Lorentzian surfaces in semi-Euclidean spaces and their Gauss map

Nurettin Cenk Turgay

Istanbul Technical University, Turkey
(TÜBİTAK Project Number: 114F199)

XIIth International Conference Geometry, Integrability and Quantization

June 5-10, 2015

Acknowledgements

Acknowledgements

We would like to present the results obtained in

- E.Ö. Canfes, M. '*On the Gauss map of minimal Lorentzian surfaces in 4-dimensional semi-Euclidean spaces*'(submitted)
- M., '*Some classifications of Lorentzian surfaces with finite type Gauss map in the Minkowski 4-space*'(accepted) J. Aust. Math. Soc
- Y. Fu and M., '*Complete classification of biconservative hypersurfaces with diagonalizable shape operator in Minkowski 4-space*'(submitted)

114F199

This work was obtained during the TÜBİTAK 1001 project (Project Name: "Y_EUCL2TIP", Project Number: 114F199).

1. Gauss map of Lorentzian surfaces
   - Pointwise 1-type Gauss map
   - Minimal Lorentzian surfaces
   - Semi-Euclidean space $\mathbb{E}^4_2$
2. Biconservative hypersurfaces
   - Biharmonic submanifolds
   - Hypersurfaces with diagonalizable shape operator
3. Other problems
   - Biconservative hypersurfaces
   - A family of hypersurfaces
   - Hypersurfaces with vanishing Gauss-Kronecker curvature

Notation

- $\mathbb{E}^m_t$ semi-Euclidean m-space with the canonical semi-Euclidean metric tensor of index t given by

$$\langle\ ,\ \rangle = \langle\ ,\ \rangle^m_t = -\sum_{i=1}^{t} dx_i^2 + \sum_{j=t+1}^{m} dx_j^2,$$

- $M \hookrightarrow \mathbb{E}^m_t$: A (semi-)Riemannian submanifold of $\mathbb{E}^m_t$,
- $\widetilde{\nabla}$ and ∇: Levi-Civita connections of $\mathbb{E}^m_s$ and M,
- h: Second fundemental form of M, $\|h\|^2$ 'squared' norm of h,
- A_ξ: Shape operator along ξ, D: Normal connection,
- R, R^D: Curvature tensor and normal curvature tensor of M,
- Δ: Laplace operator of M.

Basic Equations

- Gauss formula $\widetilde{\nabla}_X Y = \nabla_X Y + h(X, Y)$,
- Weingarten formula $\widetilde{\nabla}_X \xi = -A_\xi(X) + D_X \xi$,
- Gauss equation
  $R(X, Y, Z, W) = \langle h(Y, Z), h(X, W)\rangle - \langle h(X, Z), h(Y, W)\rangle$,
- Codazzi equation $(\bar{\nabla}_X h)(Y, Z) = (\bar{\nabla}_Y h)(X, Z)$,
- Ricci equation $\langle R^D(X, Y)\xi, \eta\rangle = \langle [A_\xi, A_\eta]X, Y\rangle$.

Section 1:

Gauss map of Lorentzian surfaces in 4-dimensional semi-Euclidean spaces

Gauss map

Now, let $M^n_r \hookrightarrow \mathbb{E}^m_t$ be oriented.

Gauss map

Now, let $M_r^n \hookrightarrow \mathbb{E}_t^m$ be oriented. Consider a local orthonormal frame field $\{e_1, e_2, \ldots, e_n\}$ of the tangent space of M.

Gauss map

Now, let $M^n_r \hookrightarrow \mathbb{E}^m_t$ be oriented. Consider a local orthonormal frame field $\{e_1, e_2, \ldots, e_n\}$ of the tangent space of M.
The (tangent) Gauss map of M is defined by

$$\begin{aligned} \nu : M &\rightarrow G(n,m) \subset \Lambda^{m,n} \cong \mathbb{E}^N_S \\ p &\mapsto \nu(p) = (e_1 \wedge e_2 \wedge \ldots \wedge e_n)(p). \end{aligned}$$

Gauss map

Now, let $M^n_r \hookrightarrow \mathbb{E}^m_t$ be oriented. Consider a local orthonormal frame field $\{e_1, e_2, \ldots, e_n\}$ of the tangent space of M.
The (tangent) Gauss map of M is defined by

$$\begin{array}{rcl} \nu : M & \rightarrow & G(n,m) \subset \Lambda^{m,n} \cong \mathbb{E}^N_S \\ p & \mapsto & \nu(p) = (e_1 \wedge e_2 \wedge \ldots \wedge e_n)(p). \end{array}$$

Note that, we have either $\underbrace{\nu(M) \subset \mathbb{S}^{N-1}_S(1)}_{r \text{ is even}}$ or $\underbrace{\nu(M) \subset \mathbb{H}^{N-1}_{S-1}(-1)}_{r \text{ is odd}}$.

Gauss map

Now, let $M_r^n \hookrightarrow \mathbb{E}_t^m$ be oriented. Consider a local orthonormal frame field $\{e_1, e_2, \ldots, e_n\}$ of the tangent space of M.
The (tangent) Gauss map of M is defined by

$$\begin{array}{rcl} \nu : M & \rightarrow & G(n,m) \subset \Lambda^{m,n} \cong \mathbb{E}_S^N \\ p & \mapsto & \nu(p) = (e_1 \wedge e_2 \wedge \ldots \wedge e_n)(p). \end{array}$$

Note that, we have either $\underbrace{\nu(M) \subset \mathbb{S}_S^{N-1}(1)}_{r \text{ is even}}$ or $\underbrace{\nu(M) \subset \mathbb{H}_{S-1}^{N-1}(-1)}_{r \text{ is odd}}$.

Remark: Codimension 1

If codimension of M is one, then one may put $(e_1 \wedge e_2 \wedge \ldots \wedge e_n) = N$ to get the definition of classical Gauss map of (hyper)surfaces, where N is the unit normal vector field of M.

Pointwise 1-type Gauss map

Let M be an oriented (semi)-Riemannian submanifold of $\mathbb{E}^m_t$ and ν its Gauss map. Then,

Pointwise 1-type Gauss map

Let M be an oriented (semi)-Riemannian submanifold of $\mathbb{E}^m_t$ and ν its Gauss map. Then,

1-type Gauss map

$\Delta\nu = \lambda(\nu + C)$

Pointwise 1-type Gauss map

Let M be an oriented (semi)-Riemannian submanifold of $\mathbb{E}^m_t$ and ν its Gauss map. Then,

1-type Gauss map

$\Delta\nu = \lambda(\nu + C)$

Pointwise 1-type Gauss map

(Proper) pointwise 1-type Gauss map if

$$\Delta\nu = f(\nu + C)$$

(non-constant) function f

Pointwise 1-type Gauss map

Let M be an oriented (semi)-Riemannian submanifold of $\mathbb{E}^m_t$ and ν its Gauss map. Then,

1-type Gauss map

$\Delta\nu = \lambda(\nu + C)$

Pointwise 1-type Gauss map

(Proper) pointwise 1-type Gauss map if

$$\Delta\nu = f(\nu + C)$$

(non-constant) function f

A pointwise 1-type Gauss map is said to be

- of the first kind if $C = 0$ ($\Delta\nu = f\nu$),

Pointwise 1-type Gauss map

Let M be an oriented (semi)-Riemannian submanifold of $\mathbb{E}^m_t$ and ν its Gauss map. Then,

1-type Gauss map

$\Delta\nu = \lambda(\nu + C)$

Pointwise 1-type Gauss map

(Proper) pointwise 1-type Gauss map if

$$\Delta\nu = f(\nu + C)$$

(non-constant) function f

A pointwise 1-type Gauss map is said to be

- of the first kind if $C = 0$ ($\Delta\nu = f\nu$),
- of the second kind if $C \neq 0$ and $f \neq 0$

Section 1.2:

Minimal Lorentzian surfaces in 4-dimensional semi-Euclidean spaces

Lorentzian surfaces

Let M be a Lorentzian surface in $\mathbb{E}^4_r$, $r = 1, 2.$. Consider a local pseudo-orthonormal frame field f_1, f_2 of the tangent space of M.

Lorentzian surfaces

Let M be a Lorentzian surface in $\mathbb{E}^4_r$, $r = 1, 2.$. Consider a local pseudo-orthonormal frame field f_1, f_2 of the tangent space of M.
$(\langle f_1, f_1\rangle = \langle f_2, f_2\rangle = 0, \quad \langle f_1, f_2\rangle = -1)$

Lorentzian surfaces

Let M be a Lorentzian surface in $\mathbb{E}^4_r$, $r = 1, 2.$. Consider a local pseudo-orthonormal frame field f_1, f_2 of the tangent space of M. $(\langle f_1, f_1\rangle = \langle f_2, f_2\rangle = 0, \quad \langle f_1, f_2\rangle = -1)$

- Mean curvature vector: $H = \frac{1}{2}\mathrm{tr} h = -h(f_1, f_2)$,

Lorentzian surfaces

Let M be a Lorentzian surface in $\mathbb{E}^4_r$, $r = 1, 2$.. Consider a local pseudo-orthonormal frame field f_1, f_2 of the tangent space of M. $(\langle f_1, f_1\rangle = \langle f_2, f_2\rangle = 0, \quad \langle f_1, f_2\rangle = -1)$

- Mean curvature vector: $H = \frac{1}{2}\mathrm{tr} h = -h(f_1, f_2)$,
- Gaussian Curvature: $K = R(f_1, f_2, f_2, f_1)$,

Lorentzian surfaces

Let M be a Lorentzian surface in $\mathbb{E}^4_r$, $r = 1, 2$.. Consider a local pseudo-orthonormal frame field f_1, f_2 of the tangent space of M. $(\langle f_1, f_1\rangle = \langle f_2, f_2\rangle = 0, \quad \langle f_1, f_2\rangle = -1)$

- Mean curvature vector: $H = \frac{1}{2}\mathrm{tr} h = -h(f_1, f_2)$,
- Gaussian Curvature: $K = R(f_1, f_2, f_2, f_1)$,
- Normal Curvature: $K^D = R^D(f_1, f_2; e_3, e_4)$,

Lorentzian surfaces

Let M be a Lorentzian surface in $\mathbb{E}^4_r$, $r = 1, 2..$ Consider a local pseudo-orthonormal frame field f_1, f_2 of the tangent space of M.
$(\langle f_1, f_1\rangle = \langle f_2, f_2\rangle = 0, \quad \langle f_1, f_2\rangle = -1)$

- Mean curvature vector: $H = \frac{1}{2}\mathrm{tr}h = -h(f_1, f_2)$,
- Gaussian Curvature: $K = R(f_1, f_2, f_2, f_1)$,
- Normal Curvature: $K^D = R^D(f_1, f_2; e_3, e_4)$,
- Laplace operator: $\Delta = f_1 f_2 + f_2 f_1 - \nabla_{f_1} f_2 - \nabla_{f_2} f_1$.

Local coordinates on Lorentzian surfaces

Theorem

[a] Let M be a Lorentzian surface in a semi-Euclidean space $\mathbb{E}^q_r$. Then, there exist local coordinates (s,t) such that the induced metric is of the form $g = -m^2(dsdt + dtds)$.

Local coordinates on Lorentzian surfaces

Theorem

[a] Let M be a Lorentzian surface in a semi-Euclidean space $\mathbb{E}^q_r$. Then, there exist local coordinates (s, t) such that the induced metric is of the form $g = -m^2(dsdt + dtds)$.Moreover,

- The Levi-Civita connection of M satisfies $\nabla_{\partial_s}\partial_t = 0$,

Local coordinates on Lorentzian surfaces

Theorem

[a] Let M be a Lorentzian surface in a semi-Euclidean space $\mathbb{E}^q_r$. Then, there exist local coordinates (s, t) such that the induced metric is of the form $g = -m^2(dsdt + dtds)$.Moreover,

- The Levi-Civita connection of M satisfies $\nabla_{\partial_s}\partial_t = 0$,
- Second fundemental form satisfies $h(\partial_s, \partial_t) = -m^2 H$.

[a] See [Chen, 'Dependence of the Gauss-Codazzi equations and the Ricci equation of Lorentz surface'].

Local coordinates on Lorentzian surfaces

Theorem

[a] Let M be a Lorentzian surface in a semi-Euclidean space $\mathbb{E}^q_r$. Then, there exist local coordinates (s,t) such that the induced metric is of the form $g = -m^2(dsdt + dtds)$.Moreover,

- The Levi-Civita connection of M satisfies $\nabla_{\partial_s}\partial_t = 0$,
- Second fundemental form satisfies $h(\partial_s, \partial_t) = -m^2 H$.

[a] See [Chen, 'Dependence of the Gauss-Codazzi equations and the Ricci equation of Lorentz surface'].

Corollary

$$H = 0 \ \Leftrightarrow \widetilde{\nabla}_{\partial_s}\partial_t = 0$$

Minimal Lorentzian surfaces

Theorem

[a] Every minimal Lorentzian surface in $\mathbb{E}^m_s$ is locally congruent to a translation surface defined by

$$x(s,t) = \alpha(s) + \beta(t),$$

where $\alpha(s)$ and $\beta(t)$ are two null curves.

[a] See [Y. Fu and Z.-H. Hou, J. Math. Anal. Appl., 371, 25–40 (2010).].

Section 1.3:

Gauss map of Lorentzian minimal surfaces in the space $\mathbb{E}^4_2$

See [M., '*Some classifications of Lorentzian surfaces with finite type Gauss map in the Minkowski 4-space*'(accepted) J. Aust. Math. Soc.].

Lorentzian Surfaces in $\mathbb{E}^4_2$

Let M^2_1 be a minimal surface in $\mathbb{E}^4_2$. Consider the local frame field $f_1, f_2; f_3, f_4$. Note that we have $h(f_1, f_2) = 0$ because of minimality of M.

Lorentzian Surfaces in $\mathbb{E}^4_2$

Let M^2_1 be a minimal surface in $\mathbb{E}^4_2$. Consider the local frame field $f_1, f_2; f_3, f_4$. Note that we have $h(f_1, f_2) = 0$ because of minimality of M.

Then its tangent Gauss map ν is defined by

$$\begin{array}{rcl} \nu : M & \rightarrow & \mathbb{H}^5_3(-1) \subset \mathbb{E}^6_4 \\ p & \mapsto & \nu(p) = (f_1 \wedge f_2)(p). \end{array}$$

Lorentzian Surfaces in $\mathbb{E}^4_2$

Let M^2_1 be a minimal surface in $\mathbb{E}^4_2$. Consider the local frame field $f_1, f_2; f_3, f_4$. Note that we have $h(f_1, f_2) = 0$ because of minimality of M.

Then its tangent Gauss map ν is defined by

$$\begin{array}{rcl} \nu : M & \rightarrow & \mathbb{H}^5_3(-1) \subset \mathbb{E}^6_4 \\ p & \mapsto & \nu(p) = (f_1 \wedge f_2)(p). \end{array}$$

The Gauss map ν of M satisfies

$$\Delta\nu = 2K\nu + 2h(f_1, f_1) \wedge h(f_2, f_2), \tag{1}$$

Lorentzian Surfaces with $\Delta\nu = f\nu$

We have $\Delta\nu = f\nu \Leftrightarrow f\nu = 2K\nu + 2h(f_1, f_1) \wedge h(f_2, f_2)$
Thus, we have

Corollary

$\Delta\nu = f\nu$ if and only if $h(f_1, f_1) \wedge h(f_2, f_2) = 0$. In this case, $f = 2K$

Lorentzian Surfaces with $\Delta\nu = f\nu$

We have $\Delta\nu = f\nu \Leftrightarrow f\nu = 2K\nu + 2h(f_1, f_1) \wedge h(f_2, f_2)$
Thus, we have

Corollary

$\Delta\nu = f\nu$ if and only if $h(f_1, f_1) \wedge h(f_2, f_2) = 0$. In this case, $f = 2K$

or, equivalently,
$(\Delta\nu = f\nu) \Leftrightarrow h(f_1, f_1) = \zeta h(f_2, f_2)$.

Lorentzian Surfaces with $\Delta\nu = f\nu$

We have $\Delta\nu = f\nu \Leftrightarrow f\nu = 2K\nu + 2h(f_1, f_1) \wedge h(f_2, f_2)$
Thus, we have

Corollary

$\Delta\nu = f\nu$ if and only if $h(f_1, f_1) \wedge h(f_2, f_2) = 0$. In this case, $f = 2K$

or, equivalently,
$(\Delta\nu = f\nu) \Leftrightarrow h(f_1, f_1) = \zeta h(f_2, f_2)$.
Note that we have 3 cases.

1. $h(f_1, f_1) = h(f_2, f_2) = 0$ (In this case M totally geodesic in $\mathbb{E}^4_2$);
2. $h(f_1, f_1) = 0$, $h(f_2, f_2) \neq 0$;
3. $h(f_1, f_1) = \zeta h(f_2, f_2)$, ζ is non-zero:
    - $h(f_1, f_1)$ and $h(f_2, f_2)$ are casual and linearly dependent;
    - $h(f_1, f_1)$ and $h(f_2, f_2)$ are light-like and linearly dependent.

Lorentzian Surfaces with $\Delta\nu = f\nu$

Case 2

$h(f_1, f_1) = 0$. In this case, we obtain M is congruent to

$$x(s,t) = s\eta_0 + \beta(t), \tag{2}$$

where η_0 is a constant light-like vector and β is a null curve in $\mathbb{E}^4_2$.

Lorentzian Surfaces with $\Delta\nu = f\nu$

Case 2

$h(f_1, f_1) = 0$. In this case, we obtain M is congruent to

$$x(s,t) = s\eta_0 + \beta(t), \tag{2}$$

where η_0 is a constant light-like vector and β is a null curve in $\mathbb{E}^4_2$.

Remark

The hypersurface given in (2) has the following property:

Lorentzian Surfaces with $\Delta\nu = f\nu$

Case 2

$h(f_1, f_1) = 0$. In this case, we obtain M is congruent to

$$x(s,t) = s\eta_0 + \beta(t), \quad (2)$$

where η_0 is a constant light-like vector and β is a null curve in $\mathbb{E}^4_2$.

Remark

The hypersurface given in (2) has the following property:
It has degenerated relative null space
$\mathcal{N}_p(M) = \{X \in T_pM | h(X,Y) = 0, \text{ for all } Y \in T_pM\}$.

Lorentzian Surfaces with $\Delta\nu = f\nu$

Case 3a

$h(f_1, f_1) = \zeta h(f_2, f_2)$ are non-vanishing and they not light-like. In this case, we obtain that M is lying on a hyperplane of $\mathbb{E}^4_2$ by the following way.

Lorentzian Surfaces with $\Delta\nu = f\nu$

Case 3a

$h(f_1, f_1) = \zeta h(f_2, f_2)$ are non-vanishing and they not light-like. In this case, we obtain that M is lying on a hyperplane of $\mathbb{E}^4_2$ by the following way.

By the assumption, we see that we may re-define s, t as $e_3 = h(\partial_s, \partial_s) = \pm h(\partial_t, \partial_t)$ for a unit normal vector field e_3.

Lorentzian Surfaces with $\Delta\nu = f\nu$

Case 3a

$h(f_1, f_1) = \zeta h(f_2, f_2)$ are non-vanishing and they not light-like. In this case, we obtain that M is lying on a hyperplane of $\mathbb{E}^4_2$ by the following way.

By the assumption, we see that we may re-define s, t as $e_3 = h(\partial_s, \partial_s) = \pm h(\partial_t, \partial_t)$ for a unit normal vector field e_3. An instance of Codazzi equation yields that e_3 is parallel. Thus, $De_4 = 0$, where e_4 is perpendicular to e_3.

Lorentzian Surfaces with $\Delta\nu = f\nu$

Case 3a

$h(f_1, f_1) = \zeta h(f_2, f_2)$ are non-vanishing and they not light-like. In this case, we obtain that M is lying on a hyperplane of $\mathbb{E}^4_2$ by the following way.

By the assumption, we see that we may re-define s, t as $e_3 = h(\partial_s, \partial_s) = \pm h(\partial_t, \partial_t)$ for a unit normal vector field e_3. An instance of Codazzi equation yields that e_3 is parallel. Thus, $De_4 = 0$, where e_4 is perpendicular to e_3. Moreover, we have $A_4 = 0$ because of $e_3 = h(\partial_s, \partial_s) = \pm h(\partial_t, \partial_t)$. Hence e_4 is constant which completes the proof.

Lorentzian Surfaces with $\Delta\nu = f\nu$

Case 3b

$h(f_1, f_1), h(f_2, f_2) \neq 0$, $h(f_1, f_1) = \zeta h(f_2, f_2)$ and they are light-like.

Lorentzian Surfaces with $\Delta\nu = f\nu$

Case 3b

$h(f_1, f_1), h(f_2, f_2) \neq 0$, $h(f_1, f_1) = \zeta h(f_2, f_2)$ and they are light-like. In this case, we obtain that M is lying on a degenerated hyperplane of $\mathbb{E}^4_2$ and it is congruent to

$$x(s,t) = \left(\phi_1(s) + \phi_2(t), \frac{\sqrt{2}}{2}(s+t), \frac{\sqrt{2}}{2}(s-t), \phi_1(s) + \phi_2(t)\right)$$

for some smooth functions ϕ_1 and ϕ_2.

Lorentzian minimal surfaces with $\Delta\nu = f\nu$

Let M be a Lorentzian minimal surface in $\mathbb{E}^4_2$. Then, they have pointwise 1-type Gauss map of the first kind iff

- A surface with degenerated relative null space given $x(s,t) = s\eta_0 + \beta(t)$

Lorentzian minimal surfaces with $\Delta\nu = f\nu$

Let M be a Lorentzian minimal surface in $\mathbb{E}^4_2$. Then, they have pointwise 1-type Gauss map of the first kind iff

- A surface with degenerated relative null space given $x(s,t) = s\eta_0 + \beta(t)$ $((h(f_1, f_1) = 0)$;

Lorentzian minimal surfaces with $\Delta\nu = f\nu$

Let M be a Lorentzian minimal surface in $\mathbb{E}^4_2$. Then, they have pointwise 1-type Gauss map of the first kind iff

- A surface with degenerated relative null space given $x(s,t) = s\eta_0 + \beta(t)$ ($(h(f_1, f_1) = 0)$);
- A minimal Lorentzian surface lying in a hyperplane $\mathbb{E}^3_2$ of $\mathbb{E}^4_2$

Lorentzian minimal surfaces with $\Delta\nu = f\nu$

Let M be a Lorentzian minimal surface in $\mathbb{E}^4_2$. Then, they have pointwise 1-type Gauss map of the first kind iff

- A surface with degenerated relative null space given $x(s,t) = s\eta_0 + \beta(t)$ ($(h(f_1, f_1) = 0)$;
- A minimal Lorentzian surface lying in a hyperplane $\mathbb{E}^3_2$ of $\mathbb{E}^4_2$ $(h(f_1, f_1)$ is time-like);

Lorentzian minimal surfaces with $\Delta\nu = f\nu$

Let M be a Lorentzian minimal surface in $\mathbb{E}^4_2$. Then, they have pointwise 1-type Gauss map of the first kind iff

- A surface with degenerated relative null space given $x(s,t) = s\eta_0 + \beta(t)$ ($(h(f_1, f_1) = 0)$);
- A minimal Lorentzian surface lying in a hyperplane $\mathbb{E}^3_2$ of $\mathbb{E}^4_2$ ($h(f_1, f_1)$ is time-like);
- A minimal Lorentzian surface lying in a hyperplane $\mathbb{E}^3_1$ of $\mathbb{E}^4_2$

Lorentzian minimal surfaces with $\Delta\nu = f\nu$

Let M be a Lorentzian minimal surface in $\mathbb{E}^4_2$. Then, they have pointwise 1-type Gauss map of the first kind iff

- A surface with degenerated relative null space given $x(s,t) = s\eta_0 + \beta(t)$ ($(h(f_1, f_1) = 0)$);
- A minimal Lorentzian surface lying in a hyperplane $\mathbb{E}^3_2$ of $\mathbb{E}^4_2$ ($h(f_1, f_1)$ is time-like);
- A minimal Lorentzian surface lying in a hyperplane $\mathbb{E}^3_1$ of $\mathbb{E}^4_2$ ($h(f_1, f_1)$ is space-like).

Lorentzian minimal surfaces with $\Delta\nu = f\nu$

Let M be a Lorentzian minimal surface in $\mathbb{E}^4_2$. Then, they have pointwise 1-type Gauss map of the first kind iff

- A surface with degenerated relative null space given $x(s,t) = s\eta_0 + \beta(t)$ ($(h(f_1, f_1) = 0)$);
- A minimal Lorentzian surface lying in a hyperplane $\mathbb{E}^3_2$ of $\mathbb{E}^4_2$ $(h(f_1, f_1)$ is time-like);
- A minimal Lorentzian surface lying in a hyperplane $\mathbb{E}^3_1$ of $\mathbb{E}^4_2$ $(h(f_1, f_1)$ is space-like).
- A minimal Lorentzian surface given by

$$x(s,t) = \left(\phi_1(s) + \phi_2(t), \frac{\sqrt{2}}{2}(s+t), \frac{\sqrt{2}}{2}(s-t), \phi_1(s) + \phi_2(t)\right)$$

Lorentzian minimal surfaces with $\Delta\nu = f\nu$

Let M be a Lorentzian minimal surface in $\mathbb{E}^4_2$. Then, they have pointwise 1-type Gauss map of the first kind iff

- A surface with degenerated relative null space given $x(s,t) = s\eta_0 + \beta(t)$ $((h(f_1,f_1) = 0)$;
- A minimal Lorentzian surface lying in a hyperplane $\mathbb{E}^3_2$ of $\mathbb{E}^4_2$ $(h(f_1,f_1)$ is time-like);
- A minimal Lorentzian surface lying in a hyperplane $\mathbb{E}^3_1$ of $\mathbb{E}^4_2$ $(h(f_1,f_1)$ is space-like).
- A minimal Lorentzian surface given by

$$x(s,t) = \left(\phi_1(s)+\phi_2(t), \frac{\sqrt{2}}{2}(s+t), \frac{\sqrt{2}}{2}(s-t), \phi_1(s)+\phi_2(t)\right)$$

$(h(f_1,f_1)$ is light-like).

Classification Theorem

Hence, we have

Theorem

Let M be a minimal Lorentzian surface properly contained by the semi-Euclidean space $\mathbb{E}^4_2$. Then, the following statements are equivalent.

Classification Theorem

Hence, we have

Theorem

Let M be a minimal Lorentzian surface properly contained by the semi-Euclidean space $\mathbb{E}^4_2$. Then, the following statements are equivalent.

(i) M has pointwise 1-type Gauss map of the first kind ($\Delta\nu = f\nu$);

(ii) M has harmonic Gauss map ($\Delta\nu = 0$);

Classification Theorem

Hence, we have

Theorem

Let M be a minimal Lorentzian surface properly contained by the semi-Euclidean space $\mathbb{E}^4_2$. Then, the following statements are equivalent.

(i) M has pointwise 1-type Gauss map of the first kind ($\Delta\nu = f\nu$);

(ii) M has harmonic Gauss map ($\Delta\nu = 0$);

(iii) M is congruent to one of following surfaces

$$x(s,t) = s\eta_0 + \beta(t),$$

$$x(s,t) = \left(\phi_1(s) + \phi_2(t), \frac{\sqrt{2}}{2}(s+t), \frac{\sqrt{2}}{2}(s-t), \phi_1(s) + \phi_2(t)\right).$$

Lorentzian Surfaces with $\Delta\nu = f(\nu + C)$

Proposition

Let M be a minimal Lorentzian surface properly contained by the semi-Euclidean space $\mathbb{E}^4_2$.

Lorentzian Surfaces with $\Delta\nu = f(\nu + C)$

Proposition

Let M be a minimal Lorentzian surface properly contained by the semi-Euclidean space $\mathbb{E}^4_2$. Then, $\Delta\nu = f(\nu + C)$ if and only if

Lorentzian Surfaces with $\Delta\nu = f(\nu + C)$

Proposition

Let M be a minimal Lorentzian surface properly contained by the semi-Euclidean space $\mathbb{E}^4_2$. Then, $\Delta\nu = f(\nu + C)$ if and only if $h(f_1, f_1)$ and $h(f_2, f_2)$ are light-like and linearly independent.

Classification of such surfaces:

$$x(s,t) = (\phi_1(s) + \phi_2(t), s + t, s + \cos c\ t + \sin c\ \phi_2(t), \\ \phi_1(s) - \sin c\ t + \cos c\ \phi_2(t))$$

Section 2:

Biconservative hypersurfaces in the Minkowski space $\mathbb{E}^4_1$

See [Fu and M., '*Complete classification of biconservative hypersurfaces with diagonalizable shape operator in Minkowski 4-space*'(submitted)].

Biharmonic submanifolds

Let M be an n-dimensional (semi-)Riemannian submanifold of a (semi-)Euclidean space.

Biharmonic submanifold

M is said to be biharmonic if $\Delta^2 x = 0$

Biharmonic submanifolds

Let M be an n-dimensional (semi-)Riemannian submanifold of a (semi-)Euclidean space.

Biharmonic submanifold

M is said to be biharmonic if $\Delta^2 x = 0$

The following formula is well-known.

Laplace-Beltrami formula

$\Delta x = \varepsilon n H$, where H is the mean curvature vector of M.

Chen's conjecture

A direct corollary of Laplace-Beltrami formula

Corollary

If M is minimal, then it is biharmonic.

Chen's conjecture

A direct corollary of Laplace-Beltrami formula

Corollary

If M is minimal, then it is biharmonic.

Thus, the following open problem arises:

Chen's Biharmonic Conjecture

Let M be a submanifold of an Euclidean space. Then, it is biharmonic if and only if it is minimal

Biconservative hypersurfaces

Now, consider a (semi-)Riemannian hypersurface M be of a (semi-)Euclidean space and let H denote its first mean curvature.

Biconservative hypersurfaces

M is said to be biconservative[a] if $(\Delta^2 x)^T = 0$

[a] Some authors have used the term 'H-Hypersurface'

Biconservative hypersurfaces

Now, consider a (semi-)Riemannian hypersurface M be of a (semi-)Euclidean space and let H denote its first mean curvature.

Biconservative hypersurfaces

M is said to be biconservative[a] if $(\Delta^2 x)^T = 0$

[a] Some authors have used the term 'H-Hypersurface'

Note that by a direct computation using Laplace-Beltrami formula, we have

Corollary

M is biconservative if and only if $S(\nabla H) = cH\nabla H$, where c is a 'constant' depending on the index and dimension of M.

Hypersurfaces in $\mathbb{E}^4_1$

It is well-known that the shape operator of a hypersurface in $\mathbb{E}^4_1$ takes one of the following 4 forms.

$$\text{Case I. } S = \begin{pmatrix} k_1 & 0 & 0 \\ 0 & k_2 & 0 \\ 0 & 0 & k_3 \end{pmatrix}, \quad \text{Case II. } S = \begin{pmatrix} k_1 & 1 & 0 \\ 0 & k_1 & 0 \\ 0 & 0 & k_3 \end{pmatrix},$$

$$\text{Case III. } S = \begin{pmatrix} k_1 & 0 & 0 \\ 0 & k_1 & 1 \\ -1 & 0 & k_1 \end{pmatrix}, \quad \text{Case IV. } S = \begin{pmatrix} k_1 & -\nu & 0 \\ \nu & k_1 & 1 \\ -1 & 0 & k_3 \end{pmatrix},$$

for some smooth functions k_1, k_2, k_3, k_4 and ν.

Biconservative hypersurfaces in $\mathbb{E}^4_1$

We have obtained the following families of biconservative hypersurfaces with diagonalizable shape operator.

Two distinct principal curvatures

- $x_1(s,t,u) = (f_1(s), s\cos t\sin u, s\sin t\sin u, s\cos u);$
- $x_2(s,t,u) = (s\sinh u\sin t, s\cosh u\sin t, s\cos t, f_2(s));$
- $x_3(s,t,u) = (s\cosh t, s\sinh t\sin u, \sinh t\cos u, f_3(s));$
- $x_4(s,t,u) =$
  $\left(\frac{1}{2}s(t^2+u^2)+s+f_4(s), st, su, \frac{1}{2}s(t^2+u^2)+f_4(s)\right).$

Biconservative hypersurfaces in $\mathbb{E}^4_1$

Zero Gauss-Kronecker Curvature

- A generalized cylinder $M^2_0 \times \mathbb{E}^1_1$ where M is a biconservative surface in $\mathbb{E}^3$;
- A generalized cylinder $M^2_0 \times \mathbb{E}^1$ where M is a biconservative Riemannian surface in $\mathbb{E}^3_1$;
- A generalized cylinder $M^2_1 \times \mathbb{E}^1$, where M is a biconservative Lorentzian surface in $\mathbb{E}^3_1$.

Biconservative hypersurfaces in $\mathbb{E}^4_1$

Three distinct principal curvatures

- $x_1(s,t,u) = (s\cosh t, s\sinh t, f_1(s)\cos u, f_1(s)\sin u)$;
- $x_2(s,t,u) = (s\sinh t, s\cosh t, f_2(s)\cos u, f_2(s)\sin u)$;
- A hypersurface in $\mathbb{E}^4_1$ given by

$$x_3(s,t,u) = \left(\frac{1}{2}s(t^2+u^2) + au^2 + s + \phi(s), st, (s+2a)u, \frac{1}{2}s(t^2+u^2) + au^2 + \phi(s)\right), \quad a \neq 0. \tag{3}$$

Section 3:

Other problems

Biconservative hypersurfaces

Problem

Classify all biconservative hypersurfaces in $\mathbb{E}^4_1$ with non-diagonalizable shape operator.

Biconservative hypersurfaces

Problem

Classify all biconservative hypersurfaces in $\mathbb{E}^4_1$ with non-diagonalizable shape operator.

In other words, classify all hypersurfaces with the shape operator

$$S = \begin{pmatrix} -\frac{9H}{4} & 1 & 0 \\ 0 & -\frac{9H}{4} & 0 \\ 0 & 0 & 3H/2 \end{pmatrix}.$$

and

Biconservative hypersurfaces

Problem

Classify all biconservative hypersurfaces in $\mathbb{E}^4_1$ with non-diagonalizable shape operator.

In other words, classify all hypersurfaces with the shape operator

$$S = \begin{pmatrix} -\frac{9H}{4} & 1 & 0 \\ 0 & -\frac{9H}{4} & 0 \\ 0 & 0 & 3H/2 \end{pmatrix}.$$

and

$$S = \begin{pmatrix} -\frac{9H}{4} & -\nu & 0 \\ \nu & -\frac{9H}{4} & 0 \\ 0 & 0 & 3H/2 \end{pmatrix}.$$

A family of hypersurfaces

Consider the hypersurface given by

$$x(s,\widetilde{t}) = \left(\frac{s|\widetilde{t}|^2}{2} + \widetilde{a}\cdot\widetilde{t} + s + \phi(s),\ s\widetilde{t},\ \frac{s|\widetilde{t}|^2}{2} + \widetilde{a}\cdot\widetilde{t} + \phi(s)\right)$$

for a smooth function ϕ, where $\widetilde{t} = (t_1, t_2, \ldots, t_n)$ and $\widetilde{a} = (a_1, a_2, \ldots, a_n)$.

A family of hypersurfaces

Consider the hypersurface given by

$$x(s,\widetilde{t}) = \left(\frac{s|\widetilde{t}|^2}{2} + \widetilde{a}\cdot\widetilde{t} + s + \phi(s),\ s\widetilde{t},\ \frac{s|\widetilde{t}|^2}{2} + \widetilde{a}\cdot\widetilde{t} + \phi(s)\right)$$

for a smooth function ϕ, where $\widetilde{t} = (t_1, t_2, \ldots, t_n)$ and $\widetilde{a} = (a_1, a_2, \ldots, a_n)$.
The shape operator of this hypersurface is

$$S = \mathrm{diag}(k_1(s), k_2(s), \ldots, k_n(s)).$$

Hypersurfaces with vanishing Gauss-Kronecker curvature

Let $\alpha(w)$ be a smooth, regular, space-like curve in $\mathbb{S}^3_1(1) \subset \mathbb{E}^4_1$ and $A(w)$, $B(w)$ form an orthogonal frame field for the normal space of α in $\mathbb{S}^3_1(1)$.

Hypersurfaces with vanishing Gauss-Kronecker curvature

Let $\alpha(w)$ be a smooth, regular, space-like curve in $\mathbb{S}^3_1(1) \subset \mathbb{E}^4_1$ and $A(w)$, $B(w)$ form an orthogonal frame field for the normal space of α in $\mathbb{S}^3_1(1)$.
Consider the hypersurface in $\mathbb{E}^4_1$ given by

$$x(s,v,w) = s\alpha(w) + c\left(\cos\frac{v}{c}A(w) + \sin\frac{v}{c}B(w)\right).$$

Hypersurfaces with vanishing Gauss-Kronecker curvature

Let $\alpha(w)$ be a smooth, regular, space-like curve in $\mathbb{S}^3_1(1) \subset \mathbb{E}^4_1$ and $A(w)$, $B(w)$ form an orthogonal frame field for the normal space of α in $\mathbb{S}^3_1(1)$.
Consider the hypersurface in $\mathbb{E}^4_1$ given by

$$x(s, v, w) = s\alpha(w) + c\left(\cos\frac{v}{c}A(w) + \sin\frac{v}{c}B(w)\right).$$

The shape operator of this hypersurface is

$$S = \mathrm{diag}(0, -1/c, k_3(s, v, w)).$$

THANK YOU

Referances I

- Chen, B.-Y., *Total Mean Curvature and Submanifold of Finite Type*, Second edition, World Scientific, 2014.
- Chen, B. Y., *Some open problems and conjectures on submanifolds of finite type*. Soochow J. Math. 17 (1991), no. 2, 169-188.
- Chen, B. Y., Morvan, J. M. and Nore, T., *Energy, tension and finite type maps*. Kodai Math. J. 9, 1986, pp. 406-418.
- Dursun, U., *Hypersurfaces with pointwise 1-type Gauss map in Lorentz-Minkowski space*, Proc. Est. Acad. Sci. 58, 2009, 146–161.

Referances II

- Dursun, U. and Bektaş, B., *Spacelike Rotational Surfaces of Elliptic, Hyperbolic and Parabolic Types in Minkowski Space* $\mathbb{E}^4_1$ *with Pointwise 1-Type Gauss Map*, Math Phys Anal Geom, 2014, 17:247–263.
- Hasanis, T., Vlachos, I., *Hypersurfaces in* E^4 *with harmonic mean curvature vector field*, Math Nachr, 172 (1995), 145–169.
- Milousheva, V., *Marginally trapped surfaces with pointwise 1-type Gauss map in Minkowski 4-space*, Int. J. Geom. 2, 2013, 34–43.

Referances III

- Turgay, N. C., *On the marginally trapped surfaces in 4-dimensional space-times with finite type Gauss map*, GenRelativGravit (2014) 46:1621
- Turgay, N. C., *H-hypersurfaces with 3 distinct principal curvatures in the Euclidean spaces* (accepted, to print) Ann. Mat. Pura Appl., DOI: 10.1007/s10231-014-0445-z.