Gauss map of real hypersurfaces in complex projective space and submanifolds in complex 2-plane Grassmannians

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- Then the Gauss map $\gamma: M o \widetilde{G}_2(\mathbb{R}^{n+2}) \cong Q^n$ is defined by
- $\gamma(p) = x(p) \wedge N_p$ (B. Palmer, 1997).

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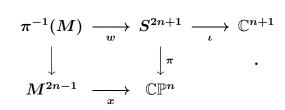
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- Also for parallel hypersurface $M_r:=\cos rx+\sin rN$ $(r\in\mathbb{R})$ of M, the Gauss image is not changed: $\gamma(M)=\gamma(M_r).$
- We define Gauss map $\gamma: M^{2n-1} \to \mathbb{G}_2(\mathbb{C}^{n+1})$ for real hypersurface M^{2n-1} in \mathbb{CP}^n .

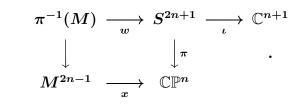
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• For $p \in M$, take a point $z_p \in \pi^{-1}(p) \subset \pi^{-1}(M)$ and let N'_p be a holizontal lift of unit normal of $M \subset \mathbb{CP}^n$ at z_p .

• If we put $\gamma(p) = \operatorname{span}_{\mathbb{C}} \{z_p, N'_p\}$, then the map $\gamma: M \to \mathbb{G}_2(\mathbb{C}^{n+1})$ is well-defined.

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- We call γ as the Gauss map of real hypersurface M in $\mathbb{CP}^n.$
- Note that for parallel hypersurface $M_r := \pi(\cos r z_p + \sin r N'_p)$ of M,image of the Gauss map $\gamma: M^{2n-1} \to \mathbb{CP}^n$ is not changed: $\gamma(M) = \gamma(M_r)$.

Hopf hypersurfaces in Kähler manifold

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- For a real hypersurface M^{2n-1} in Kähler manifold (\widetilde{M}^n, J) and a unit normal vector N,
- a vector $\xi := -JN$ tangent to M is called the structure vector of M.
- And when ξ is an eigenvector of the shape operator A of M, we call M a Hopf hypersurface in \widetilde{M} .

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- Conversely, if a Hopf hypersurface M in $\mathbb{CP}^n(4)$ satisfies $A\xi = \mu\xi$ (μ is necessarily constant), and for $r \in (0, \pi/2)$ with $\mu = 2 \cot 2r$, $r \in (0, \pi/2)$, if rank of the focal map $\phi_r : M \to \mathbb{CP}^n$ is constant, then

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- $\phi_r(M)$ is a complex submanifold of $\mathbb{CP}^n(4)$ and M lies on a tube over $\phi_r(M)$. (Cecil-Ryan, 1982).

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- In this talk, we will give a characterization of Hopf hypersurface M in \mathbb{CP}^n by using the Gauss map $\gamma: M \to \mathbb{G}_2(\mathbb{C}^{n+2}).$

- Here, \tilde{g} is a Riemannian metric of M, Q is a subbundle of End $T\widetilde{M}$ with rank 3, satisfying:

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- For each $p \in M$, there exists a neighborhood $U \ni p$, such that there exists local frame field $\{\tilde{I}_1, \tilde{I}_2, \tilde{I}_3\}$ of Q.

Quaternionic Kähler manifold

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• For each
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, \tilde{g} is invariant, i.e.,
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- Vector bundle Q is parallel with respect to the Levi-Civita connection of ğ at End TM.

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- with respect to the induced metric, (M, I) is an almost Hermitian manifold.

Totally complex submanifold of Q.K. manifold

• In particular, when almost Hermitian submanifold (M, \overline{g}, I) is Kähler, we call M a Kähler submanifold of quaternionic Kähler manifold \widetilde{M} .

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- Similarly, an almost Hermitian submanifold (M, \bar{g}, I) is called totally complex submanifold if at each point $p \in M$, with respect to $\tilde{L} \in Q_p$ which anti-commute with \tilde{I}_p , $\tilde{L}T_pM \perp T_pM$ hold.

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- In quaternionic K\u00e4hler manifold, a submanifold is totally complex if and only if it is K\u00e4hler (Alekseevsky-Marchiafava, 2001).

• Theorem (K., Diff. Geom. Appl. 2014) Let M^{2n-1} be a real hypersurface in complex projective space \mathbb{CP}^n , and let $\gamma: M \to \mathbb{G}_2(\mathbb{C}^{n+1})$ be the Gauss map.

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- Theorem (K., Diff. Geom. Appl. 2014) Let M²ⁿ⁻¹ be a real hypersurface in complex projective space CPⁿ, and let γ : M → G₂(Cⁿ⁺¹) be the Gauss map.
- If M is not Hopf, then the Gauss map γ is an immersion.
- If M is a Hopf hypersurface, then the image $\gamma(M)$ is a half-dimensional totally complex submanifold of $\mathbb{G}_2(\mathbb{C}^{n+1})$.
- And a Hopf hypersurface M in CPⁿ is a total space of a circle bundle over a Kähler manifold such that the fibration is nothing but the Gauss map γ : M → γ(M).

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- Then, for each point p in Σ , if we assign $ilde{I}_p \in Q_{arphi(p)}$,
- then we have a submanifold $\tilde{I}(\Sigma)$ of the twistor space $\mathcal{Z} = \{\tilde{I} \in Q | \ \tilde{I}^2 = -1\}$ of $\mathbb{G}_2(\mathbb{C}^{n+1})$ (natural lift).

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- Since Σ is a totally complex submanifold of G₂(Cⁿ⁺¹), *Ĩ*(Σ) is a Legendrian submanifold of the twistor space Z with respect to a complex contact structure (Alekseevsky-Marchiafava, 2004).

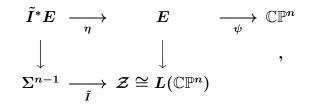
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- its parallel hypersurface $\phi_r(\tilde{I}^*E)$ gives Hopf hypersurface with $A\xi = 2 \tan 2r\xi$ (on open subset of regular points of $M = \tilde{I}^*E$).

Remarks

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- Recently K. Tsukada proved that conormal bundle of any complex submanifold in CPⁿ is realized as a half dimensional totally complex submanifold in G₂(Cⁿ⁺¹).
- For real hypersurfaces in complex hyperbolic space \mathbb{CH}^n , we define Gauss map $\gamma: M \to \mathbb{G}_{1,1}(\mathbb{C}_1^{n+1})$, and
- we obtain similar results for Hopf hypersurfaces in CHⁿ by using para-quaternionic Kähler structure (J.T. Cho and M.K., Topol. Appl. 2015).

Split-quaternions

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$$\widetilde{\mathbb{H}} = C(2,0) = C(1,1)$$
, Split-quaternions (or
coquaternions, para-quaternions):
 $q = q_0 + iq_1 + jq_2 + kq_3$, $i^2 = -1$, $j^2 = k^2 = 1$,
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- Introduced by James Cockle in 1849.

Para-quaternionic structure

• $\{I_1, I_2, I_3\}$, $I_1^2 = -1$, $I_2^2 = I_3^2 = 1$, $I_1I_2 = -I_2I_1 = -I_3$, $I_2I_3 = -I_3I_2 = I_1$, $I_3I_1 = -I_1I_3 = -I_2$ gives para-quaternionic structure,

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 gives para-quaternionic structure,
• $\tilde{V} = \{aI_1 + bI_2 + cI_3 | a, b, c \in \mathbb{R}\} \cong \mathfrak{su}(1, 1) \cong \mathbb{R}^3_1, and$

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- $ilde{V}=\{aI_1+bI_2+cI_3|~a,b,c\in\mathbb{R}\}\cong\mathfrak{su}(1,1)\cong\mathbb{R}^3_1,$ and
- $Q_+ = \{I \in \tilde{V} | I^2 = 1\} \cong S_1^2$: de-Sitter space, $Q_- = \{I \in \tilde{V} | I^2 = -1\} \cong H^2$: hyperbolic space, $Q_0 = \{I \in \tilde{V} | I^2 = 0, I \neq 0\} \cong$ lightcone.



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- Suppose M is a Hopf hypersurface with $|\mu|>2$ (resp. $0\leq |\mu|<2$).
- Then g(M) is a real (2n-2)-dimensional submanifold of $\mathbb{G}_{1,1}(\mathbb{C}_1^{n+1})$, and

• There exist sections $\tilde{I}_{\tilde{1}}$, $\tilde{I}_{\tilde{2}}$ and \tilde{I}_{3} of the bundle $\tilde{Q}|_{g(M)}$ of the para-quaternionic Kähler structure such that

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• such that $dg_x(T_xM)$ is invariant under \tilde{I}_1 and $\tilde{I}_2 dg_x(T_xM), \tilde{I}_3 dg_x(T_xM)$ are orthogonal to $dg_x(T_xM)$.

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• When $|\mu| > 2$, p and q are both even.

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$$p = \sum_{ert \lambda ert > 1} \dim \{ X ert \ AX = \lambda X, \ X \perp \xi \},$$
 $q = \sum_{ert \lambda ert < 1} \dim \{ X ert \ AX = \lambda X, \ X \perp \xi \}.$

- When $|\mu| > 2$, p and q are both even.
- When $0 \leq |\mu| < 2$, we have p = q.

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- ullet then the induced metric of g(M) is non-degenerate and
- g(M) is a pseudo-Kähler (resp. para-Kähler) submanifold of C_{1,1}(Cⁿ⁺¹).



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- Suppose M is a Hopf hypersurface with $|\mu| = 2$.
- Then g(M) is a real (2n-2)-dimensional submanifold of $\mathbb{G}_{1,1}(\mathbb{C}_1^{n+1})$, and

• There exist sections $\tilde{I}_{ ilde{1}}$ and $\tilde{I}_{ ilde{2}}$ of the bundle $\tilde{Q}|_{g(M)}$ of the para-quaternionic Kähler structure such that

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• such that $ilde{I}_{1}dg_{x}(T_{x}M), ilde{I}_{2}dg_{x}(T_{x}M)$ are orthogonal to $dg_{x}(T_{x}M)$.

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• satisfies $p+q \leq n-1$.