Painlevé Test and the Resolution of Singularities for Integrable Equations

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Cauchy-Kovalevskaya Theorem

- Painlevé test and Painlevé property.
- Regularise principal balance.
- Hamiltonian system.
- Use regularisation to prove Painlevé property.
- Further problems.

Cauchy-Kovalevskaya Theorem

The initial value problem

$$\mathrm{P_I}\colon\ u'' = 6u^2 + t, \quad u(t_0) = u_0, \quad u(t_1) = u_1$$

has the convergent power series solution

$$u = u_0 + u_1(t - t_0) + \left(3u_0^2 + \tfrac{t_0}{2}\right)(t - t_0)^2 + \left(2u_0 u_1 + \tfrac{1}{6}\right)(t - t_0)^3 + \cdots .$$

Moreover, u is analytically dependent on t_0, u_0, u_1.

(1842) Augustin Cauchy: ?? Comptes. rendus.
(1875) Sophie von Kowalevsky: *Zur Theorie der partiellen Differentialgleichungen*, J. Reine Angew. Math.

Painlevé Test

What about Laurent series solutions?

$$\mathrm{P_I}: u'' = 6u^2 + t, \quad u = u_0(t-t_0)^{-k} + u_1(t-t_0)^{-k+1} + \cdots, \quad k > 0.$$

- Determine k by dominant balance:

$$u \sim u_0(t-t_0)^{-k} \implies k(k+1)u_0 = 6u_0^2 \implies k = 2, u_0 = 1.$$

- Get coefficients from recursive relation

$$u = (t-t_0)^{-2} - \frac{t_0}{10}(t-t_0)^2 - \frac{1}{6}(t-t_0)^3 + r(t-t_0)^4 + \frac{t_0^2}{18}(t-t_0)^5 + \cdots$$

Painlevé Test

$$u = u_0 + u_1(t-t_0) + \left(3u_0^2 + \frac{t_0}{2}\right)(t-t_0)^2 + \left(2u_0u_1 + \frac{1}{6}\right)(t-t_0)^3 + \cdots$$
$$u = (t-t_0)^{-2} - \frac{t_0}{10}(t-t_0)^2 - \frac{1}{6}(t-t_0)^3 + r(t-t_0)^4 + \frac{t_0^2}{18}(t-t_0)^5 + \cdots$$

- Power series is real. Laurent series is formal.
- Power series has $n = 2$ initial parameters. Laurent series has $n - 1 = 1$ resonance parameter r of resonance $4 - (-2) = 6$.
- t_0 is a movable singularity, which is a resonance parameter of resonance -1.
- t_0 and r are comparable to u_0 and u_1.

Painlevé Test

$$u = u_0 + u_1(t - t_0) + \left(3u_0^2 + \frac{t_0}{2}\right)(t - t_0)^2 + \left(2u_0 u_1 + \frac{1}{6}\right)(t - t_0)^3 + \cdots$$

$$u = (t - t_0)^{-2} - \frac{t_0}{10}(t - t_0)^2 - \frac{1}{6}(t - t_0)^3 + t_1(t - t_0)^4 + \frac{t_0^2}{18}(t - t_0)^5 + \cdots$$

- Changing u_0 and u_1 gives all the analytic solutions.
- Changing t_0 and t_1 should give all the singular solutions.
- The movable pole singularities should all be included in Laurent series solution and are therefore single valued.
- Integrability should be detected by the Painlevé property: All movable singularities are single valued.

Painlevé test: Checking all movable pole singularities have enough number of resonance parameters.

Kovalevski Top

Differential equation for the spinning top.

$$\alpha' = r\beta - q\gamma, \qquad Ap' + (C - B)qr = Mg(y_0\gamma - z_0\beta)$$
$$\beta' = p\gamma - r\alpha, \qquad Bp' + (A - C)rp = Mg(z_0\alpha - x_0\gamma)$$
$$\gamma' = q\alpha - p\beta, \qquad Cp' + (B - A)pq = Mg(x_0\beta - y_0\alpha)$$

The system is integrable for the following cases.

- $A = B = C$.
- Euler Top: $x_0 = y_0 = z_0$.
- Lagrange Top: $A = B$, $x_0 = y_0 = 0$.
- Kovalevski Top: $A = B = 2C$, $z_0 = 0$.

(1889) *Sur le probléme de la rotation d'un corps solide autour d'un point fixé*, Acta Math.

Painlevé Property

53 families of second order ODEs satisfying the Painlevé property.
6 exceptional ODEs.

$$u'' = 6u^2 + t,$$
$$u'' = 2u^3 + tu + \alpha,$$
$$u'' = u^{-1}u'^2 - t^{-1}u' + t^{-1}(\alpha u^2 + \beta) + u^{-1}(\gamma u^4 + \delta),$$
$$u'' = \frac{1}{2u}u'^2 + \frac{3}{2}u^3 + 4tu^2 + 2(t^2 - \alpha)u + \frac{\beta}{u},$$
$$u'' = \left(\frac{1}{2u} + \frac{1}{u-1}\right)u'^2 - \frac{1}{t}u' + \frac{(u-1)^2}{t^2}\left(\alpha u + \frac{\beta}{u}\right) + \gamma\frac{u}{t} + \delta\frac{u(u+1)}{u-1},$$
$$u'' = \frac{1}{2}\left(\frac{1}{u} + \frac{1}{u-1} + \frac{1}{u-t}\right)u'^2 - \frac{1}{2}\left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{u-t}\right)u'$$
$$+ \frac{u(u-1)(u-t)}{t^2(t-1)^2}\left(\alpha + \beta\frac{t}{u^2} + \gamma\frac{t-1}{(u-1)^2} + \delta\frac{t(t-1)}{(u-t)^2}\right).$$

(1902) *Sur les équations différentielles du second ordre et d'ordre supérieur dont l'intégrale générale est uniforme*, Acta Math.

Painlevé Test

(1980) Ablowitz-Ramani-Segur: *A connection between nonlinear evolution equations and ordinary differential equations of P-type I, II*, J. Math. Phys.
(1989) Ercolani-Siggia: *Painlevé property and geometry*, Phys. D
(1989) Adler-van Moerbeke: *The complex geometry of the Kowalewski-Painlevé analysis*, Invent. Math.
(2008) Conte-Musette: The Painlevé handbook. Springer

Gap between the Painlevé test and Painlevé property: The test is heuristic and formal. Further rigorous and analytical argument is needed to achieve the Painlevé Property.

Question: Analytical aspect of the Painlevé Test.

Principal Balance

n-th order ODE

$$\boldsymbol{u}' = \boldsymbol{f}(t, \boldsymbol{u}), \quad \boldsymbol{u}(t) \in \mathbb{R}^n.$$

Balance: Formal Laurent series solution (t stands for $t - t_0$)

$$\begin{aligned}\boldsymbol{u} = \boldsymbol{a}_0(t_0)t^{-\boldsymbol{k}} + \boldsymbol{a}_1(t_0)t^{1-\boldsymbol{k}} + \cdots \\ + \boldsymbol{a}_{r_1}(t_0, \boldsymbol{t}_1)t^{r_1-\boldsymbol{k}} + \boldsymbol{a}_{r_1+1}(t_0, \boldsymbol{t}_1)t^{r_1+1-\boldsymbol{k}} + \cdots \\ + \cdots\cdots + \\ + \boldsymbol{a}_{r_s}(t_0, \boldsymbol{t}_1, \ldots, \boldsymbol{t}_s)t^{r_s-\boldsymbol{k}} + \boldsymbol{a}_{r_s+1}(t_0, \boldsymbol{t}_1, \ldots, \boldsymbol{t}_s)t^{r_s+1-\boldsymbol{k}} + \cdots .\end{aligned}$$

The resonance parameter $\boldsymbol{t}_j$ has resonance r_j, and the resonance matrix has full rank

$$R = \left(\frac{\partial \boldsymbol{a}_0}{\partial t_0}, \frac{\partial \boldsymbol{a}_1}{\partial \boldsymbol{t}_1}, \ldots, \frac{\partial \boldsymbol{a}_s}{\partial \boldsymbol{t}_s} \right)$$

Principal Balance: $n = 1 + \dim \boldsymbol{t}_1 + \cdots + \dim \boldsymbol{t}_s$.

Principal Balance

Gelfand-Dikii hierarchy with 2 degrees of freedom

$$H = -q_1 p_2^2 - 2p_1 p_2 + 3q_1^2 q_2 - q_1^4 - q_2^2.$$

One formal Laurent series solution is (note leading 0 for q_2)

$$\begin{aligned}
q_1 &= t^{-2} + t_2 - 3t_2^2 t^2 - 4t_3 t^3 - 10t_2^3 t^4 - 6t_2 t_3 t^5 - 2t_4 t^6 + \dots, \\
q_2 &= 0t^{-4} + 0t^{-3} + 3t_2 t^{-2} - 6t_2^2 - t_3 t - 9t_2^3 t^2 + \left(-\tfrac{33}{2} t_2^4 + 9t_4\right) t^4 + \cdots \\
p_1 &= -t^{-5} + 2t_2 t^{-3} + t_3 - 4t_2^3 t - 15t_2 t_3 t^2 + \left(22t_2^4 - 22t_4\right) t^3 + \cdots, \\
p_2 &= t^{-3} + 3t_2^2 t + 6t_3 t^2 + 20t_2^3 t^3 + 15t_2 t_3 t^4 + 6t_4 t^5 + \dots.
\end{aligned}$$

Leading exponent $\boldsymbol{k} = (2, 4, 5, 3)$.
Leading coefficient $\boldsymbol{a}_0 = (1, 0, -1, 1)$.

Resonance $\boldsymbol{r} = (-1, 2, 5, 8)$. Matrix $R = \begin{pmatrix} 2 & 1 & -4 & -2 \\ 0 & 3 & -6 & 9 \\ -5 & 2 & 1 & -22 \\ 3 & 0 & 6 & 6 \end{pmatrix}$.

Main Result

Theorem
A regular system of ordinary differential equations passes the Painlevé test if and only if there is a triangular change of variable, such that the system is converted to another regular system, and the Laurent series solutions produced by the Painlevé test are converted to power series solutions.

Main Result

Explanation

- Regular means complex analytic. We need rational function on the right in order for "formal Laurent series solution" to make sense.
- Pass Painlevé test here only means all balances are principal. Not yet studied more general compatible system of balances.
- Triangular change of variable means (up to permuting u_i)

$$u_1 = \tau^{-k},$$
$$u_i = a_i(t, \tau, \rho_2, \dots, \rho_{i-1}) + b_i(t, \tau, \rho_2, \dots, \rho_{i-1})\rho_i, \quad 1 < i \leq n,$$

a_i, b_i meromorphic in τ and analytic in others, and $b_i \neq 0$.

Main Result

Significance

- Simple invertible transformation to regularise both the solution and the equation. (Link between Kovalevskaya's two contributions)
- Laurent series solution in the principal balance must converge.
- No dominant balance, no Kowalevskian matrix needed.
- Natural with respect to Hamiltonian structure.
- Local result.

Resolution of Singularity for $\mathrm{P_I}$

$$\mathrm{P_I}: \; u'' = 6u^2 + t.$$

A system for u and u'. Has principal balance

$$\begin{aligned} u &= (t-t_0)^{-2} - \tfrac{t_0}{10}(t-t_0)^2 - \tfrac{1}{6}(t-t_0)^3 \\ &\quad + t_1(t-t_0)^4 + \tfrac{t_0^2}{18}(t-t_0)^5 + \cdots, \\ u' &= -2(t-t_0)^{-3} - \tfrac{t_0}{5}(t-t_0) - \tfrac{1}{2}(t-t_0)^2 \\ &\quad + 4t_1(t-t_0)^3 + \tfrac{5t_0^2}{18}(t-t_0)^4 + \cdots. \end{aligned}$$

Step 1: $u \sim (t-t_0)^{-2} \implies$ indicial normalization $u = \tau^{-2}$.

$$2\tau\tau'' - 6\tau'^2 + 6 + t\tau^4 = 0.$$

Resolution of Singularity for $\mathrm{P_I}$

Step 2: Find expansion of u' in terms of τ.
Substitute formal τ-series

$$\tau' = a_0(t) + a_1(t)\tau + a_2(t)\tau^2 + \cdots$$

into differential equation for τ and get recursive relation

$$6a_0^2 = 6, \quad 2(6a_0 - n)a_n = t\delta_{4,n} + 2a'_{n-1} - 6 \sum_{\substack{i+j=n \\ i,j\neq 0}} a_i a_j.$$

Solve recursive relation to get formal τ-series

$$\tau' = 1 + \tfrac{t}{4}\tau^4 + \tfrac{1}{4}\tau^5 + a_6\tau^6 - a_6'\tau^7 + \cdots ,$$

where a_6 is an arbitrary function of t. Then

$$u' = -2\tau^{-3}\tau' = -2\tau^{-3} - \tfrac{t}{2}\tau - \tfrac{1}{2}\tau^2 - 2a_6\tau^3 + 2a_6'\tau^4 + \cdots .$$

Resolution of Singularity for $\mathrm{P_I}$

Step 3: Truncate the τ-expansion of u'.
(Painlevé's) change of variable

$$u = \tau^{-2},$$
$$u' = -2\tau^{-3} - \tfrac{t}{2}\tau - \tfrac{1}{2}\tau^2 + \rho\tau^3.$$

This converts $\mathrm{P_I}$ to a regular system

$$\tau' = 1 + \tfrac{t}{4}\tau^4 + \tfrac{1}{4}\tau^5 - \tfrac{1}{2}\rho\tau^6,$$
$$\rho' = \tfrac{t^2}{8}\tau + \tfrac{3t}{8}\tau^2 + \left(\tfrac{1}{4} - t\rho\right)\tau^3 - \tfrac{5}{4}\rho\tau^4 + \tfrac{3}{2}\rho^2\tau^5,$$

and converts the formal Laurent series to formal power series

$$\tau = (t - t_0) + \tfrac{t_0}{20}(t - t_0)^5 + \cdots ,$$
$$\rho = 7t_1 + \tfrac{t_0^2}{16}(t - t_0)^2 + \cdots .$$

Resolution of Singularity for Hamiltonian System

Theorem

If a regular Hamiltonian system of ordinary differential equations passes the Painlevé test in the Hamiltonian way, then there is a canonical triangular change of variable, such that the system is converted to another regular Hamiltonian system, and the Laurent series solutions are converted to power series solutions.

- Autonomous system: The new Hamiltonian function is obtained by substituting the new variables.
- Non-autonomous system: The new Hamiltonian function is obtained by substituting the new variables and then dropping the singular terms.

Resolution of Singularity for Hamiltonian System

Hamiltonian way

$$q_i = a_{i,0}(t_0)(t-t_0)^{-k_i} + \cdots + a_{i,j}(t_0, r_2, \ldots, r_{n_l})(t-t_0)^{j-l_i} + \cdots$$
$$p_i = b_{i,0}(t_0)(t-t_0)^{-l_i} + \cdots + b_{i,j}(t_0, r_2, \ldots, r_{n_l})(t-t_0)^{j-k_i} + \cdots$$

Leading coefficients $k_1, \ldots, k_n, l_1, \ldots, l_n$.
Resonances $r_1, \ldots, r_n, s_1, \ldots, s_n$.

Definition
The balance is Hamiltonian principal, if the resonance vectors form a simplectic basis of $\mathbb{R}^{2n}$, and there is d, such that $k_i + l_i = d - 1$, and $r_j + s_j = d - 1$ for the resonances of simplectically conjugate resonance vectors.

Note: May arrange to have
$-1 = r_1 < r_2 \leq \cdots \leq r_n \leq s_n \leq s_{n-1} \leq s_1$.

Resolution of Singularity for Hamiltonian System

Gelfand-Dikii hierarchy with 2 degrees of freedom

$$H = -q_1 p_2^2 - 2p_1 p_2 + 3q_1^2 q_2 - q_1^4 - q_2^2.$$

Principal balance

$$\begin{aligned}
q_1 &= t^{-2} + t_2 - 3t_2^2 t^2 - 4t_3 t^3 - 10t_2^3 t^4 - 6t_2 t_3 t^5 + 2t_4 t^6 + \cdots ,\\
q_2 &= 0t^{-4} + 0t^{-3} + 3t_2 t^{-2} - 6t_2^2 - 6t_3 t - 9t_2^3 t^2 + \left(-\tfrac{33}{2} t_2^4 - 9t_4\right) t^4 + \cdots ,\\
p_1 &= -t^{-5} + 2t_2 t^{-3} + t_3 - 4t_2^3 t - 15t_2 t_3 t^2 + \left(22t_2^4 + 22t_4\right) t^3 + \cdots ,\\
p_2 &= t^{-3} + 3t_2^2 t + 6t_3 t^2 + 20t_2^3 t^3 + 15t_2 t_3 t^4 - 6t_4 t^5 + \cdots .
\end{aligned}$$

Resonance matrix

$$R = (R_{-1}\ R_2\ R_5\ R_8) = \begin{pmatrix} 2 & 1 & -4 & 2 \\ 0 & 3 & -6 & -9 \\ -5 & 2 & 1 & 22 \\ 3 & 0 & 6 & 6 \end{pmatrix}$$

$-1 + 8 = 2 + 5$, and $(R_{-1}\ R_2\ \frac{1}{81} R_8\ \frac{1}{9} R_5)$ is a symplectic matrix.

Resolution of Singularity for Hamiltonian System

Indicial normalization $q_1 = \tau^{-2} = \tau_1{}^{-2}$. Then

$$\begin{aligned}
\tau &= t - \tfrac{1}{2}t_2 t^3 + \tfrac{15}{8}t_2^2 t^5 + 2t_3 t^6 + \tfrac{39}{16}t_2^3 t^7 + \left(-\tfrac{133}{128} - t_4\right)t^9 + \cdots,\\
t &= \tau + \tfrac{1}{2}t_2\tau^3 - \tfrac{9}{8}t_2^2\tau^5 - 2t_3\tau^6 - \tfrac{135}{16}t_2^2\tau^7 - 9t_2t_3\tau^8 + \left(t_4 - \tfrac{2037}{128}t_1^4\right)\tau^9 + \cdots,\\
q_2 &= 0t^{-4} + 0t^{-3} + 3t_2 t^{-2} - 6t_2^2 - 6t_3 t - 9t_2^3 t^2 + \left(-\tfrac{33}{2}t_2^4 - 9t_4\right)t^4 + \cdots\\
&= 3t_2\tau^{-2} - 9t_2^2 - 6t_3\tau + 9t_2t_3\tau^3 + \left(\tfrac{27}{2}t_2^4 - 9t_4\right)\tau^4 + \cdots.
\end{aligned}$$

Introduce $q_2 = \tau_2\tau^{-2}$. Then

$$\begin{aligned}
\tau_2 &= 3t_2 - 9t_2^2\tau^2 - 6t_3\tau^3 + 9t_2t_3\tau^5 + \left(\tfrac{27}{2}t_2^4 - 9t_4\right)\tau^6 + \cdots,\\
t_2 &= \tfrac{1}{3}\tau_2 + \tfrac{1}{3}\tau_2^2\tau + 2t_3\tau^3 + \tfrac{2}{3}\tau^3\tau^4 + \left(\tfrac{29}{18}\tau_2^4 + 3t_4\right)\tau^6 + \cdots,\\
p_2 &= \tau^{-3} - \tfrac{1}{2}\tau_2\tau^{-1} + \tfrac{3}{8}\tau_2^2\tau + 9t_3\tau^2 + \tfrac{35}{16}\tau_2^3\tau^3 + 18\tau_2t_3\tau^4\\
&\quad + \left(\tfrac{867}{128}\tau_2^4 - \tfrac{27}{2}t_4\right)\tau^5 + \cdots.
\end{aligned}$$

Introduce $p_2 = \tau^{-3} - \tfrac{1}{2}\tau_2\tau^{-1} + \tfrac{3}{8}\tau_2^2\tau + \rho_2\tau^2$.

Resolution of Singularity for Hamiltonian System

Introduce $p_2 = \tau^{-3} - \frac{1}{2}\tau_2\tau^{-1} + \frac{3}{8}\tau_2^2\tau + \rho_2\tau^2$. Then

$$
\begin{aligned}
p_2 &= \tau^{-3} - \tfrac{1}{2}\tau_2\tau^{-1} + \tfrac{3}{8}\tau_2^2\tau + 9t_3\tau^2 + \tfrac{35}{16}\tau_2^3\tau^3 + 18\tau_2 t_3\tau^4 \\
&\quad + \left(\tfrac{867}{128}\tau_2^4 - \tfrac{27}{2}t_4\right)\tau^5 + \cdots , \\
\rho_2 &= 9t_3 + \tfrac{35}{16}\tau_2^3\tau + 18\tau_2 t_3\tau^2 + \left(\tfrac{867}{128}\tau_2^4 - \tfrac{27}{2}t_4\right)\tau^3 + \cdots , \\
9t_3 &= \rho_2 - \tfrac{35}{16}\tau_2^3\tau - 2\tau_2\rho_2\tau^2 + \left(-\tfrac{307}{128}\tau_2^4 + \tfrac{3}{2}t_4\right)\tau^3 + \cdots , \\
p_1 &= -\tau^{-5} + \tfrac{3}{2}\tau_2\tau^{-3} + \tfrac{1}{8}\tau_2^2\tau^{-1} - \tfrac{5}{16}\tau_2^3\tau - \tau_2\rho_2\tau^2 + \left(\tfrac{189}{128}\tau_2^4 + \tfrac{81}{2}t_4\right)\tau^3,
\end{aligned}
$$

Introduce $p_1 = -\tau^{-5} + \frac{3}{2}\tau_2\tau^{-3} + \cdots + \rho_1\tau^3$. Then

$$\rho_1 = \frac{189}{128}\tau_2^4 + \frac{81}{2}t_4 + o(\tau).$$

Resolution of Singularity for Hamiltonian System

Triangular change of variable

$$
\begin{aligned}
q_1 &= {\tau_1}^2, \\
q_2 &= \tau_2 \tau_1^{-2}, \\
p_1 &= -\tau_1^{-5} + \tfrac{3}{2}\tau_2\tau_1^{-3} + \tfrac{1}{8}\tau_2^2\tau_1^{-1} - \tfrac{5}{16}\tau_2^3\tau_1 - \tau_2\rho_2\tau_1^2 + \rho_1\tau_1^3, \\
p_2 &= \tau_1^{-3} - \tfrac{1}{2}\tau_2\tau_1^{-1} + \tfrac{3}{8}\tau_2^2\tau_1 + \rho_2\tau_1^2,
\end{aligned}
$$

Resonance parameters become initial values of new variables

$$
\begin{aligned}
\tau_1 &= t - \tfrac{1}{2}t_2 t^3 + \cdots, \\
\tau_2 &= 3t_2 - 9t_2^2\tau^2 + \cdots, \\
\rho_2 &= 9t_3 + \tfrac{35}{16}\tau_2^3\tau + \cdots, \\
\rho_1 &= \tfrac{81}{2}t_4 + \tfrac{189}{128}\tau_2^4 + o(\tau).
\end{aligned}
$$

Resolution of Singularity for Hamiltonian System

By

$$dq_1 \wedge dp_1 + dq_2 \wedge dp_2 = -2d\tau_1 \wedge d\rho_1 + d\tau_2 \wedge d\rho_2.$$

We introduce $Q_1 = \tau_1, Q_2 = \tau_2, P_1 = -2\rho_1, P_2 = \rho_2$.

$$\begin{aligned}
q_1 &= Q_1^{-2},\\
q_2 &= Q_1^{-2} Q_2,\\
p_1 &= -Q_1^{-5} + \tfrac{3}{2} Q_1^{-3} Q_2 + \tfrac{1}{8} Q_1^{-1} Q_2^2 - \tfrac{5}{16} Q_1 Q_2^3 - Q_1^2 Q_2 P_2 - \tfrac{1}{2} Q_1^3 P_1,\\
p_2 &= Q_1^{-3} - \tfrac{1}{2} Q_1^{-1} Q_2 + \tfrac{3}{8} Q_1 Q_2^2 + Q_1^2 P_2.
\end{aligned}$$

New Hamiltonian system

$$\begin{aligned}
H = P_1 - \tfrac{35}{64} Q_2^4 - 2Q_1 Q_2^2 P_2 + Q_1^2 \left(-P_2^2 - \tfrac{1}{2} Q_2 P_1 + \tfrac{15}{64} Q_2^5\right)\\
+ \tfrac{11}{8} Q_1^3 Q_2^3 P_2 + Q_1^4 \left(2Q_2 P_2^2 + \tfrac{3}{8} Q_2^2 P_1\right) + Q_1^5 P_1 P_2.
\end{aligned}$$

Resolution of Singularity for Hamiltonian System

A non-autonomous example: $H = p_1 q_2 + p_2(2q_1^3 + tq_1 + \alpha)$ hastwo principal balances. For the balance with $q_1 \sim (t - t_0)^{-1}$, we get canonical triangular change of variable

$$
\begin{aligned}
q_1 &= Q_1^{-1}, \\
q_2 &= -Q_1^{-2} - \tfrac{t}{2} - \tfrac{1}{2}(1 + 2\alpha)Q_1 + Q_2 Q_1^2, \\
p_1 &= 2P_2 Q_1^{-3} - \tfrac{1}{2}(1 + 2\alpha)P_2 + 2P_2 Q_2 Q_1 - P_1 Q_1^2, \\
p_2 &= P_2 Q_1^{-2}.
\end{aligned}
$$

Substituting into H above, we have

$$
\begin{aligned}
H = & -\tfrac{1}{2}P_2 Q_1^{-2} + P_1 + \tfrac{1}{4}(1 + 2\alpha)tP_2 + \tfrac{1}{4}((1 + 2\alpha)^2 - 4tQ_2)P_2 Q_1 \\
& + \tfrac{1}{2}\left[tP_1 - 3(1 + 2\alpha)P_2 Q_2\right] Q_1^2 + \tfrac{1}{2}\left[(1 + 2\alpha)P_1 + 4P_2 Q_2^2\right] Q_1^3 - P_1 Q_2 Q_1^4.
\end{aligned}
$$

The new system has Hamiltonian function

$$
\begin{aligned}
\bar{H} = & P_1 + \tfrac{1}{4}(1 + 2\alpha)tP_2 + \tfrac{1}{4}\left[(1 + 2\alpha)^2 - 4tQ_2\right] P_2 Q_1 \\
& + \tfrac{1}{2}\left[tP_1 - 3(1 + 2\alpha)P_2 Q_2\right] Q_1^2 + \tfrac{1}{2}\left[(1 + 2\alpha)P_1 + 4P_2 Q_2^2\right] Q_1^3 - P_1 Q_2 Q_1^4.
\end{aligned}
$$

Application: Painlevé Property for $\mathrm{P_I}$

Theorem

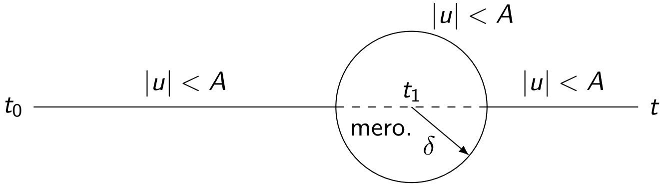
The only movable singularities of

$$\mathrm{P_I}\colon u'' = 6u^2 + t$$

are poles of second order. In particular, all (complex) solutions are single values.

Application: Painlevé Property for $\mathrm{P_I}$

Key idea: Starting with finite initial values $u(t_0)$ and $u'(t_0)$. Try to extend the solution along a straight line $[t_0, t_1]$.



For a prescribed big A, we need to show that, if

$$|u(t_1)| = A \text{ and } |u| \leq A \text{ along } [t_0, t_1],$$

then for a specific δ ($\delta = 3A^{-\frac{1}{2}}$), u extends to a meromorphic function on $B(t_1, \delta)$, such that $|u| < A$ on the boundary.
The argument continues by replacing the straight line inside the disk by the half boundary circle.

Application: Painlevé Property for $\mathrm{P_I}$

Estimating u on $B(t_1, \delta)$ by using triangular change

$$
\begin{aligned}
u &= \tau^{-2}, \\
u' &= -2\tau^{-3} - \frac{t}{2}\tau - \frac{1}{2}\tau^2 + \rho\tau^3,
\end{aligned}
$$

to convert $\mathrm{P_I}$ to

$$
\begin{aligned}
\tau' &= 1 + \frac{t}{4}\tau^4 + \frac{1}{4}\tau^5 - \frac{1}{2}\rho\tau^6, \\
\rho' &= \frac{t^2}{8}\tau + \frac{3t}{8}\tau^2 + \left(\frac{1}{4} - t\rho\right)\tau^3 - \frac{5}{4}\rho\tau^4 + \frac{3}{2}\rho^2\tau^5.
\end{aligned}
$$

Big $u(t_1)$ should imply bounds for $\tau(t_1)$ and $\rho(t_1)$. The bounds should give specific estimation on the range and bound for τ and ρ on $B(t_1, \delta)$.

Application: Painlevé Property for $\mathrm{P_I}$

$$|t_1| \le B,\ |u(t_1)| = A,\ |u| \le A \text{ along } [t_0, t_1] \implies |\tau(t_1)| = A^{-\frac{1}{2}}.$$

Then estimate $\rho(t_1)$ by using $\tau' = 1 + \frac{t}{4}\tau^4 + \frac{1}{4}\tau^5 - \frac{1}{2}\rho\tau^6$. We have

$$u'^2 = 4u^3 + 2tu - 2\int_{t_0}^{t} u\,dt + a, \quad \tau'^2 = 1 + \frac{t}{2}\tau^4 - \frac{1}{2}\tau^6 \int_{t_0}^{t} u\,dt + \frac{a}{4}\tau^6,$$

where $a = u'(t_0)^2 - 4u(t_0)^3 - 2t_0u(t_0)$ is constant. Then

$$\left|\tau'(t_1)^2 - \left(1 + \tfrac{t_1}{4}\tau(t_1)^4 + \tfrac{1}{4}\tau(t_1)^5\right)^2\right|$$
$$\le \left|\tau'(t_1)^2 - 1 - \tfrac{t_1}{2}\tau(t_1)^4\right| + \left|-\tfrac{1}{2}\tau(t_1)^5 + \left(\tfrac{t_1}{4}\tau(t_1)^4 + \tfrac{1}{4}\tau(t_1)^5\right)^2\right| < cBA^{-2}.$$

So $\tau'(t_1)^2$ is close to 1, and we may choose $\tau'(t_1)$ to be close to 1. Then the estimation implies $|\rho(t_1)| < cBA$.

Application: Painlevé Property for $\mathrm{P_I}$

Lemma (Enhanced existence and uniqueness for ODE solution)
Initial value problem ($f = (f_1, \ldots, f_n)$ and $w = (w_1, \ldots, w_n)$)

$$w' = f(t, w), \quad w(t_0) = w_0.$$

Suppose there are positive ϵ ($= 3A^{-\frac{1}{2}}$), ρ_i ($= c_1 A^{-\frac{1}{2}}$ and $c_2 BA$), L_i ($= 2$ and $c(c_2)B^2A^{-\frac{1}{2}}$), M_i, N_{ij}, a_i (both $= 1$), such that

1. If $|t - t_0| \le \epsilon$ and $|w_i - w_{i0}| \le \rho_i$, then

$$|f_i(t, w)| \le L_i, \quad |\partial_t f_i(t, w)| \le M_i, \quad |\partial_{w_j} f_i(t, w)| \le N_{ij};$$

2. $\epsilon L_i \le \rho_i$ (determine c_1, c_2), $\epsilon(\sum_i a_i N_{ij}) < a_j$ (determine A).

Then the IVP has a unique solution $w(t)$ for $|t - t_0| \le \epsilon$ and

$$|w_i(t) - w_{i0} - f_i(t_0, w_0)(t - t_0)| \le \frac{1}{2}(M_i + L_1 N_{i1} + \cdots + L_n N_{in})|t - t_0|^2.$$

Application: Painlevé Property for $\mathrm{P_I}$

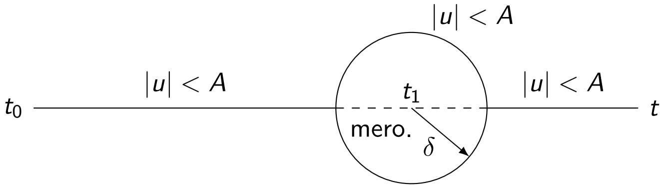
Conclusion: τ and ρ are analytic on $B(t_1, 3A^{-\frac{1}{2}})$ and satisfy

$$|\tau(t) - \tau(t_1) - \tau'(t_1)(t - t_1)| < cBA^{-3/2}|t - t_1|^2.$$

Since $\tau'(t_1)$ is very close to 1 and $|\tau(t_1)| = A^{-1/2}$, we have the following on the circle $|t - t_1| = 3A^{-1/2}$

$$|\tau(t)| \geq |\tau'(t_1)|3A^{-1/2} - A^{-1/2} - cBA^{-3/2}(3A^{-1/2})^2 > A^{-1/2}.$$

For $u = \tau^{-2}$, this means $|u| < A$ on the circle.



Application: Painlevé Property for $\mathrm{P_I}$

The similar idea works for $\mathrm{P_{II}}$ and $\mathrm{P_{IV}}$. But we had difficulty with th other Painlevé transcendents.

(2003) Hu-Yan: *An elementary and direct proof of the Painlevé property for the Painlevé equations I, II and IV*, J. Anal. Math.

Hinkkanen-Laine: J. Anal. Math.
(1999) *Solutions of the first and second Painlevé equations are meromorphic*
(2001) *Solutions of a modified third Painlevé equation are meromorphic*
(2004) *The meromorphic nature of the sixth Painlevé transcendents*

Our approach is the only conceptual and systematic one. Further study needed . . . .

Non-principal Balances

The function $u = \frac{1}{t-a} + \frac{1}{t-b}$ is all the solutions of

$$u'' + 3uu' + u^3 = 0.$$

The equation has one principal balance $(t_1 = c^{-1})$

$$u = \frac{1}{t-t_0} + \frac{1}{t-t_0+c} = (t-t_0)^{-1} + c^{-1} - c^{-2}(t-t_0) + -c^{-3}(t-t_0)^2 + \cdots,$$

and one non-principal balance

$$u = 2(t-t_0)^{-1}.$$

The non-principal balance is the "boundary" of the principal balance, and we get compatible tree of balances.

Further study needed ....

Final Remarks

Further Work

- Systematic way of proving the Painlevé property by using regularisation.
- Extend the regularisation to system of compatible balances.
- Multivariable meromorphic function?
- PDE.

Philosophical

- Functions v.s. Differential equations.
- Partial integrability.