1 Differentials and Tensor-valued Differentials

$f(x^1, x^2, \ldots, x^n)$

—-

$dx^i (i = 1, 2, \ldots, n)$

$$dx^i \wedge 1 = dx^i,\ dx^i \wedge a = a \wedge dx^i \ \ (a \in r)(i = 1, 2, \ldots, n)$$
$$dx^i \wedge dx^k + dx^k \wedge dx^i = 0 \ \ (i, k = 1, 2, \ldots, n)$$

—-

$$u = a + a_i \cdot dx^i + \frac{1}{2} a_{ik} \cdot \wedge dx^k + \ldots$$

... $a, a_i, a_{i,k}, \ldots \in r$

$$u = u_0 + u_1 + u_2 + \ldots + u_n$$

$$\eta u = \sum_{p=0}^{n} (-1)^p \cdot u_p, \quad \zeta u = \sum_{p=0}^{n} (-1)^{\binom{p}{2}} \cdot u_p$$

$$e_l u^{j_1 \ldots j_p}_{i_1 \ldots i_p} = \sum_{h=0}^{n} a^{j_1 \ldots j_q l k_1 \ldots k_h}_{i_1 \ldots i_p} \wedge \ldots \wedge dx^{k_h}$$

$$u^{j_1 \ldots j_p}_{i_1 \ldots i_p} = \sum_{h=0}^{n} a^{j_1 \ldots j_q k_1 \ldots k_h}_{i_1 \ldots i_p} \wedge \ldots \wedge dx^{k_h}$$

$$(du)^{j_1 \ldots j_q}_{i_1 \ldots i_p} = dx^l \wedge d_l u^{j_1 \ldots j_q}_{i_1 \ldots i_p}$$

$dx^l \wedge \frac{\partial u}{\partial x^l}$

$$\omega^k_i = \Gamma^k_{ij} \cdot dx^j$$

$$(du)^{j_1\dots j_q}_{i_1\dots i_p} = dx^l \wedge \frac{\partial}{\partial x^l} u^{j_1\dots j_q}_{i_1\dots i_p} + \omega^{j_1}_m \wedge u^{mj_2\dots j_q}_{i_1\dots i_p} + \ldots +$$
$$- \omega^m_{i_1} \wedge u^{j_1\dots j_q}_{mi_2\dots i_p} - \ldots$$

$$\Omega^k_i = d(\o^k_i) - \o^m_i \wedge \omega^k_m = \frac{1}{2} R^k_{ijl} \cdot dx^j \wedge dx^l$$

—

$$(d\Omega)^k_i = 0.$$

$$e_l(u \wedge v) = e_l u \wedge v + \eta u \wedge e_l v, \tag{1.1}$$

$$d(u \wedge v) = du \wedge v + \eta u \wedge dv$$

—-

$e_l\zeta = \eta\zeta e_l$, $e_l d + d e_l = d_l$, $\zeta d = d\eta\zeta$,
— $dx^i, dx^i \wedge dx^k$, g_{ik}, $dx_i = g_{ik} \cdot dx^k$, $dx_i \wedge dx_k$

$$d_l u = 0, \qquad (l = 1, 2, \ldots, n)$$

—-

$$z = \sqrt{|g_{ik}|} \cdot dx^1 \wedge dx^2 \wedge \ldots \wedge dx^n.$$

2 Interior Multiplication and Differentiation

$$u \vee v = \sum_{m=0}^{n} (-1)^{\binom{m}{2}} \frac{\eta^m}{m!} e_{i_1 \ldots e_{i_m}} u \wedge e^{i_1} \ldots e^{i_m} v, \quad (e^i = g^{ik} e_k)$$

—- $(u \vee v) \vee w = u \vee (v \vee w)$.

$$u \wedge v = \eta^p v \wedge u$$

—

$$u \vee v = \sum_{m=0}^{n} (-1)^{\binom{m}{2}} \frac{2^m}{m!} e_{i_1} \ldots e_{i_m} \eta^{p+m} v \vee e^{i_1} \ldots e^{i_m} u$$

—- $dx^i \vee dx^k \vee \ldots \vee dx^l$

$$dx^i \vee dx^k + dx^k \vee dx^i = 2g^{ik}$$

–

$$e_l(u \vee v) = e_l u \vee v + \eta u \vee e_l v.$$

— $*u = u \vee z$. – $du = dx^l \wedge d_l u$

$$\delta u = dx^l \vee d_l u$$

$$\delta u = du + e^l d_l u$$

–

$$\delta u = du + d^* u \quad d^* u = *^{-1} d * u = (-1)^{\binom{n}{2}} d(u \vee z) \vee z$$

—

$$\delta(u \vee v) = \delta u \vee v + \eta u \vee \delta v + 2e^n u \vee d_h v \tag{2.1}$$

—-

$$\zeta \delta \zeta u = d_l u \vee dx^i .$$

—

$$\delta e_l + e_l \delta = d_l$$

3 Scalar Products

$$(u, v) = (\zeta u \vee v) \wedge z,$$

— $\sum_{m=0}^{n} (\zeta u \vee v)_m$.... $(\zeta u \vee v)_m$ —-

$$(u, v) = (v, u) = (\eta u, \eta v) = (\zeta u, \zeta v) = (u \vee z, v \vee z) = (z \vee u, z \vee v),$$

$$(u \vee w, v) = (u, v \vee \zeta w),$$
$$(w \vee u, v) = (u, \zeta w \vee v).$$

— $u = \sum_{m=0}^{n} u_m$, $v = \sum_{m=0}^{n} v_m$ —

$$(u, v) = \sum_m (u_m, v_m) = \sum_m (\zeta u_m \wedge *v_m), \quad (*v_m = v_m \vee z)$$

$$(u, v)_p = \frac{1}{p!} e_{i_1} \ldots e_{i_p} (dx^{i_p}) \vee \ldots \vee dx^{i_1} \vee u, v)$$

—

$$(v, u)_p = (-1)^{\binom{p}{2}} (u, v)_p, \; (\eta u, \eta v)_p = (-1)^p \cdot (u, v)_p, \; (u \vee w, v)_p = (u, v \vee \zeta w)_p,$$

—

$$(u, v)_1 = e_i (dx^i \vee u, v) = (\zeta u \vee dx^i \vee v)_0 \cdot e_i z$$

—

$$d(u, v)_1 = (u, \delta v) + (v, \delta u).$$

4 Lie Operators and Differentials

$$A = \alpha^i (x^1, \ldots, x^n) \frac{\partial}{\partial x^i}$$
$$Au = a^i \frac{\partial u}{\partial x^i} + d(\alpha^i) \wedge e_i u$$

$$Au = \alpha^i \cdot d_i u + (d\alpha)^i \wedge e_i u$$

—-

$$A(u \wedge v) = Au \wedge v + u \wedge Av, \quad dAu = Adu, \tag{4.1}$$

— $\alpha = \alpha_i \cdot dx^i = g_{ik} \cdot \alpha^k \cdot dx^i$.

$$d_i \alpha_k + d_k \alpha_i = 0$$

$\alpha^i = 0(i < n)$, $\alpha^n = 1$

$$Au = \alpha^i \cdot d_i u + \frac{1}{2}(d\alpha \vee u - u \vee d\alpha)$$

—

$$A(u \vee v) = Au \vee v + u \vee Av, \quad \delta Au = A\delta u.$$

——-

$$d\gamma = \frac{1}{4}(d\alpha \vee d\beta - d\beta \vee d\alpha) + 2\alpha^i \cdot \beta^k \cdot \Omega_{ik}.$$

5 Dirac Equations

$$\delta u = a \vee u$$

$$\delta v = -\zeta a \vee v,$$

$$d(u, v)_1 = 0,$$

6 Spherical Differentials

$$w = dx^1 \vee dx^2 \vee dx^3 \quad w_i = dx^i \vee w = w \vee dx^i$$

— $X_i u = x^k \frac{\partial u}{\partial x^l} x^l \frac{\partial u}{\partial x^k} + \frac{1}{2} w_i \vee u - \frac{1}{2} u \vee w_i,$ –

$$X_k X_l X_l X_k = -X_i$$

—

$$(K+1)u = \sum_i X_i u \vee w_i$$

$$X_1^2 + X_2^2 + X_3^2 = -K - K^2$$

—

$$K(u \vee w) = Ku \vee w,$$

–

$$(K+1)u = -\zeta\delta\zeta u \vee r dr + \sum_{i=1}^{3} x^i \frac{\partial u}{\partial x^i} + \frac{3}{2}(u - \eta u) + g\eta u$$

——

$$X_1 u = X_2 u = X_3 u = 0$$

—

$$\delta u = 0,$$

—- $P_k^m = P_{-k-1}^m$

$$Y_k^m = P_k^m(\cos\vartheta) \cdot e^{im\varphi}$$

–

$$S_k^m = r^{1-k} \cdot d(r^k \cdot Y_k^m),$$

—-

$$\delta S_k^m = \frac{1-k}{r} dr \vee S_k^m$$

—-

$$K S_k^m = k \cdot S_k^m.$$

$$X_3 S_k^m = im \cdot S_k^m,$$

—

$$delta(R \vee S_k^m) = (\delta R + \eta\zeta R \vee \frac{1-k}{r} dr) \wedge S_k^m$$

—-

$$\delta u = 0$$

—

$$u = \sum R_k^m \vee S_k^m$$

—

$$a \cdot r^{k-1} + a' \cdot r^{k-1} w + a'' \cdot r^{-k-1} \cdot dr + a''' \cdot r^{-k-1} \cdot dr \vee w$$

– a, a', a'', a'''

7 Dirac Equation in Space and Time

$$(dx^1)2 + (dx^2)^2 + (dx^3)^2 - c^2 \cdot (dt)^2$$

—

$$dx^i \vee dt = -dt \vee dx^i = dx^i \wedge dt, \;\; dt \vee dt = -c^{-2}.$$

$$z = dx^1 \vee dx^2 \vee dx^3 \vee icdt = w \vee icdt = w \wedge icdt$$

$- \, \varepsilon^{\pm} \vee \varepsilon^{\pm} = \varepsilon^{\pm}, \; \varepsilon^{\pm} \vee \varepsilon^{\mp} = 0, \; \varepsilon^{+} + \varepsilon^{-} = 1$

$$e^{\pm} = \frac{1}{2} \mp \frac{ic}{2} dt$$

— $\delta u = a \vee u, \; a = \alpha + \beta \vee icdt$

$\frac{\partial a}{\partial t} = 0$

$$Hu = -\frac{h}{2\pi i} \frac{\partial u}{\partial t}$$

——

$$u = p \vee T^{\pm}, \quad T^{\pm} = \varepsilon^{\pm} \cdot e^{-\frac{2\pi i}{h} E \cdot t}$$

$$\delta p = \alpha \vee p \mp (\frac{2\pi}{hc} E + \beta) \vee \eta p \tag{7.1}$$

8 Spherically symmetric Dirac Equation

$$p = R \vee S$$

—

$$\delta R = \alpha \vee R \pm (\frac{2\pi}{hc}E + \beta) \vee \eta R + \frac{k-1}{r} dr \vee \eta\zeta R$$

— $R = f_0(r) + f_1(r) \cdot dr + f_2(r) \cdot w + f_3(r) \cdot dr \vee w$ –

$$\begin{aligned}
&\frac{df_1}{dr} + \frac{1+k}{r} f_1 \mp \frac{2\pi}{hc} E \cdot f_0 \\
&= (\alpha_0 \pm \beta_0) f_0 + (\alpha_1 \mp \beta_1) f_1 + (-\alpha_2 \pm \beta_2) f_2 + (-\alpha_3 \mp \beta_3) f_3, \\
&\frac{df_0}{dr} + \frac{1-k}{r} f_0 \pm \frac{2\pi}{hc} E \cdot f_1 \\
&= (\alpha_1 \pm \beta_1) f_0 + (\alpha_0 \mp \beta_0) f_1 + (-\alpha_3 \pm \beta_3) f_2 + (-\alpha_2 \mp \beta_3) f_3, \\
&\frac{df_3}{dr} + \frac{1+k}{r} f_3 \pm \frac{2\pi}{hc} E \cdot f_2 \\
&= (\alpha_2 \pm \beta_2) f_0 + (\alpha_3 \mp \beta_3) f_1 + (\alpha_0 \mp \beta_0) f_2 + (\alpha_1 \pm \beta_1) f_3, \\
&\frac{df_2}{dr} + \frac{1-k}{r} f_2 \mp \frac{2\pi}{hc} E \cdot f_3 \\
&= (\alpha_3 \pm \beta_3) f_0 + (\alpha_2 \mp \beta_2) f_1 + (\alpha_1 \mp \beta_1) f_2 + (\alpha_0 \pm \beta_0) f_3,
\end{aligned} \tag{8.1}$$

——

$$\begin{aligned}
\alpha &= \alpha_0(r) + \alpha_1(r) \cdot dr + \alpha_2(r) \cdot w + \alpha_3(r) \cdot dr \vee w, \\
\beta &= \beta_0(r) + \beta_1(r) \cdot dr + \beta_2(r) \cdot w + \beta_3(r) \cdot dr \vee w,
\end{aligned}$$

9 The Dirac Equation of the Electron

$$\omega = A_1 \cdot dx^1 + A_2 \cdot dx^2 + A_3 \cdot dx^3 - c \cdot \Phi \cdot dt$$

—

$$d\omega = \Theta$$

—-

$$\begin{aligned}
\Theta = H_1 \cdot dx^2 \wedge dx^3 + H_2 \cdot dx^3 \wedge dx^1 + H_3 \cdot dx^1 \wedge dx^2, \\
c \cdot E_1 \cdot dx^1 \wedge dt + c \cdot E_2 \cdot dx^2 \wedge dt + c \cdot E_3 \cdot dx^3 \wedge dt
\end{aligned}$$

$$d\Theta = 0, \quad \delta\Theta = 0$$

$$d\omega = \delta\omega$$

$$\frac{h}{2\pi i}\delta u = \frac{1}{c}(iE_0 + e\omega) \vee u$$

$$u \vee \varepsilon^- = u, \quad u \vee \varepsilon^+ = 0,$$

$$u \vee \varepsilon^+ = u, \quad u \vee \varepsilon^- = 0$$

$$|e| \cdot (u, \eta\overline{u})_1 = \rho \cdot w - (i_1 \cdot w_1 + i_2 \cdot w_2 + i_2 \cdot w_3) \wedge dt,$$

$$d(u, \eta\overline{v})_1 = 0,$$

$$v = e^{\frac{2\pi i}{hc} ef} u$$

—

$$u \to e^{\frac{2\pi i}{hc} ef} u$$

——

$$u = R \vee S \vee T,$$

—- $\alpha = \alpha_0 = -\frac{2\pi}{hc} E_0,\ \alpha_k = 0 (k \neq 0),\ \beta = \beta_0 = \frac{2\pi}{hc} \frac{Ze^2}{r},\ \beta_k = 0 (k \neq 0)$

— $\frac{h}{2\pi i} X_k, (k = 1, 2, 3)$

– $\frac{h}{2\pi i} (X_k + \frac{1}{2} w_j)$