8\, \lambda + k_1^2] + 4\, x\right), \quad \xi = \frac{k_1^3}{8}(t + \frac{4\, x}{k_1^2}). \tag{132}$$

Example: Taking $\mu = -6$, $\nu = 1.5$, and $k_1 = 1.5$,, and changing λ as $a)\,\lambda = 0$, $b)\,\lambda = 0.2$, in Eqs. (125) - (127), we get the surface given in Figure 4.

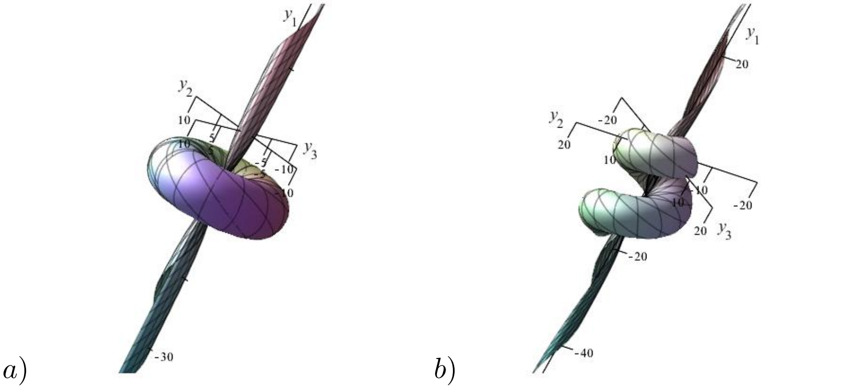


Figure: (a)-(b) $(x,t) \in [-8,8] \times [-8,8]$

Example: Taking $\mu = 1.5$, $\nu = 0.1$, $k_1 = 1.7$, and $\lambda = 0.1$, in Eqs. (125) - (127), we get the surface given in Figure 5.

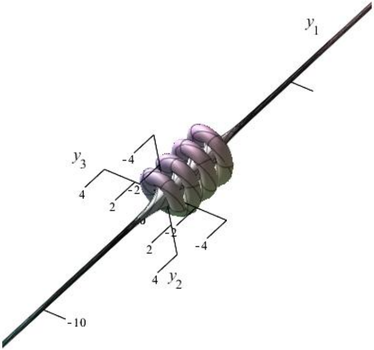
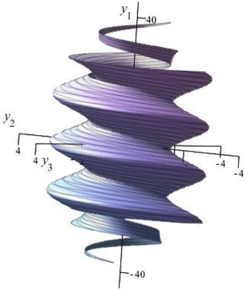
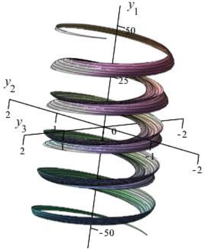


Figure: $(x, t) \in [-15, 15] \times [-15, 15]$

Example: Taking $\mu = 3$, $\nu = -1$, and $k_1 = 1$,, and changing λ as $a)\,\lambda = 1,\ b)\,\lambda = -4,$ in Eqs. (125) - (127), we get the surface given in Figure 6.



$a)$



$b)$

Figure: (a)-(b) $(x, t) \in [-8, 8] \times [-8, 8]$

Example: Taking $\mu = -3$, $\nu = -1$, $k_1 = 1$, and $\lambda = -0.2$, in Eqs. (125) - (127), we get the surface given in Figure 7.

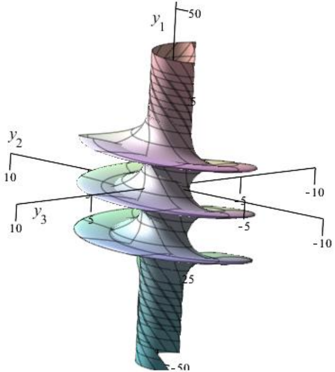


Figure: $(x, t) \in [-30, 30] \times [-30, 30]$

In this section we obtain surfaces corresponding the sine-Gordon (SG) [Ceyhan et. al., 2000], [Tek, 2007].
Let $u(x,t)$ satisfy the SG equation

$$u_{xt} = \sin u. \tag{133}$$

The Lax pairs U and V of the KdV equation in Eq. (133) are given as

$$U = \frac{i}{2}\begin{pmatrix} \lambda & -u_x \\ -u_x & -\lambda \end{pmatrix}, \tag{134}$$

$$V = \frac{1}{2\,\lambda}\begin{pmatrix} -i\,\cos u & \sin u \\ -\sin u & i\,\cos u \end{pmatrix}, \tag{135}$$

were λ is a spectral constant.

In the following proposition, we obtain SG surfaces using spectral parameter deformation of U and V.

Proposition

Let u satisfy the SG equation given in Eq. (133) and $\mathfrak{su}(2)$ valued Lax pairs U and V are defined by Eqs. (134) and (135), respectively. The matrices A and B defined as spectral parameter deformations of the Lax pairs U and V, respectively

$$A = \mu \frac{\partial U}{\partial \lambda} = \frac{i\,\mu}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{136}$$

$$B = \mu \frac{\partial V}{\partial \lambda} = \frac{\mu}{2\,\lambda} \begin{pmatrix} i\,\cos u & -\sin u \\ \sin u & -i\,\cos u \end{pmatrix}, \tag{137}$$

where μ is a constant and λ is a spectral parameter.

Proposition

Then the first and second fundamental forms of the surface S are given as

$$(ds_I)^2 \equiv g_{jk}\, dx^j\, dx^k = \frac{\mu^2}{4}\Big(dx^2 + \frac{2}{\lambda^2}\, \cos u\, dx\, dt + \frac{1}{\lambda^4}\, dt^2\Big),$$
$$(ds_{II})^2 \equiv h_{jk}\, dx^j\, dx^k = -\frac{\mu}{\lambda} \sin u\, dx\, dt, \qquad (138)$$

and the Gaussian and mean curvatures are given as

$$K = -\frac{4\,\lambda^2}{\mu^2}, \quad H = \frac{2\,\lambda}{\mu} \cot u, \qquad (139)$$

where $x^1 = x$, $x^2 = t$.

In the following proposition, we use spectral and Gauge deformation to obtain SG surfaces.

Proposition

Let u satisfy the SG equation given in Eq. (133) and $\mathfrak{su}(2)$ valued Lax pairs U and V are defined by Eqs. (134) and (135), respectively. The matrices A and B defined as

$$A = \mu \frac{\partial U}{\partial \lambda} + \frac{\mathrm{i}\nu}{2}[\sigma_1, U] = \frac{1}{2} \begin{pmatrix} \mathrm{i}\mu & \nu\lambda \\ -\nu\lambda & -\mathrm{i}\mu \end{pmatrix}, \tag{140}$$

$$\begin{aligned} B &= \mu \frac{\partial V}{\partial \lambda} + \frac{\mathrm{i}\nu}{2}[\sigma_1, V] \\ &= \frac{1}{2\,\lambda^2} \begin{pmatrix} \mathrm{i}(\mu\cos u - \lambda\nu\sin u) & -\mu\sin u - \lambda\nu\cos u \\ \mu\sin u + \lambda\nu\cos u & -\mathrm{i}(\mu\cos u - \lambda\nu\sin u) \end{pmatrix} \end{aligned} \tag{141}$$

where μ is a constant and λ is a spectral parameter.

Proposition

Then the first and second fundamental forms of the surface S are given as

$$(ds_I)^2 \equiv g_{jk}\, dx^j\, dx^k, \quad and \quad (ds_{II})^2 \equiv h_{jk}\, dx^j\, dx^k \tag{142}$$

where

$$g_{11} = \frac{1}{4}(\mu^2 + \lambda^2\, \nu^2), \tag{143}$$

$$g_{12} = g_{21} = \frac{1}{4\lambda^2}\Big[(\mu^2 - \lambda^2\, \nu^2)\cos u - 2\mu\, \nu \lambda \sin u\Big], \tag{144}$$

$$g_{22} = \frac{1}{4\lambda^2}(\mu^2 + \lambda^2\, \nu^2),\; h_{11} = \frac{1}{2}\lambda^2 \nu, \tag{145}$$

$$h_{12} = h_{21} = -\frac{1}{2\lambda}(\mu \sin u + \lambda\, \nu \cos u),\; h_{22} = \frac{\nu}{2\lambda^2}. \tag{146}$$

Proposition

The Gaussian and mean curvatures are given as

$$K = \frac{L_1 \cos^2 u + L_2 \sin u \cos u - L_1}{L_3 \cos^2 u + L_4 \sin u \cos u + L_5}, \tag{147}$$

$$H = \frac{L_6 \cos^2 u + L_7 \sin u \cos u + L_8}{L_3 \cos^2 u + L_4 \sin u \cos u + L_5}, \tag{148}$$

where

$$L_1 = 4\lambda^2(\lambda^2 \nu^2 - \mu^2),\ L_2 = 8\,\mu\,\nu\lambda^3, \tag{149}$$

$$L_3 = \mu^4 + \lambda^2 \nu^2(\lambda^2 \nu^2 - 6\mu^2), \tag{150}$$

$$L_4 = 4\mu\,\lambda\,\nu(\lambda^2 \nu^2 - \mu^2), \tag{151}$$

$$L_5 = -\mu^4 - \lambda^2 \nu^2(\lambda^2 \nu^2 - 2\mu^2), \tag{152}$$

$$L_6 = 2\nu\,\lambda^2(\lambda^2 \nu^2 - 3\,\mu^2), \tag{153}$$

$$L_7 = 2\,\mu\,\lambda(3\lambda^2\nu^2 - \mu^2),\ L_8 = 2\nu\,\lambda^2(\mu^2 - \lambda^2 \nu^2). \tag{154}$$

The following proposition gives SG surfaces belongs to Weingarten surfaces [Ceyhan et. al., 2000].

Proposition

Let u satisfy the SG equation given in Eq. (133) *and S be the surface obtained using spectral parameter deformation. Then the surface S is a Weingarten surface that has the following algebraic relation between Gaussian and mean curvatures of the surface*

$$(\mu^2 + \lambda^2 \nu^2) K - 4\nu\lambda^2 H + 4\lambda^2 = 0. \tag{155}$$

Proposition

Let u satisfy Eq. (133). $\mathfrak{su}(2)$ valued Lax pairs U and V of the SG equation are given by Eqs. (134) and (135), respectively. $\mathfrak{su}(2)$ valued matrices A and B are defined as

$$A = U'\varphi = -\frac{i}{2}\varphi_x\sigma_1, \quad B = V'\varphi = \frac{i}{2\lambda}\varphi(\cos u\sigma_2 + \sin u\sigma_3) \tag{156}$$

where λ is constant and $\sigma_1, \sigma_2, \sigma_3$ are the Pauli sigma matrices. Here primes denote Frèchet differentiation and φ is a symmetry of (133), i.e. φ is a solution of

$$\varphi_{xt} = \varphi \cos u \tag{157}$$

Proposition

Then the surface S has the following first and second fundamental forms

$$(ds_I)^2 \equiv g_{jk} dx^j\, dx^k = \frac{1}{4}\Big(\varphi_x^2\, dx^2 + \frac{1}{\lambda^2}\,\varphi^2 dt^2\Big), \tag{158}$$

$$(ds_{II})^2 \equiv g_{jk} dx^j\, dx^k = \frac{1}{2}\Big(\lambda\,\varphi_x\, \sin u\, dx^2 + \frac{1}{\lambda}\,\varphi\, u_t\, dt^2\Big), \tag{159}$$

and the Gaussian and mean curvatures are given as

$$K = \frac{4\lambda^2 u_t \sin u}{\varphi\varphi_x},\;\; H = \frac{\lambda(\varphi_x u_t + \varphi \sin u)}{\varphi\varphi_x} \tag{160}$$

Indeed, Eq. (157) has infinitely many explicit solutions in terms of u and its derivatives. The following corollary gives the surfaces corresponding to $\varphi = u_x$ which is special case of Proposition 9.12.

Corollary

Let $\varphi = u_x$ in the previous Proposition, then the surface turns out to be a sphere with first and second fundamental forms

$$(ds_I)^2 = \frac{1}{4}\Big(\sin^2 u\, dx^2 + \frac{1}{\lambda^2}\, u_t^2 dt^2\Big), \tag{161}$$

$$(ds_{II})^2 = \frac{1}{2}\Big(\lambda\, \sin^2 u\, dx^2 + \frac{1}{\lambda}\, u_t^2\, dt^2\Big), \tag{162}$$

and the corresponding Gaussian and mean curvatures are

$$K = 4\,\lambda^2,\;\; H = 2\,\lambda. \tag{163}$$

For the following solutions (Generalized symmetries) of the symmetry equation $\varphi_{xt} = \varphi \cos u$

$$\varphi = u_x, \varphi = u_{3x} + \frac{u_x^3}{2}, \tag{164}$$

$$\varphi = u_{3t} + \frac{u_t^3}{2}, \tag{165}$$

$$\varphi = u_{5x} + \frac{5}{2}\, u_x^2\, u_{3x} + \frac{5}{2}\, u_x\, u_{2x}^2 + \frac{3}{8}\, u_x^5 \tag{166}$$

We obtain different SG surfaces with the Gaussian and mean curvatures of the surfaces which are constructed previously.

In this section we obtain surfaces in $\mathbb{R}^3$ corresponding nonlinear Schrödinger (NLS) equation [Ceyhan et. al., 2000], [Tek, 2015].

Let complex function $u(x,t) = r(x,t) + is(x,t)$ satisfy the NLS equation

$$r_t = s_{xx} + 2s(r^2 + s^2), \tag{167}$$
$$s_t = -r_{xx} - 2r(r^2 + s^2), \tag{168}$$

where r, s are real functions.

By changing the variables r and s as

$$r = q\cos\phi, \quad s = q\sin\phi, \tag{169}$$

and NLS given in Eqs. (167) and (167) take the following form

$$q\phi_t = -q_{xx} - 2q^3 + q\phi_x^2, \tag{170}$$
$$q_t = q\phi_{xx} + 2q_x\phi_x. \tag{171}$$

The Lax pairs U and V of these equations are given as

$$U = \frac{i}{2} \begin{pmatrix} -2\lambda & 2\,q\,(\sin\phi - i\,\cos\phi) \\ 2\,q\,(\sin\phi + i\,\cos\phi) & 2\lambda \end{pmatrix}, \tag{172}$$

$$V = -\frac{i}{2} \begin{pmatrix} -2\left(2\,\lambda^2 - q^2\right) & z_1 + i z_2 \\ z_1 - i z_2 & 2\left(2\,\lambda^2 - q^2\right) \end{pmatrix}, \tag{173}$$

where

$$z_1 = 2\,(q_x + 2\,\lambda\,q)\cos\phi - 2\,q\,\phi_x \sin\phi, \tag{174}$$

$$z_2 = 2\,(q_x + 2\,\lambda\,q)\sin\phi - 2\,q\,\phi_x \cos\phi, \tag{175}$$

and λ is a constant.

In the following proposition we obtain the NLS surfaces using spectral deformation.

Proposition

Let q and ϕ satisfy NLS equation given in Eqs. (170) and (171). The Lax pairs U and V of the NLS equation are given by Eqs. (172) and (173), respectively. $\mathfrak{su}(2)$ valued matrices A and B are defined as

$$A = \mu \frac{\partial U}{\partial \lambda} = \frac{i}{2} \begin{pmatrix} -2\mu & 0 \\ 0 & 2\mu \end{pmatrix}, \tag{176}$$

$$B = \mu \frac{\partial V}{\partial \lambda} = -\frac{i}{2} \begin{pmatrix} -8\lambda\mu & 4\mu\, q(\cos\phi - i\,\sin\phi) \\ 4\mu\, q(\cos\phi + i\,\sin\phi) & 8\lambda\mu \end{pmatrix} \tag{177}$$

where λ is spectral parameter and μ is a constant.

Proposition

Then the surface S has the following first and second fundamental forms $(j, k = 1, 2)$

$$(ds_I)^2 \equiv g_{jk}\, dx^j\, dx^k = \mu^2 \Big([dx - 4\lambda\, dt]^2 + 4\, q^2\, dt^2 \Big), \tag{178}$$

$$(ds_{II})^2 \equiv h_{jk}\, dx^j\, dx^k = -2\, \mu\, q \left(dx - [2\, \lambda - \phi_x] dt \right)^2 + 2\, \mu\, q_{2x}\, dt^2 \tag{179}$$

The Gaussian and mean curvatures of S are obtained as

$$K = -\frac{q_{xx}}{\mu^2\, q}, \quad H = \frac{q_{xx} - q\, (\phi_x + 2\, \lambda)^2 - 4\, q^3}{4\, \mu\, q^2}, \tag{180}$$

where $x^1 = x$, $x^2 = t$.

Let $\phi = \alpha\, t$ and $q = q(x)$ satisfies the following equation

$$q_{xx} = -2q^3 - \alpha\, q. \tag{181}$$

When we multiply Eq. (181) by q_x and integrate the resultant equation, $q(x)$ satisfy the following equation

$$q_x^2 = -q^4 - \alpha\, q^2. \tag{182}$$

The following proposition gives a class of NLS surfaces which are Willmore-like.

Proposition

Let $\phi = \alpha\, t$ *and* $q = q(x)$ *satisfy the equation given in Eq.* (182) *and* S *be the surface obtained in Proposition 9.15. Then the surface* S *is called a Willmore-like surface. This means that* K *and* H *satisfy the following equation*

$$\nabla^2 H + aH^3 + bH\,K = 0, \tag{183}$$

where a, b, *and* α *have the following form*

$$a = \frac{4}{3}, \quad b = 0, \quad \alpha = -2\,\lambda^2, \tag{184}$$

and λ *is an arbitrary constant.*

The following proposition contains the Weingarten surfaces.

Proposition

Let S be the surface obtained in Proposition 9.15, $\theta = \alpha\, t$ and $q = q(x)$ satisfy Eq. (182). Then the surface S is a Weingarten surface that has the following algebraic relation between K and H

$$8\,\mu^2 H^2\left(K\mu^2 - \alpha\right) = \left(3\,K\mu^2 - 2\,\alpha + 4\,\lambda^2\right)^2, \tag{185}$$

where α, μ, and λ are constants. This surface S is a Weingarten surface.
When $\alpha = -4\lambda^2$, the surface S reduces to a quadratic Weingarten surface

$$K - \frac{8}{9}\,H^2 + 4\,\frac{\lambda^2}{\mu^2} = 0. \tag{186}$$

Proposition

Let $\theta = \alpha\, t$ and $q = q(x)$ satisfy the equation given in Eq. (182) and S be the surface in Proposition 9.15. Then there are NLS surfaces satisfying the generalized shape equation

$$(\nabla^2 + 4H^2 - 2K)\frac{\partial \mathcal{E}}{\partial H} + 2(\nabla \cdot \bar{\nabla} + 2KH)\frac{\partial \mathcal{E}}{\partial K} - 4H\mathcal{E} + 2p = 0, \quad (187)$$

where the Lagrange function $\mathcal{E}$ is a polynomial of Gaussian and mean curvatures of the surface S.

We now give some examples of $\mathcal{E}$ for the NLS surfaces that solve the Euler-Lagrange equation given in Eq. (105) and provide the constraints [Tek, 2015].

Example

Let $\deg(\mathcal{E}) = N$, then

i) for $N = 3$:
$\mathcal{E} = a_1\, H^3 + a_2\, H^2 + a_3\, H + a_4 + a_5\, K + a_6\, K\, H,$

- $\alpha = -2\,\lambda^2,\;\; a_1 = -\dfrac{p\,\mu^4}{18\,\lambda^4},\;\; a_2 = a_4 = 0,\; a_3 = \dfrac{p\,\mu^2}{16\,\lambda^2},\;\; a_6 = \dfrac{p\,\mu^4}{8\,\lambda^4},$
where $\lambda \neq 0$, and μ, p, and a_5 are arbitrary constants;

ii) for $N = 4$:
$\mathcal{E} = a_1 H^4 + a_2 H^3 + a_3 H^2 + a_4 H + a_5 + a_6 K + a_7 K\, H + a_8 K^2 + a_9 K\, H^2,$

- $\alpha = -2\,\lambda^2,\;\; a_1 = -\dfrac{8}{189}\,(8\,a_8 + 15\,a_9),\;\; a_2 = -\dfrac{p\,\mu^4}{18\,\lambda^4},$
- $a_3 = \dfrac{2\,\lambda^2}{7\,\mu^2}\,(32\,a_8 + 25\,a_9),\; a_4 = \dfrac{p\,\mu^2}{16\,\lambda^2},$
- $a_5 = -\dfrac{2\,\lambda^4}{21\,\mu^4}\,(38\,a_8 + 45\,a_9)\;\; a_7 = \dfrac{p\,\mu^4}{8\,\lambda^4},$
where $\lambda \neq 0$, $\mu \neq 0$, and p, a_6, a_8, and a_9 are arbitrary constants;

Example

iii) for $N = 5$:

$$\mathcal{E} = a_1\, H^5 + a_2\, H^4 + a_3\, H^3 + a_4\, H^2 + a_5\, H + a_6 + a_7\, K + a_8\, K\, H + a_9\, K^2 + a_{10}\, K\, H^2 + a_{11}\, K^2\, H + a_{12}\, K\, H^3,$$

For general $N \geq 3$, from the above examples, the polynomial function $\mathcal{E}$ takes the form

$$\mathcal{E} = \sum_{n=0}^{N} H^n \sum_{l=0}^{\lfloor \frac{(N-n)}{2} \rfloor} a_{nl} K^l,$$

where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x and a_{nl} are constants.

Let $q = 2\eta\,\text{sech}\xi$ and $\theta(t) = \rho$ be solution of NLS equation, where $\xi = 2\eta x - \kappa$ and $\rho = -4\eta^2 t$.

In order to find the position vector first we solve the Lax equation given in Eq. (51).

The solution of Lax equation is a 2×2 matrix Φ

$$\Phi = \begin{pmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{pmatrix}, \tag{188}$$

where Φ_{11}, Φ_{12}, Φ_{21}, Φ_{122} are given as

$$\begin{aligned}\Phi_{11} = \frac{i}{\eta(\sin\rho + i\cos\rho)} \Big(& C_1 \mathrm{e}^{2\,i\left(\lambda^2+2\,\eta^2\right)t} (\eta\tanh\xi + i\lambda)(\tanh\xi + 1)^{-i\lambda/4} \\ & \times (\tanh\xi - 1)^{i\lambda/4\eta} - D_1 \mathrm{e}^{-2\,i\lambda^2 t}\eta^2 \text{sech}\,\xi(\tanh\xi + 1)^{i\lambda/4\eta} \\ & \times (\tanh\xi - 1)^{-i\lambda/4\eta}\Big),\end{aligned} \tag{189}$$

$$\Phi_{12} = \frac{i}{\eta(\sin\rho + i\cos\rho)} \Big(C_2 e^{2i(\lambda^2+2\eta^2)t} (\eta\tanh\xi + i\lambda)(\tanh\xi + 1)^{-i\lambda/4\eta} \times (\tanh\xi - 1)^{i\lambda/4\eta} - D_2 e^{-2i\lambda^2 t}\eta^2 \operatorname{sech}\xi (\tanh\xi + 1)^{i\lambda/4\eta} \times (\tanh\xi - 1)^{-i\lambda/4\eta} \Big), \quad (190)$$

$$\Phi_{21} = C_1 e^{2i(\lambda^2+2\eta^2)t} \operatorname{sech}\xi\,(\tanh\xi + 1)^{-i\lambda/4\eta}(\tanh\xi - 1)^{i\lambda/4\eta} + D_1 e^{-2i\lambda^2 t}(\eta\tanh\xi - i\lambda)(\tanh\xi + 1)^{i\lambda/4\eta}(\tanh\xi - 1)^{-i\lambda/4\eta}, \quad (191)$$

$$\Phi_{22} = C_2 e^{2i(\lambda^2+2\eta^2)t} \operatorname{sech}\xi\,(\tanh\xi + 1)^{-i\lambda/4\eta}(\tanh\xi - 1)^{i\lambda/4\eta} + D_2 e^{-2i\lambda^2 t}(\eta\tanh\xi - i\lambda)(\tanh\xi + 1)^{i\lambda/4\eta}(\tanh\xi - 1)^{-i\lambda/4\eta}. \quad (192)$$

Here the determinant of the solution of the Lax equation Φ is constant and it has the following form

$$\det(\Phi) = \frac{(\eta^2 + \lambda^2)(C_1 D_2 - C_2 D_1)}{\eta} \neq 0. \quad (193)$$

We use the following equation

$$F_x = \Phi^{-1} A \Phi, \; F_t = \Phi^{-1} B \Phi. \tag{194}$$

in order to find the immersion function F. We obtain F as

$$F = \mathrm{e}_1 y_1 + \mathrm{e}_2 y_2 + \mathrm{e}_3 y_3 \tag{195}$$

where y_1, y_2, and y_3 are given as

$$y_1 = -\frac{1}{\eta\left(\mathrm{e}^{2\xi}+1\right)} W_8 \left(\Omega_4\left(\mathrm{e}^{2\xi}+1\right) - 2\eta\right), \tag{196}$$

$$y_2 = -W_8 \operatorname{sech}(\xi) \sin(\Omega_5), \tag{197}$$

$$y_3 = W_8 \operatorname{sech}(\xi) \cos(\Omega_5), \tag{198}$$

where

$$W_8 = \frac{\mu\eta}{\eta^2+\lambda^2}, \; \Omega_4 = (4\lambda t - x)(\eta^2+\lambda^2), \tag{199}$$

$$\Omega_5 = \frac{1}{\eta}\Big(4\eta(\eta^2+\lambda^2)t - \lambda(2\eta x - \kappa)\Big), \; \xi = 2\eta x - \kappa. \tag{200}$$

Example: If we take $\lambda = 2$, $\mu = 3$, $\kappa = 10$ and changing η as $a)\, \eta = 0.5,\ b)\, \eta = 0.75, \text{and}\, c)\, \eta = 1,$ in Eqs.(196) - (198), we get the surface given in (Figure 8)

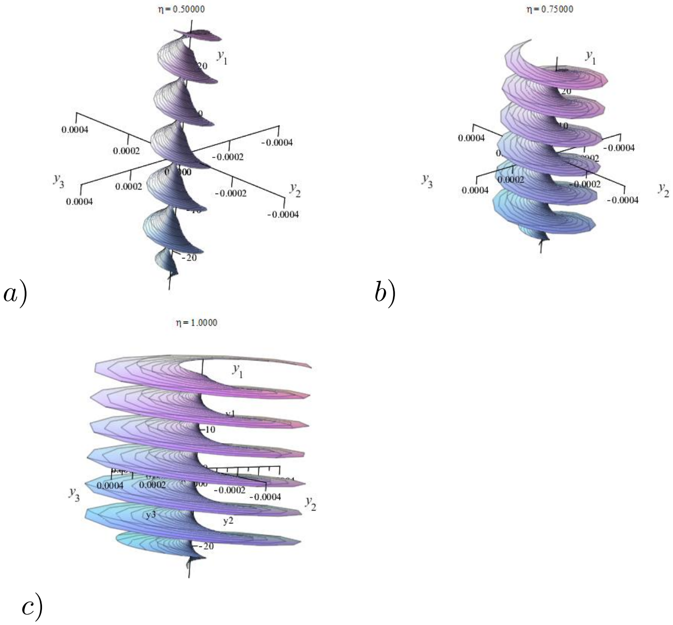


Figure: (a)-(c) $(x, t) \in [-1, 1] \times [-1, 1]$

Example: If we take $\lambda = 0$, $\mu = 0.2$, $\eta = 0.3$ and $\kappa = 4$ in Eqs.(196) - (198), we get the surface given in (Figure 9)

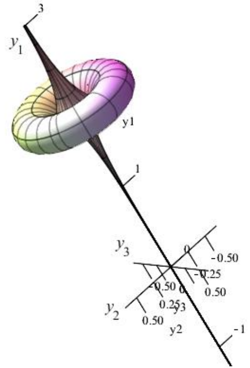


Figure: $(x, t) \in [-15, 15] \times [-15, 15]$

Example: If we take $\lambda = 0.5$, $\mu = 1$, $\eta = 2$ and $\kappa = 2$ in Eqs.(196) - (198), we get the surface given in (Figure 10)

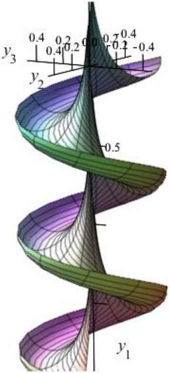


Figure: $(x, t) \in [-0.5, 0.5] \times [-0.5, 0.5]$

Example: If we take $\lambda = 0.5$, $\mu = 1$, $\eta = 2$ and $\kappa = 0$ in Eqs.(196) - (198), we get the surface given in (Figure 11)

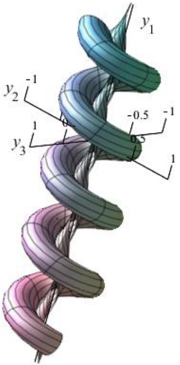


Figure: $(x, t) \in [-0.8, 0.8] \times [-0.8, 0.8]$

In this section, we develop surfaces in three dimensional Minkowski space using the similar techniques that we used in previous sections.

Consider the isometric immersion $F : \mathcal{U} \to M_3$.

Here $\mathcal{U} \in M_2$ is the domain of the immersion, M_2 and M_3 are two and three dimensional Minkowski spaces.

To investigate the surfaces in M_3, the Lie group G that we use is $SL(2, \mathbb{R})$, the corresponding Lie algebra $\mathfrak{g}$ is $\mathfrak{sl}(2, \mathbb{R})$.

The base 2×2 matrices of $\mathfrak{sl}(2, \mathbb{R})$ are

$$\mathrm{e}_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \mathrm{e}_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \mathrm{e}_3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{201}$$

The inner product defined on $\mathfrak{sl}(2, \mathbb{R})$ is given as

$$\langle X, Y \rangle = \frac{1}{2}\, \mathrm{trace}(XY), \tag{202}$$

for $X, Y \in \mathfrak{sl}(2, \mathbb{R})$.

In this section, we obtain surfaces corresponding KdV equation using spectral parameter deformation [Gürses and Tek, 2014].

Let $u(x,t)$ satisfy the KdV equation

$$u_t = \frac{1}{4} u_{xxx} + \frac{3}{2} u u_x. \tag{203}$$

The Lax pairs U and V of the KdV equation in Eq. (203) are given as

$$U = \begin{pmatrix} 0 & 1 \\ \lambda - u & 0 \end{pmatrix}, \tag{204}$$

$$V = \begin{pmatrix} -\frac{1}{4} u_x & \frac{1}{2} u + \lambda \\ -\frac{1}{4} u_{xx} + \frac{1}{2} (2\lambda + u)(\lambda - u) & \frac{1}{4} u_x \end{pmatrix}, \tag{205}$$

where λ is the spectral parameter.

In the following proposition, we obtain KdV surfaces using spectral parameter deformation of U and V.

Proposition

Let u satisfy the KdV equation given in Eq. (203) and $\mathfrak{sl}(2,\mathbb{R})$ valued Lax pairs U and V are defined by Eqs. (204) and (205), respectively. The matrices A and B defined as spectral parameter deformations of the Lax pairs U and V, respectively

$$A = \mu \frac{\partial U}{\partial \lambda} = \begin{pmatrix} 0 & 0 \\ \mu & 0 \end{pmatrix}, \tag{206}$$

$$B = \mu \frac{\partial V}{\partial \lambda} = \begin{pmatrix} 0 & \mu \\ \frac{\mu}{2}(4\lambda - u) & 0 \end{pmatrix}, \tag{207}$$

where λ is spectral parameter, and μ is a constant.

Proposition

Then the first and second fundamental forms of the surface S are given as

$$(ds_I)^2 \equiv g_{ij} dx^i\, dx^j = \mu^2 dx\, dt + \frac{\mu^2}{2}(4\lambda - u) dt^2, \tag{208}$$

$$(ds_{II})^2 \equiv h_{ij} dx^i dx^j = -\mu\, dx^2 - \mu(2\,\lambda + u) dx\, dt - \frac{\mu}{4}\Big(u_{xx} + (u + 2\,\lambda)^2\Big) dt^2, \tag{209}$$

and the Gaussian and mean curvatures are given as

$$K = -\frac{u_{xx}}{\mu^2}, \quad H = \frac{2(\lambda - u)}{\mu}, \tag{210}$$

where $x^1 = x$, $x^2 = t$.

When we use traveling wave ansatz $u_t + u_t/c = 0$ in KdV equation, we obtain the following form of the KdV equation

$$u_{xx} = -3u^2 - \frac{4}{c}u + 4\beta, \tag{211}$$

where c and β are constants.

In the following proposition, we give quadratic Weingarten surfaces.

Proposition

Let u be a traveling wave solution of the KdV equation given in Eq. (211) and S be the surface obtained using spectral parameter deformation in Proposition 10.1. Then the surface S is a Weingarten surface that has the following algebraic relation between K and H

$$4\,c\,\mu^2\,K + 4\,\mu\,(2+3\,c\,\lambda)\,H - 3\,c\,\mu^2\,H^2 - 4\,(3\,c\,\lambda^2 + 4\,\lambda - 4\,\beta\,c) = 0, \tag{212}$$

where c and β are constants; $\mu \neq 0$ and $c \neq 0$.

When we multiply the KdV equation in Eq. (211) by u_x and integrate the resultant equation, we obtain the following form of the KdV equation

$$u_x^2 = -2u^3 + 4\alpha u^2 + 8\beta u + 2\gamma, \tag{213}$$

where $\alpha = -1/c$, $c \neq 0$.

The following proposition contains Willmore-like surfaces.

Proposition

Let u satisfy the KdV equation given in Eq. (213) and S be the surface S obtained in Proposition 10.1. Then the surface S is called a Willmore-like surface. This means that the K and H satisfy the following partial differential equation

$$\nabla^2 H + aH^3 + bH\,K = 0, \tag{214}$$

where a, b, β, and γ have the following form

$$a = \frac{7}{4}, b = 1, \tag{215}$$

$$\beta = \frac{1}{20}\left(28\lambda\alpha - 16\alpha^2 - 21\lambda^2\right), \tag{216}$$

$$\gamma = \frac{1}{5}\left(16\alpha^3 - 56\lambda\alpha^2 + 70\alpha\lambda^2 - 28\lambda^3\right). \tag{217}$$

Here $\alpha = -1/c$ $(c \neq 0)$, λ and c are arbitrary constants.

In the following proposition we give KdV surfaces that solve the Euler-Lagrange equation given in Eq. (105).

Proposition

Let u satisfy Eq. (213) and S be the surface in Proposition 10.1. Then there are KdV surfaces satisfying the following generalized shape equation

$$(\nabla^2 + 4H^2 - 2K)\frac{\partial \mathcal{E}}{\partial H} + 2(\nabla \cdot \bar{\nabla} + 2KH)\frac{\partial \mathcal{E}}{\partial K} - 4H\mathcal{E} + 2p = 0, \quad (218)$$

where Lagrange functions are polynomials of Gaussian and mean curvatures of the surface S.

Let us now give some examples of polynomial Lagrange functions of H and K that solve the Euler-Lagrange equation given in Eq. (105) and provide the constraints [Gürses and Tek, 2014].

Example

i) for $N=3: \mathcal{E}=a_1 H^3+a_2 H^2+a_3 H+a_4+a_5 K+a_6 K H$,

$$a_1=-\frac{11\, p\, \mu^4}{64\, \Xi_1},\ a_2=-\frac{15}{32\, \Xi_1}\, p\, \mu^3 \big(2\, \alpha-3\, \lambda\big),$$

$$a_3=-\frac{p\, \mu^2}{16\, \Xi_1}\big(33\, \lambda^2-44\, \alpha\, \lambda+8\, \alpha^2-20\, \beta\big),$$

$$a_4=\frac{p\, \mu}{8\, \Xi_1}\left(47\, \lambda^3-94\, \alpha\, \lambda^2+4\,(10\, \alpha^2-17\, \beta)\, \lambda+40\, \alpha\, \beta-2\, \gamma\right),$$

$$a_6=\frac{7\, p\, \mu^4}{16\, \Xi_1}$$

where

$$\Xi_1=12\, \lambda^4-32\, \alpha\, \lambda^3+(20\, \alpha^2-36\, \beta)\lambda^2+(40\, \alpha\, \beta-3\, \gamma)\lambda+2\, \alpha\, \gamma+16\, \beta^2,$$

$\mu \neq 0$, $p \neq 0$, λ, α, β, γ and a_5 are arbitrary constants, but λ, α, β and γ cannot be zero at the same time.

Example

ii) for $N=4$:
$\mathcal{E}=a_1H^4+a_2H^3+a_3H^2+a_4H+a_5+a_6K+a_7KH+a_8K^2+a_9KH^2$,
a_1,a_2,a_3,a_4,a_5,a_7 can be written in terms of a_8,a_9, α, β, γ, μ, p and λ.

iii) for $N=5$:
$\mathcal{E}=a_1\,H^5+a_2\,H^4+a_3\,H^3+a_4\,H^2+a_5\,H+a_6+a_7\,K+a_8\,K\,H+a_9\,K^2+a_{10}\,K\,H^2+a_{11}\,K^2\,H+a_{12}\,K\,H^3$,
$a_1,a_2,a_3,a_4,a_5,a_6,a_8$ can be written in terms of a_9,a_{10},a_{11},a_{12}, α, β, γ, μ, p and λ.

For general $N\geq 3$, from the above examples, the polynomial function $\mathcal{E}$ takes the form

$$\mathcal{E}=\sum_{n=0}^{N}H^n\sum_{l=0}^{\lfloor\frac{(N-n)}{2}\rfloor}a_{nl}K^l,$$

where $\lfloor x\rfloor$ denotes the greatest integer less than or equal to x and a_{nl} are constants.

In this section, we find the position vector of the KdV surfaces using the solution of KdV equation and its the Lax pairs. We will consider two different solutions of the KdV equation.

Example

Consider the constant solution

$$u = u_0 = \frac{2}{3}(\alpha \pm \sqrt{\alpha^2 + 3\beta}) \tag{219}$$

of the integrated form of the KdV equation given in Eq. (213)], where $\alpha = -1/c$, $c \neq 0$.

Using this solution and corresponding matrix Lax pairs U and V given by Eqs. (204) and (205) of KdV equation, we solve the Lax equations given in Eq. (51).

The solution of Lax equation is a 2×2 matrix Φ

$$\Phi = \begin{pmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{pmatrix}. \tag{220}$$

We find these components as

$$\Phi_{11} = C_1 \,\mathrm{e}^{m(nt+x)} + D_1 \,\mathrm{e}^{-m(nt+x)}, \tag{221}$$
$$\Phi_{12} = C_2 \,\mathrm{e}^{m(nt+x)} + D_2 \,\mathrm{e}^{-m(nt+x)}, \tag{222}$$
$$\Phi_{21} = m(C_1 \,\mathrm{e}^{m(nt+x)} - D_1 \,\mathrm{e}^{-m(nt+x)}), \tag{223}$$
$$\Phi_{22} = m(C_2 \,\mathrm{e}^{m(nt+x)} - D_2 \,\mathrm{e}^{-m(nt+x)}) \tag{224}$$

where $\lambda - u_0 = m^2$, $(2\lambda + u_0)/2 = n$, C_1, C_2, D_1 and D_2 are arbitrary constants.

Here we find that $\det(\Phi) = 2m(C_2 D_1 - C_1 D_2) \neq 0$.

By using A, B, and Φ, we solve Eq. (52) and write F as

$$F = \Phi^{-1} \frac{\partial \Phi}{\partial \lambda} = y_1 \mathrm{e}_1 + y_2 \mathrm{e}_2 + y_3 \mathrm{e}_3, \tag{225}$$

where $\mathrm{e}_1, \mathrm{e}_2, \mathrm{e}_3$ are basis elements of $\mathfrak{sl}(2, \mathbb{R})$ and

$$y_1 = -\left(\frac{D_1 C_2 + C_1 D_2}{D_1 C_2 - C_1 D_2} \right) \frac{(4\lambda - u_0)t + x}{2\sqrt{\lambda - u_0}}, \tag{226}$$

$$y_2 = \left(\frac{D_1 C_1 - D_2 C_2}{D_1 C_2 - D_2 C_1} \right) \frac{(4\lambda - u_0)t + x}{2\sqrt{\lambda - u_0}}, \tag{227}$$

$$y_3 = -\left(\frac{D_1 C_1 + D_2 C_2}{D_1 C_2 - D_2 C_1} \right) \frac{(4\lambda - u_0)t + x}{2\sqrt{\lambda - u_0}}. \tag{228}$$

Hence we find the position vector $\mathbf{Y} = (y_1(x,t), y_2(x,t), y_3(x,t))$ of KdV surfaces in M_3 using the constant solution given in Eq. (219).

The components y_1, y_2 and y_3 of the position vector the KdV surfaces are given by Eqs. (226)-(228), respectively. This surface is plane in M_3.

Example

Consider the one soliton solution

$$u = 2\,k^2\,c^2 \text{sech}^2 k(t - cx) \tag{229}$$

of the KdV equation, where $k^2 = -1/c^3$.

We solve the Lax equations given in Eq. (51) using one soliton solution and corresponding matrix Lax pairs U and V given by Eqs. (204) and (205) of KdV equation. Here we denote $k(t - cx) = \xi$ and let $\lambda = k^2 c^2$.

The solution of Lax equation is a 2×2 matrix Φ

$$\Phi = \begin{pmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{pmatrix}, \tag{230}$$

where Φ_{11}, Φ_{12}, Φ_{21}, Φ_{122} are given as

$$\Phi_{11} = B_1\left(2kt\,\mathrm{sech}\,\xi + \sinh\xi + \xi\mathrm{sech}\,\xi\right) + C_1\mathrm{sech}\,\xi, \tag{231}$$

$$\Phi_{12} = B_2\left(2kt\,\mathrm{sech}\,\xi + \sinh\xi + \xi\mathrm{sech}\,\xi\right) + C_2\mathrm{sech}\,\xi, \tag{232}$$

$$\Phi_{21} = kc\Bigg[B_1\Bigg(2kt\,\mathrm{sech}\,\xi\tanh\xi - \cosh\xi - \mathrm{sech}\,\xi + \xi\mathrm{sech}\,\xi\tanh\xi\Bigg) + C_1\mathrm{sech}\,\xi\tanh\xi\Bigg], \tag{233}$$

$$\Phi_{22} = kc\Bigg[B_2\Bigg(2kt\,\mathrm{sech}\,\xi\tanh\xi - \cosh\xi - \mathrm{sech}\,\xi + \xi\mathrm{sech}\,\xi\tanh\xi\Bigg) + C_2\mathrm{sech}\,\xi\tanh\xi\Bigg], \tag{234}$$

where B_1, B_2, C_1 and C_2 are arbitrary constants. The determinant of the matrix Φ is a constant, we find it as

$$\det(\Phi) = 2\,k\,c(C_2B_1 - C_1B_2) \neq 0. \tag{235}$$

We use the following equation

$$F_x = \Phi^{-1} A \Phi,\ F_t = \Phi^{-1} B \Phi. \tag{236}$$

in order to find the immersion function F. When we solve the consequent equations, we acquire the immersion function F as

$$F = y_1 \mathrm{e}_1 + y_2 \mathrm{e}_2 + y_3 \mathrm{e}_3, \tag{237}$$

where

$$y_1 = \frac{2W_9}{\zeta_1}\Big[\Big(\Omega_6\zeta_2 + \Omega_7\zeta_1 + \zeta_3\Big)W_{10} + \Omega_8\zeta_2 W_{11} + W_{12}\Big], \tag{238}$$

$$y_2 = \frac{W_9}{\zeta_1}\Big[\Big(\Omega_9\zeta_2 + \Omega_7\zeta_1 + \zeta_4\Big)W_{13} + \Big(\Omega_{10}\zeta_2 + \Omega_{11}\Big)W_{14} + W_{15}\Big], \tag{239}$$

$$y_3 = \frac{W_9}{\zeta_1}\Big[\Big(\Omega_9\zeta_2 + \Omega_7\zeta_1 + \zeta_4\Big)W_{16} + \Big(\Omega_{10}\zeta_2 + \Omega_{11}\Big)W_{17} + W_{18}\Big], \tag{240}$$

and e_1, e_2, e_3 are basis elements of $\mathfrak{sl}(2,\mathbb{R})$.

Here ζ_i, $i=1,2,3,4$, Ω_j, $j=6,7,...,10$, and W_l, $l=9,10,...,18$ are given as

$$\zeta_1=1+\mathrm{e}^{-2\xi}, \zeta_2=\mathrm{e}^{-2\xi}-1, \zeta_3=c^3(\mathrm{e}^{-4\xi}-1-2\sinh(2\xi)), \quad (241)$$
$$\zeta_4=\zeta_3+288t^2, \Omega_6=-8(cx+3t)^2, \Omega_7=4kc^3(9t-cx), \quad (242)$$
$$\Omega_8=8kc^3(3t-cx), \Omega_9=-8(c^2x^2-6tcx-9t^2), \quad (243)$$
$$\Omega_{10}=-16kc^3(cx+3t), \Omega_{11}=-192kc^3t, \quad (244)$$
$$W_9=\mu/32c^2\,(B_1C_2-B_2C_1), W_{10}=B_1B_2, W_{11}=C_1B_2+C_2B_1, \quad (245)$$
$$W_{12}=-16c^3C_1C_2, W_{13}=B_2^2-B_1^2, W_{14}=B_2C_2-B_1C_1, \quad (246)$$
$$W_{15}=16c^3\left(C_1^2-C_2^2\right), W_{16}=B_1^2+B_2^2, W_{17}=B_1C_1+B_2C_2, \quad (247)$$
$$W_{18}=-16c^3(C_1^2+C_2^2), \quad (248)$$

where ζ_i, $i=1,2,3,4$ and Ω_j, $j=6,7,...,10$ are functions of x and t, and W_l, $l=9,10,...,18$ are constants given in terms of arbitrary constants B_1, B_2, C_1, and C_2.

Example: Taking $\mu = 1$, $k = 1$, $c = 1$, $B_1 = -1$, $B_2 = 1$, $C_1 = 1$, $C_2 = 1$ in Eqs. (238) - (240), we get the surface given in Figure 12.

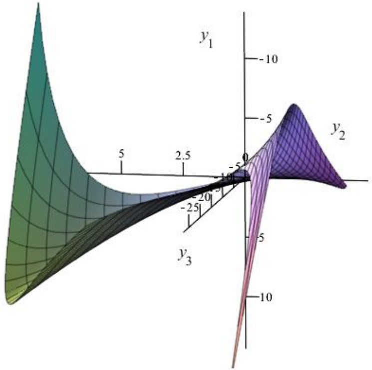


Figure: $(x, t) \in [-1.7, 1.7] \times [-1.7, 1.7]$

Example: Taking $\mu = 1$, $k = 1$, $c = 3$, $B_1 = -1$, $B_2 = 1$, $C_1 = 1$, $C_2 = 1$ in Eqs. (238) - (240), we get the surface given in Figure 13.

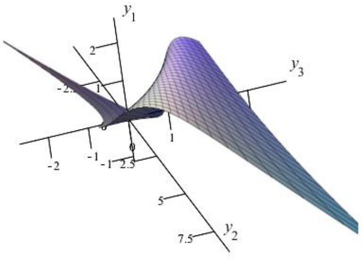


Figure: $(x, t) \in [-2, 2] \times [-2, 2]$

In this section, we develop KdV surfaces using spectral-Gauge deformation.

Proposition

Let u satisfy the KdV equation given in Eq. (203) and $\mathfrak{sl}(2,\mathbb{R})$ valued Lax pairs U and V are defined by Eqs. (204) and (205), respectively. $\mathfrak{sl}(2,\mathbb{R})$ valued matrices A and B are defined as

$$A = \mu_1 \frac{\partial U}{\partial \lambda} + \mu_2 [e_1, U], \tag{249}$$

$$B = \mu_1 \frac{\partial V}{\partial \lambda} + \mu_2 [e_1, U] \tag{250}$$

$$= \begin{pmatrix} 0 & \mu_2(2\lambda + u) + \mu_1 \\ \frac{\mu_2}{2}(u_{xx} - 2(2\lambda - u)(u + \lambda)) + \frac{\mu_1}{2}(4\lambda - 4) & 0 \end{pmatrix}$$

where μ_1 and μ_2 are arbitrary constants.

Proposition

First and second fundamental forms of the surface S are given as

$$(ds_I)^2 \equiv g_{ij}dx^i\,dx^j = 2\,\mu_2\Big(2\mu_2(u-\lambda)+\mu_1\Big)dx^2 \tag{251}$$

$$+\Big(\mu_2\Big[\mu_2\,u_{2x} - 2(u+2\,\lambda)(\mu_1 - 2\,\mu_2\,[\lambda-u])\Big] + \mu_1^2\Big)dxdt$$

$$-\frac{1}{2}\Big(2[u+2\lambda]+\mu_1\Big)\Big(2\mu_2[u+2\lambda][\lambda-u] - \mu_1[4\lambda-u] - \mu_2 u_{xx}\Big)dt^2,$$

$$(ds_{II})^2 \equiv h_{ij}dx^i\,dx^j = \Big(4\mu_2(\lambda-u)-\mu_1\Big)dx^2 \tag{252}$$

$$-\Big(\mu_2\,u_{2x} + [\mu_1 - 4\,\mu_2\,(\lambda-u)]\,[2\,\lambda+u]\Big)dx\,dt$$

$$-\frac{1}{4}\Big(\big[\mu_1+2\,\mu_2(2\,\lambda+u)\big]u_{2x} + \big[\mu_1 - 4\,\mu_2(\lambda-u)\big]\,\big[u+2\,\lambda\big]^2\Big)dt^2,$$

Proposition

and the corresponding Gaussian and mean curvatures are

$$K_1 = \frac{u_{xx}}{\mu_2^2 u_{xx} + \mu_1\left(4\,\mu_2[\lambda - u] - \mu_1\right)}, \tag{253}$$

$$H_1 = \frac{2\mu_1(\lambda - u) + \mu_2 u_{xx}}{\mu_2^2 u_{xx} + \mu_1\left(4\,\mu_2[\lambda - u] - \mu_1\right)}, \tag{254}$$

where $x^1 = x$, $x^2 = t$.

In this section we obtain surfaces in M_3 corresponding Harry Dym (HD) equation [Tek, 2007], [Tek, 2009].
Let $u(x,t)$ satisfy the HD equation

$$u_t = -u^3\, u_{xxx}. \tag{255}$$

The Lax pairs U and V of the HD equation in Eq. (255) are given as

$$U = \begin{pmatrix} 0 & 1 \\ \dfrac{\lambda^2}{u^2} & 0 \end{pmatrix}, \tag{256}$$

$$V = 2\,\lambda^2 \begin{pmatrix} u_x & -2\,u \\ u_{xx} - \dfrac{2\,\lambda^2}{u} & -u_x \end{pmatrix}, \tag{257}$$

were λ is a spectral parameter.

In the following proposition, we develop HD surfaces using spectral deformation of the Lax pairs U and V.

Proposition

Let u satisfy the HD equation given in Eq. (255) and $\mathfrak{sl}(2,\mathbb{R})$ valued Lax pairs U and V are defined by Eq. (256) and (257), respectively. The matrices A and B are defined as

$$A = \mu \frac{\partial U}{\partial \lambda} = 2\,\mu\,\lambda \begin{pmatrix} 0 & 0 \\ \frac{1}{u^2} & 0 \end{pmatrix}, \tag{258}$$

$$B = \mu \frac{\partial V}{\partial \lambda} = 4\,\mu\,\lambda \begin{pmatrix} u_x & -2\,u \\ u_{xx} - \frac{4\,\lambda}{u} & -u_x \end{pmatrix} \tag{259}$$

where μ is a constant and λ is a spectral parameter.

Proposition

The first and second fundamental forms of the surface S are given as

$$(ds_I)^2 \equiv g_{jk}\, dx^j\, dx^k = -16\,\mu^2\,\lambda^2 \Big(\frac{1}{u} dx\, dt + [u_x^2 - 2\,u\,u_{xx} + 8\,\lambda^2] dt^2\Big), \tag{260}$$

$$(ds_{II})^2 \equiv h_{jk}\, dx^j\, dx^k = -\frac{2\,\mu\,\lambda}{u^2}\Big(dx^2 - 8\,\lambda^2\,u\,dx\,dt + 2\,u^2\,\big[2\,u^2\,u_x\,u_{xxx} + u^3\,u_{4x} + 8\,\lambda^4\big]\,dt^2\Big). \tag{261}$$

Gaussian and mean curvatures of the surface S are given as

$$K = -\frac{u^2}{8\,\mu^2\,\lambda^2}\,\big(2\,u_x\,u_{xxx} + u\,u_{xxxx}\big), \tag{262}$$

$$H = \frac{1}{4\,\mu\,\lambda}\big(u_x^2 - 2\,u\,u_{xx} + 4\,\lambda^2\big), \tag{263}$$

where $x^1 = x$, $x^2 = t$.

When we use traveling wave ansatz $u_t - \alpha\, u_x = 0$ in HD equation given by Eq. (255), we get

$$u_{xx} = \frac{\alpha}{2}\,\frac{1}{u} - C_1. \tag{264}$$

where α and C_1 are arbitrary constants.

When we multiply the HD equation in Eq. (264) by u_x and integrate the resultant equation, we obtain the following form of the HD equation

$$u_x^2 = -\alpha\frac{1}{u} - 2\,C_1\,u + 2\,C_2. \tag{265}$$

In the following proposition we give HD surfaces belong to Willmore-like surfaces.

Proposition

Let u satisfy the equation given in Eq. (265) and S be the surface obtained in Proposition 10.9. Then the surface S is called a Willmore-like surface. This means that K and H satisfy the following partial differential equation

$$\nabla^2 H + aH^3 + bH\,K = 0, \tag{266}$$

where a, b, C_1, and C_2 have the following form

$$a = -2, \quad b = 6, \quad C_1 = \frac{16\,\lambda^4}{\alpha}, \quad C_2 = -6\,\lambda^2 \tag{267}$$

and λ is an arbitrary constant.

The following proposition gives HD surfaces belongs to Weingarten surfaces.

Proposition

Let u be a travelling wave solution of the HD equation given in Eq. (265) and S be the surface obtained using spectral parameter deformation in Proposition 10.9. Then the surface S is a Weingarten surface that has the following algebraic relation between Gaussian and mean curvatures of the surface

$$4\mu^2\lambda^2(4K - 3H^2) + (24\mu\lambda^3 + 4\mu\lambda C_2)H + C_3 = 0, \tag{268}$$

where $C_3 = -4\lambda^2(3\lambda^2 - C_2) - 2\alpha C_1 + C_2^2$.

In the following proposition we obtain HD surfaces arise from a variational principle in another words solve the Euler-Lagrange equation [Eq. (105)].

Proposition

Let u satisfy the equation given in Eq. (265) and S be the surface in Proposition 10.9. Then there are HD surfaces satisfying the following generalized shape equation

$$(\nabla^2 + 4H^2 - 2K)\frac{\partial \mathcal{E}}{\partial H} + 2(\nabla \cdot \bar{\nabla} + 2KH)\frac{\partial \mathcal{E}}{\partial K} - 4H\mathcal{E} + 2p = 0, \quad (269)$$

where the Lagrange function $\mathcal{E}$ is a polynomial of K and H.

We now give some examples of $\mathcal{E}$ for the HD surfaces that solve the Euler-Lagrange equation given in Eq. (105) and provide the constraints [Tek, 2009].

Example

i) for $N=3$:
$\mathcal{E} = a_1\, H^3 + a_2\, H^2 + a_3\, H + a_4 + a_5\, K + a_6\, K\, H,$

- $a_1 = -\dfrac{11\,\mu\,a_2}{30\,\lambda},\; a_3 = -\dfrac{4\,\lambda\,a_2}{15\,\mu},\; a_6 = \dfrac{14\,\mu\,a_2}{15\,\lambda},$
- $a_4 = 0, C_1 = p = 0, C_2 = 2\,\lambda,$
  where $\lambda \neq 0$, μ, and a_5 are arbitrary constants.

Example

ii) for $N = 4$:

$\mathcal{E} =$

$$a_1\,H^4+a_2\,H^3+a_3\,H^2+a_4\,H+a_5+a_6\,K+a_7\,K\,H+a_8\,K^2+a_9\,K\,H^2,$$

- $a_1 = -\frac{1}{64}\,(15\,a_8 + 34\,a_9),$
- $a_2 = \frac{1}{480\,\mu\,\lambda}\,\left(\lambda^2\,[358\,a_9 - 7\,a_8] - 176\,\mu^2\,a_3\right),$
- $a_4 = \frac{4\,\lambda}{15\,\mu^3}\,\left(\lambda^2\,[13\,a_8 + 8\,a_9] - \mu^2\,a_3\right),$
- $a_5 = -\frac{3\,\lambda^4}{4\,\mu^4}\,(3\,a_8 + 2\,a_9),$
- $a_7 = \frac{1}{120\,\mu\,\lambda}\,\left(\lambda^2\,[359\,a_8 + 154\,a_9] + 112\,\mu^2\,a_3\right),$
- $C_1 = p = 0, C_2 = 2\,\lambda,$

  where $\lambda \neq 0$, $\mu \neq 0$, and a_6 are arbitrary constants.

Example

iii) for $N = 5$:
$\mathcal{E} = a_1\, H^5 + a_2\, H^4 + a_3\, H^3 + a_4\, H^2 + a_5\, H + a_6 + a_7\, K + a_8\, K\, H + a_9\, K^2 + a_{10}\, K\, H^2 + a_{11}\, K^2\, H + a_{12}\, K\, H^3$,

iv) for $N = 6$:
$\mathcal{E} = a_1\, H^6 + a_2\, H^5 + a_3\, H^4 + a_4\, H^3 + a_5\, H^2 + a_6\, H + a_7 + a_8\, K + a_9\, K\, H + a_{10}\, K^2 + a_{11}\, K\, H^2 + a_{12}\, K^2\, H + a_{13}\, K\, H^3 + a_{14}\, K^3 + a_{15}\, K^2\, H^2 + a_{16}\, K\, H^4$,

For general $N \geq 3$, from the above examples, the polynomial function $\mathcal{E}$ takes the form

$$\mathcal{E} = \sum_{n=0}^{N} H^n \sum_{l=0}^{\lfloor \frac{(N-n)}{2} \rfloor} a_{nl} K^l,$$

where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x and a_{nl} are constants.

In this section, we find the position vector of the HD surfaces that we obtained using spectral parameter deformation in Proposition 10.9.

Consider a solution

$$u = -(\alpha/2)\, 18^{1/3}\, \xi^{2/3}, \tag{270}$$

of the HD equation, where $\xi = t + x/\alpha$ and $\alpha \neq 0$ is a constant.

In order to find the position vector first we solve the Lax equation given in Eq. (51) and the solution of Lax equation 2×2 matrix Φ

$$\Phi = \begin{pmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{pmatrix}, \tag{271}$$

where Φ_{11}, Φ_{12}, Φ_{21}, Φ_{122} are given as

$$\Phi_{11} = \frac{1}{\lambda^{3/2}}\Big(A_1\,[18^{1/3} - 6\,\lambda\,\xi^{1/3}]\exp\{4\,\lambda^3\,t + \lambda\,18^{2/3}\,\xi^{1/3}/3\} + B_1\,[18^{1/3} + 6\,\lambda\,\xi^{1/3}]\exp\{-4\,\lambda^3\,t - \lambda\,18^{2/3}\,\xi^{1/3}/3\}\Big) \tag{272}$$

$$\Phi_{21} = -\frac{2\,\sqrt{\lambda}\,18^{2/3}}{3\,\alpha\,\xi^{1/3}}\Big(A_1\,\exp\{4\,\lambda^3\,t + \lambda\,18^{2/3}\,\xi^{1/3}/3\} + B_1\,\exp\{-4\,\lambda^3\,t - \lambda\,18^{2/3}\,\xi^{1/3}/3\}\Big) \tag{273}$$

$$\Phi_{12} = \frac{1}{\lambda^{3/2}}\Big(A_2\,[18^{1/3} - 6\,\lambda\,\xi^{1/3}]\exp\{4\,\lambda^3\,t + \lambda\,18^{2/3}\,\xi^{1/3}/3\} + B_2\,[18^{1/3} + 6\,\lambda\,\xi^{1/3}]\exp\{-4\,\lambda^3\,t - \lambda\,18^{2/3}\,\xi^{1/3}/3\}\Big) \tag{274}$$

$$\Phi_{22} = -\frac{2\,\sqrt{\lambda}\,18^{2/3}}{3\,\alpha\,\xi^{1/3}}\Big(A_2\,\exp\{4\,\lambda^3\,t + \lambda\,18^{2/3}\,\xi^{1/3}/3\} + B_2\,\exp\{-4\,\lambda^3\,t - \lambda\,18^{2/3}\,\xi^{1/3}/3\}\Big), \tag{275}$$

where $\xi = t + x/\alpha$, and A_1, A_2, B_1, B_2, and $\alpha \neq 0$ are constants. Here the determinant of the solution of the Lax equation Φ is constant and it has the following form

$$\det(\Phi) = \frac{8 \cdot 18^{2/3}}{\alpha} \left(A_1 B_2 - A_2 B_1\right) \neq 0. \tag{276}$$

We use the following equation

$$F = \mu \, \Phi^{-1} \, \frac{\partial \Phi}{\partial \lambda}, \tag{277}$$

in order to find the immersion function F.

We obtain F as

$$F = e_1 y_1 + e_2 y_2 + e_3 y_3 \tag{278}$$

where y_1, y_2, and y_3 are given as

$$y_1 = \Omega_{12}\Big(\Omega_{13}\,W_{19} + \Omega_{14}\,W_{20} + \Omega_{15}\,W_{21}\Big), \tag{279}$$

$$y_2 = \frac{\Omega_{12}}{2}\Big(\Omega_{13}\,W_{22} + \Omega_{14}\,W_{23} + \Omega_{15}\,W_{24}\Big), \tag{280}$$

$$y_3 = \frac{\Omega_{12}}{2}\Big(\Omega_{13}\,W_{25} + \Omega_{14}\,W_{26} + \Omega_{15}\,W_{27}\Big), \tag{281}$$

where

$$\Omega_{12} = \frac{\mu}{3\,\alpha^{2/3}\,\lambda^2\,(A_1\,B_2 - A_2\,B_1)\,(\alpha\,t + x)^{1/3}}, \tag{282}$$

$$\Omega_{13} = \frac{1}{2}\Big(3\,\lambda\,\alpha^{2/3}(\alpha\,t + x)^{1/3} + \alpha\,18^{1/3}\Big)\ \exp\{-2\,\lambda\Big(12\,\lambda^2\,\alpha^{1/3}t + 18^{2/3}\,(\alpha\,t + x)^{1/3}\Big)/(3\,\alpha^{1/3})\} \tag{283}$$

$$\Omega_{14} = \frac{1}{2}\Big(-3\,\lambda\,\alpha^2/3(\alpha\,t + x)^{1/3} + \alpha\,18^{1/3}\Big)\ \exp\{2\,\lambda\Big(12\,\lambda^2\,\alpha^{1/3}t + 18^{2/3}\,(\alpha\,t + x)^{1/3}\Big)/(3\,\alpha^{1/3})\} \tag{284}$$

$$\Omega_{15} = 2\,\lambda^2\,18^{2/3}\,\alpha^{1/3}\,(\alpha\,t + x)^{2/3} + 72\,\lambda^4\,\alpha^{2/3}\,t\,(\alpha\,t + x)^{1/3} + 18^{1/3}\,\alpha,$$

$$W_{19} = B_1\,B_2,\ W_{20} = A_1\,A_2,\ W_{21} = \frac{1}{2}\,(A_1\,B_2 + A_2\,B_1), \tag{285}$$

$$W_{22} = B_2^2 - B_1^2,\ W_{23} = A_2^2 - A_1^2,\ W_{24} = A_2\,B_2 - A_1\,B_1, \tag{286}$$

$$W_{25} = B_2^2 + B_1^2,\ W_{26} = A_2^2 + A_1^2,\ W_{27} = A_2\,B_2 + A_1\,B_1. \tag{287}$$

Example: Taking $\mu = 1$, $\alpha = 1$, $\lambda = 1$, $A_1 = 1$, $A_2 = -1$, $B_1 = -1$, $B_2 = -1$, in Eqs. (279) - (281), we get the surface given in Figure 14.

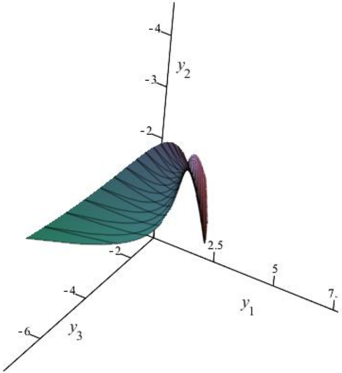
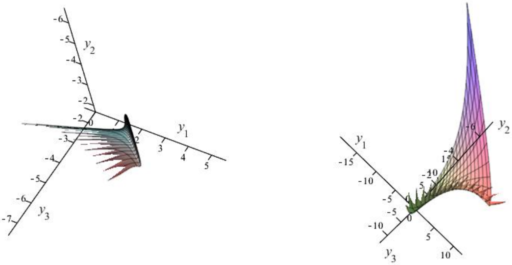


Figure: $(x, t) \in [-0.2, 0.2] \times [-0.2, 0.2]$

Example: Taking $\mu = 1$, $\alpha = 0.2$, $\lambda = 0.7$, $A_1 = 1$, $A_2 = -1$, $B_1 = -1$, $B_2 = -1$, in Eqs. (279) - (281), we get the surface given in Figure 15.



$a)$ $b)$

Figure: (a) $(x, t) \in [-0.2, 0.2] \times [-0.2, 0.2]$, (b) $(x, t) \in [-0.5, 0.5] \times [-0.5, 0.5]$

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