FROM GRASSMANN TO KÄHLER VIA CLIFFORD AND É. CARTAN

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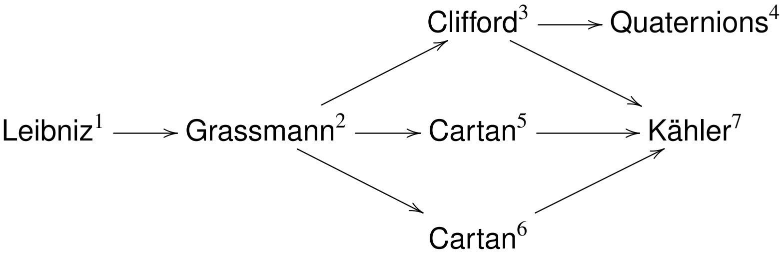
- Do not attempt to understand everything
- Big mess of wonderful ideas.
- Too advanced for his time.
- No present consensus.
- What to take home
- I am a physicist... Issue of rigor
- Look at what I can offer, not at my defficiencies
- My style is E. Cartan's & Kähler's
- My mathematical specialities:
  1. Finsler bundles: connections on affine-Finsler and metric-Finsler bundles.
  2. Application of Kaehler calculus to complex variable, Helmholz-Hodge theorems, geometrization of the unit imaginary in quantum mechanics, canonical main-bundle Kaluza-Klein geometry, solutions with symmetry of exterior systems, high energy physics.


Clifford3
Quaternions4
Leibniz1
Grassmann2
Cartan5
Kähler7
Cartan6

1. Geometric Calculus
2. Calculus of Extension, Unparalleled insight in math history
3. Retrospectively: cleans up algebraic mess See 4.
4. Example quaternions: Clifford vs Grassmann
5. Modern calculus: differential forms.
6. Tensor product of algebras
7. Integrates Cartan & Clifford Concept of differential form Solves major problems in physics. Another tragedy is developing!

Grassmann opinion on Leibniz (I) in his "Geometric Analysis"

1a. *Leading Intellect:* ability to comprehend and develop ideas towards which the age strives.

1b. *Still Greater Intellect*: has ideas ahead of his time and anticipates their course for centuries.

2a. *L.Is.*: ideas developed simultaneously by leading intelects

2b. *SGI:* ideas are property of an individual, whose intellect is appreciated by only the few who discern the wealth of developments to follows in future times.

3a. LIs: powerful approval; great response and elaboration: high point of the time

3b. SGI: ideas die away in their day, only understood by a few and perhaps by nobody in its entirety. Often only after centuries, they sow a rich...

I.1 Grassmann on Leibniz

1. On Leibniz: his idea of geometric analysis is sublime and prophetic. It shared fate (3b). Ideas remained hidden until there was related analysis from various directions. Idea, in the form given it by Leibniz, invigorates the new analysis.
2. Leibniz idea was excluded by the process of historical development for more than a century. It has now appeared and contains a potent seed whose right to merge with the historical development can no longer be denied. Newer analysis proceed from another viewpoints – afforded by the progress of time.
3. Leibniz distinguished on the one hand his goal of a completely geometric analysis, whose formulation floated before his eyes as a distant goal and, on the other hand his attempt to realize a geometric analysis. The latter lags infinitely behinds its goal and to which he did not ascribe the merit which he would ascribe to geometric analysis in general.

I.2 GRASSMANN

Dieudonné	Exterior algebra Grassmannian Jungle of modern algebraic developments
Cartan:	Multiple products A computational system: Exterior Progressive Interior Progressive Multiple applications
Clifford:	Exterior and Interior
Rota et al.	Progressive $\rightarrow$ Joint Regressive $\rightarrow$ Meet

$a \times b \downarrow\downarrow\downarrow$ Exists only in dimensions 3 and 7

$$ab \equiv \frac{1}{2}(ab+ba) + \frac{1}{2}(ab-ba)$$

Specialization $a \cdot b = \frac{1}{2}(ab+ba)$

$$ab = a \cdot b + a \wedge b = a \vee b$$

Example of computation with orthonormal basis:

$$a_i a_j = a_i \wedge a_j = -a_j \wedge a_i \text{ if } i \neq j$$
$$a_i^2 = a_i \cdot a_i = 1$$

$(a_1 a_2 a_3)(a_1 a_2 + a_1 a_4 + a_4 a_5) = -a_3 + a_2 a_3 a_4 + a_1 a_2 a_3 a_4 a_5$

$A_3 B_2 = (AB)_1 + (AB)_3 + (AB)_5$

Basis in algebra:

$$1, a_1, a_2, a_3, a_1 \wedge a_2, a_2 \wedge a_3, a_3 \wedge a_1, a_1 \wedge a_2 \wedge a_3.$$

In general

$$2^n = 1 - \binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{0}$$

$dd = 0$ for scalar-value differential forms

$$d(a_{ijk}\ldots dx \wedge dx^d \wedge dx^k \wedge) = \sum_l a_{ijk,\ldots,l} dx^l \wedge dx^i \wedge dx^j \wedge dx^k \wedge$$

$$\int_R d\alpha = \int_{\partial R} \alpha$$

Affine and Euclidean Spaces

$$d\boldsymbol{P} = dx^i \boldsymbol{a}_i \cdot \omega^i \boldsymbol{e}_i ; d\boldsymbol{e}_i = \omega_i^j \boldsymbol{e}_j$$

$$\Omega \equiv dd\boldsymbol{P} = (d\omega^i - \omega^j \wedge \omega_i^j)\boldsymbol{e}_j = 0$$

$$\Omega_i^j \boldsymbol{e}_j = d(d\boldsymbol{e}_i) = (d\omega_i^j - \omega_i^k \wedge \omega_k^j)\boldsymbol{e}_j = 0$$

$$d = dx^\mu \wedge d_\mu$$

1908 • Periodicity of Clifford algebras
• Progressive, regressive and (Grassmann's) interior product
• Assignment of importance, description and application of Grassmann's progressive, regressive and interior products

1913 Spinors

1923 Tensor product of algebras

1925 Triality of spin

1938 Pure spinors

KÄHLER ALGEBRA

$$dx^\mu \vee dx^\nu + dx^\mu \vee dx^\nu = 2\delta^{\mu\nu}$$

(i.e. associative) algebra defined as total tensor algebra modulo

$$dx^\mu dx^\nu + dx^\nu dx^\mu - \delta\eta^{\mu\nu}$$

KÄHLER CALCULUS

$\partial \stackrel{dt}{=} dx^\mu \vee d_\mu = dx^\mu \wedge d_\mu + dx^\mu \cdot d_\mu$

$\partial = d + 2$ (coderivative if LC)

$U^{\lambda_1 \dots \lambda_l}_{\mu_1 \dots \mu_m \pi_1 \dots \pi_p dx^{\pi_1} \wedge \dots \wedge dx^{\pi_p}}$

V : vector-valued differential 0-form (vector field)

dv : vector-valued differential 1-form

ddv : vector-valued differential 2-form (3 indices)

$$= v^i (d\omega_j^k \wedge \omega_j^l \wedge \omega_l|^k) \boldsymbol{e}_k$$

$\Omega = R^k_{ipq} \phi^i \boldsymbol{P}_k \omega^p \wedge \omega^p$

$\boldsymbol{U}(\boldsymbol{v}) = \boldsymbol{U}|\boldsymbol{v} = dd\boldsymbol{v}$

Definition

Suppose E is an open set in R^n. A *differential form of order* $k \geq 1$ in E (briefly, a k-form in E) is a function ω, symbolically represented by the sum

$$\omega = \sum a_{i_1} \ldots a_{i_k}(x) dx_{i_1} \wedge \ldots \wedge dx_{i_k} \tag{1}$$

(the indices $i_1, \ldots, i_k$ range independently from 1 to n), which assigns to each k-surface Φ in E a number $\omega(\Phi) = \int_\Phi \omega$, according to the rule

$$\int_\Phi = \int_D \sum a_{i_1 \ldots i_k}(\Phi(u)) \frac{\partial(x_{1_1, \ldots, x_n})}{\partial(u_1, \ldots, u_k)} du, \tag{2}$$

where D is the parameter domain of Φ.

II GRASSMANN'S CONTRIBUTIONS

II.1 His own version

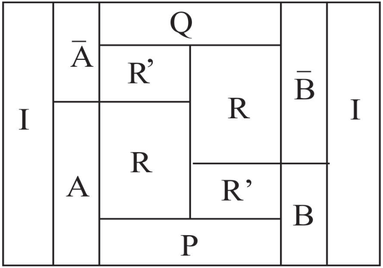
- Products of sides, not of lengths but of displacements.
- Realization that ptoduct of the sum of two displacements by a third displacement is the same as âĂę (Distributive property).
- The new analysis proceeds by abstraction (like arithmetic and combinatorial theory, and unlike geometry), without outside principles.
- It is coordinate free and not restricted to $n = 3$.
- Representation of intersections of lines, planes, etc by products of lines, planes, etc.
- Leave unchanged the present line that reads: Rotations etc.
- Qquotients of vectors. Remove the parenthesis and its contents.

II.2 Dieudonné's report (mainly of 1844)

- Major: exterior algebra, Grassmannian, products II.1.3
- Idea of vector, space whose elements are functions
- Linear and multilinear functions without coordinates
- Applies his method to Pfaff problem of classification of differential forms beyond results of Clebsh.
  Conclusion substantially equivalent to Darboux Theorem.
- Intrinsic conceptions only now familiar to most mathematicians
- In 1862 reproduces 1844 results through operations with elements of vector bases
- Proves exchange theorem of Steinitz (1910)
- Deduces invariance of dimension
- 1846 Scalar product (announced in 1844)
- Attemps to define $\boldsymbol{v} \cdot (\boldsymbol{a} \wedge \boldsymbol{b})$, later expanded
- Tangent mapping $f'(x)$ as endomorphism
- Endomorphism $\boldsymbol{e}_1 \wedge \ldots \wedge \boldsymbol{e}_n = \frac{\boldsymbol{e}_1 \wedge \ldots \boldsymbol{e}_n}{\boldsymbol{a}_1 \wedge \ldots \wedge \boldsymbol{a}_n} \boldsymbol{a}_1 \wedge \ldots \wedge \boldsymbol{a}_n$

Orthonormal complement $\overline{A}$ of subspaces A

$$A \wedge \overline{A} = I \equiv (\boldsymbol{a}_1 \wedge \boldsymbol{a}_2 \wedge \ldots \wedge \boldsymbol{a}_h) \wedge (\boldsymbol{a}_{h+1} \wedge \ldots \wedge \boldsymbol{a}_n)$$



$A \cap \overline{A} = 0$; $P \cap R = 0$; $R \cap R' = 0$, etc.

$A \cup B = P + R + R'$; $A \cap B = P$

$A \cup \overline{B} = P + Q + R$; $A \cap \overline{B} = R$

$\overline{A} \cup \overline{B} = Q + R + R'$; $\overline{A} \cap \overline{B} = Q$

Simple multivectors A_h, B_k If $h+k \leq n$, Grassman defines progressive product

If $h+k > n$, Grassman defines regressive product

Progressive: $A \wedge B \rightarrow A \cup B$ iff $A \cap B = 0$

Regressive $\overline{\overline{A} \wedge \overline{B}} \rightarrow A \cap B$ iff $\overline{A} \cap \overline{B} = 0$

Grade of progressive $h+k$

Grade of regressive: $n - [(n-h)+(n-k)] = h+k-n \geq 0; < n$

Iff $A \cap B = 0, \qquad A \wedge B \rightarrow A \cup B, \quad P = 0$
Iff $\overline{A} \cap \overline{B} = 0, \qquad Q = 0 \ \ \overline{A} \wedge \overline{B} = R' \wedge R, \quad \overline{\overline{A} \wedge \overline{B}} = P \rightarrow A \cap B$

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Grassmann on Leibniz (II)

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Cartan's Report on Grassmann

II.3.1 Cartan vs. Study on Grassmann

Study $\rightarrow$ Cartan 1908
Cartan's section on Grassmann
Role of "Hodge Operation"
Complete Works (Ed. Engel)

II.3.2 Multiplicity of Products

A. Symmetric Multiplication

B. Circular Multiplications

Among them	Complex Interior

C. Linear Multiplications

Among them	Algebraic Exterior	$e_r e_s = e_s e_r$

D. Non-Algebraic System

Progressive Regressive	Interior

II.3.3 Complementary Units (Cartan's Report)

Definition: *Complementary unit* of a given unit E of order h is the *product* E' *of the n-h primitive unit that do not enter* E*,* this product being preceded by the sign "$+$" or the sign "$-$" depending on whether EE' *equals* $+1$ *or* $-1\ldots$ One refers to the complementary unit of E by IE. One thus has

$$|E = (EE') \cdot E';$$

for example if $n = 5$, one has

$$|\boldsymbol{e}_2\boldsymbol{e}_4\boldsymbol{e}_5 = -\boldsymbol{e}_1\boldsymbol{e}_2$$

HORRIBLE MESS!

II.3.3 SAME HORRIBLE MESS WITH ANOTHER EXAMPLE

Dieudonné: One could also form the "interior product" $a \wedge (|b)$, $a(p+n-q)$-vector. However, for $p = q = 1$, Grassmann wanted that product to be, not an n-vector, but a *number*, ... he unfortunately *identified* n-vectors and scalars....

Notice the use of inverted commas

Dieudonné starts the next paragraph with: This is the beginning of the debacle . Faces with...., and of course gets nowehere, except in very particular cases....

Modern Concept of Interior Product

- It is zero if one of them does nor contain the other.
- It is the complement of one to the other (orthogonal complement) otherwise
  - It is of dimension $|h-k|$
  - It amounts to scalar product for $h=k=1$

Grassmann's Progressive Interior Product (PIP) of A_h and B_k

- PIP of A_h and B_k is the product of A_h and $(\overline{B})_{n-k}$ when $h+(n-k) \leq n$ i.e. $h \leq k$
- For $A \wedge (\overline{B})_{n-k}$ to be different from zero, R must be empty. Then $A = P \subset B$
- $A \wedge \overline{B} = P \wedge Q$, i.e. complement of R' in I
- But R' is the modern concept of interior product
- Hence PIP is the complement of the modern concept of interior product
- Grade = write $h+(n-k)$ as $n+(h-k)$

Grassmann's Regressive Interior Product (RIP) of A_h and B_k

- RIP of A_h and B_k is the product of A_h and $(\overline{B})_{n-k}$ when $h+(n-k)>n$, i.e. $h>k$
- It, therefore, is $\overline{\overline{A}\wedge B}$
- For $\overline{A}\times B$ to be different from zero, R' must be empty. Then $B=P\subset A$
- $\overline{A}\wedge B=P\wedge Q$, i.e. the complement of R in I
- Hence $\overline{A}\times B=R$
- But R and, therefore, RIP of A and B is the modern concept of interior product