Space-like Surfaces of Pseudo-Hyperbolic Space $\mathbb{H}^4_1(-1)$

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Introduction

- In late 1970's B.Y. Chen introduced the notion of **finite type submanifold** of a Euclidean space.
- Since then the **finite type submanifolds** of **Euclidean spaces** or **pseudo-Euclidean spaces** have been studied extensively, and many important results have been obtained ([2],[3],[6], etc.).

- In [5], Chen and Piccinni extended the notion of finite type to differentiable maps, in particular, to Gauss map of submanifolds.
- A smooth map ϕ from a compact Riemannian manifold M into a Euclidean space $\mathbb{E}^m$ is said to be of **finite type** if ϕ can be expressed as a finite sum of $\mathbb{E}^m$-valued eigenfunctions of the Laplacian Δ of M, that is,

$$\phi = \phi_0 + \phi_1 + \phi_2 + \cdots + \phi_k, \qquad (1)$$

where ϕ_0 is a constant map, $\phi_1, \ldots, \phi_k$ non-constant maps such that $\Delta\phi_i = \lambda_{p_i}\phi_i,\ \lambda_{p_i} \in \mathbb{R},\ i = 1, \ldots, k$.

- If $\lambda_{p_1}, \ldots, \lambda_{p_k}$ are mutually distinct, then the map ϕ is said to be of **k-type**.
- If ϕ is an isometric immersion, then M is called **a submanifold of finite type** (or of k-type).
- In the spectral decomposition of an immersion ϕ on a compact manifold, the constant vector ϕ_0 is the center of mass.

- In [6], Chen introduced the notion of a map of finite type on **a non-compact manifold**.
- When M is **non-compact** the vector ϕ_0 in the spectral decomposition in (1) is not necessary a constant vector.

Classical Gauss map

- Let $\mathbf{x} : M^n \to \mathbb{E}^m$ be an isometric immersion from a Riemannian n-manifold M^n into a Euclidean m-space $\mathbb{E}^m$.
- Let $G(n, m)$ denote the Grassmannian manifold consisting of linear n-subspaces of $\mathbb{E}^m$.
- The **classical Gauss map**

$$\nu^c : M^n \to G(n, m)$$

associated with $\mathbf{x}$ is the map which carries each point $p \in M$ to the linear subspace of $\mathbb{E}^m$ obtained by parallel displacement of the tangent space T_pM to the origin of $\mathbb{E}^m$.

- Since $G(n,m)$ can be canonically imbedded in the vector space $\bigwedge^n \mathbb{E}^m = \mathbb{E}^N$, $N = \binom{m}{n}$, obtained by the exterior products of n-vectors in $\mathbb{E}^m$, **the classical Gauss map** gives rise to a well-defined map from M^n into the Euclidean N-space $\mathbb{E}^N$ where $N = \binom{m}{n}$.

Obata's sense generalized Gauss map

- Let $\mathbf{x} : M^n \to \widetilde{M}^m$ be an isometric immersion from a Riemannian n-manifold M^n into a simply-connected complete m-space $\widetilde{M}^m$ of constant curvature.
- In [11], Obata studied the generalized Gauss map which assigns to each $p \in M$ the totally geodesic n-space tangent to $\mathbf{x}(M)$ at $\mathbf{x}(p)$.
- In the case $\widetilde{M}^m = \mathbb{S}^m$, the generalized Gauss map is also called the spherical Gauss map.
- If $\widetilde{M}^m = \mathbb{H}^m$, the generalized Gauss map is called the hyperbolic Gauss map.

- In [8], Chen and Lue studied spherical submanifolds with finite type spherical Gauss map. They obtained several results in this respect.

- In [10], we investigated submanifolds of hyperbolic spaces with **finite type hyperbolic Gauss map**.
- We characterized and classified submanifolds of the hyperbolic m-space $\mathbb{H}^m(-1)$ with finite type hyperbolic Gauss map.

Basic notations and formulas

- Let $\mathbb{E}_t^m$ denote the pseudo-Euclidean m-space with the canonical pseudo-Euclidean metric of index t given by

$$g_0 = \sum_{i=1}^{t} dx_i^2 - \sum_{j=t+1}^{m} dx_j^2, \qquad (2)$$

- where $(x_1, x_2, \ldots, x_m)$ is a rectangular coordinate system of $\mathbb{E}_t^m$.

$$\mathbb{S}_t^{m-1}(x_0, c) = \{x \in \mathbb{E}_t^m | \langle x - x_0, x - x_0 \rangle = c^{-1} > 0, c > 0\}$$
$$\mathbb{H}_t^{m-1}(x_0, -c) = \{x \in \mathbb{E}_{t+1}^m | \langle x - x_0, x - x_0 \rangle = -c^{-1} < 0, c > 0\},$$

- $\mathbb{S}_t^{m-1}(x_0, c)$ and $\mathbb{H}_t^{m-1}(x_0, -c)$ are complete pseudo-Riemannian manifolds with index t of constant curvature c and $-c$.

- An n-dimensional submanifold M of $\mathbb{H}^m_t(-1) \subset \mathbb{E}^{m+1}_{t+1}$ is said to be **space-like** if the metric induced on M from the ambient space $\mathbb{H}^m_t(-1)$ is positive definite.
- The mean curvature vector H of M in $\mathbb{E}^m_t$ is defined by

$$H = \frac{1}{n} \sum_{r=n+1}^{m} \varepsilon_r tr A_r e_r. \tag{3}$$

- If $H = 0$ holds identically, we call M is **maximal**.
- The **scalar curvature** S of M in $\mathbb{H}^{m-1}_{t-1}(-c)$, $c > 0$ is given

$$S = -cn(n-1) + n^2|\hat{H}|^2 - \|\hat{h}\|^2. \tag{4}$$

- A submanifold M is said to be **totally geodesic** if the second fundamental form h of M vanishes identically.
- M is called **totally umbilical** if its second fundamental form satisfies
$$h(X,Y) = \langle X, Y \rangle H$$
for vectors X and Y tangent to M.

Pseudo-hyperbolic Gauss map

- Let $\mathbf{x} : M^n \longrightarrow \mathbb{H}^{m-1}_s(-1) \subset \mathbb{E}^m_{s+1}$ be an isometric immersion from a space-like oriented Riemannian n-manifold M^n into a pseudo-hyperbolic $m-1$-space $\mathbb{H}^{m-1}_s(-1) \subset \mathbb{E}^m_{s+1}$. The Obata's map can be written as
$\hat{\nu} : M^n \to G(n+1, m)$

$$\hat{\nu}(p) = (\mathbf{x} \wedge e_1 \wedge e_2 \wedge \cdots \wedge e_n)(p).$$

- Considering the natural inclusion of $G(n+1, m)$ into $\mathbb{E}^N_q$, **the pseudo-hyperbolic Gauss map** $\tilde{\nu}$ associated with $\mathbf{x}$ is thus given by

$$\tilde{\nu} = \mathbf{x} \wedge e_1 \wedge e_2 \wedge \cdots \wedge e_n : M^n \to G(n+1, m) \subset \mathbb{E}^N_q, \quad (5)$$

where $N = \binom{m}{n+1}$.

- The Laplacian formula is given by

$$\triangle\tilde{\nu} = \sum_{i=1}^{n}(\nabla_{e_i}e_i - e_ie_i)\tilde{\nu}. \tag{6}$$

- Then we have

$$\begin{aligned}\Delta\tilde{\nu} &= \|\hat{h}\|^2\tilde{\nu} + n\hat{H}\wedge e_1\wedge\cdots\wedge e_n\\ &- n\sum_{k=1}^{n}\mathbf{x}\wedge e_1\wedge\cdots\wedge \underbrace{D_{e_k}\hat{H}}_{k-th}\wedge\cdots\wedge e_n\\ &+ \sum_{j,k=1}^{n}\sum_{\substack{r,s=n+1\\ s<r}}^{m-1}\varepsilon_r\varepsilon_s R^r_{sjk}\mathbf{x}\wedge e_1\wedge\cdots\wedge\underbrace{e_s}_{k-th}\wedge\cdots\wedge\underbrace{e_r}_{j-th}\wedge\cdots\wedge e_n,\end{aligned} \tag{7}$$

where $R^r_{sjk} = R^D(e_j, e_k; e_r, e_s)$.

- In [4], Chen investigated non-compact finite type pseudo-Riemannian submanifolds of a pseudo-Euclidean spaces.
- He gave the defition for the finite type submanifolds of the pseudo-Riemannian sphere $\mathbb{S}_t^{m-1}$ or the pseudo-hyperbolic space $\mathbb{H}_{t-1}^{m-1}$.
- In [9], Dursun constructed the definition for a smooth map as the following:

- A smooth map

$$\phi : M_q \longrightarrow \mathbb{H}^{m-1}_{t-1}(-1) \subset \mathbb{R}^m_t$$

from a pseudo-Riemannian manifold M_q into a pseudo-hyperbolic space $\mathbb{H}^{m-1}_{t-1}(-1)$ is called of k-type in $\mathbb{H}^{m-1}_{t-1}(-1)$ if the map ϕ has the following form:

$$\phi = \phi_1 + \phi_2 + \cdots + \phi_k, \quad \Delta\phi_i = \lambda_i \phi_i, \quad \lambda_i \in \mathbb{R}, i = 1, \ldots, k, \tag{8}$$

such that $\lambda_1, \ldots, \lambda_k$ are all distinct.

- Moreover, according to definition one of the component in the spectral decomposition may be constant.

First, we state the following Proposition.

Proposition

Let $\mathbf{x} : (M^n, g) \longrightarrow \mathbb{H}_1^{m-1}(-1) \subset \mathbb{E}_2^m$ be an isometric immersion from a space-like manifold M^n in an anti-de Sitter space $\mathbb{H}_1^{m-1}(-1) \subset \mathbb{E}_2^m$. Then, one of the following cases occurs:

(a) the Obata's map $\hat{\nu} : (M^n, g) \longrightarrow G(n+1, m)$ is a harmonic map if and only if $\mathbf{x} : (M^n, g) \longrightarrow \mathbb{H}_1^{m-1}(-1)$ is a maximal immersion;

(b) the pseudo-hyperbolic Gauss map $\tilde{\nu} : (M^n, g) \longrightarrow \mathbb{E}_S^N$ with $N = \binom{m}{n+1}$ and $S = 2\binom{m-2}{n}$ is a harmonic map if and only if

(b.1) $\mathbf{x} : (M^n, g) \longrightarrow \mathbb{H}_1^{m-1}(-1) \subset \mathbb{E}_2^m$ is a totally geodesic immersion, or

(b.2) M^n is maximal, has flat normal bundle and scalar curvature should satisfy the following equality $S = -n(n-1)$.

In this section, we classify space-like surfaces in $\mathbb{H}^4_1(-1) \subset \mathbb{E}^5_2$ with 1-type pseudo-hyperbolic Gauss map.

Theorem 1

A space-like surface M of $\mathbb{H}^4_1(-1) \subset \mathbb{E}^5_2$ has 1-type pseudo-hyperbolic Gauss map if and only if M is a maximal surface of $\mathbb{H}^4_1(-1)$ with M has constant scalar curvature and flat normal bundle.

Proof here

We obtain the following corollaries as an immediate consequnece of Theorem 1.

Corollary 1

Let M be a space-like surface in an anti-de Sitter space $\mathbb{H}^3_1(-1) \subset \mathbb{E}^4_2$. Then, M has 1-type pseudo-hyperbolic Gauss map if and only if it is maximal surface of $\mathbb{H}^3_1(-1) \subset \mathbb{E}^4_2$ with constant scalar curvature.

Corollary 2

A totally geodesic hyperboloid $\mathbb{H}^2(-1)$ in $\mathbb{H}^4_1(-1)$ has biharmonic pseudo-hyperbolic Gauss map which is of 1-type.

Example 1

Maximal space-like surface in $\mathbb{H}^3_1(-1)$

Let $\mathbf{x} : M = \mathbb{H}^1(-a^{-2}) \times \mathbb{H}^1(-b^{-2}) \longrightarrow \mathbb{H}^3_1(-1) \subset \mathbb{E}^4_2$ be an isometric immersion from M into $\mathbb{H}^3_1(-1)$ defined by

$$\mathbf{x}(u, v) = (a \sinh u, b \sinh v, a \cosh u, b \cosh v)$$

with $a^2 + b^2 = 1$. If we put $e_1 = \frac{1}{a}\frac{\partial}{\partial u}, \quad e_2 = \frac{1}{b}\frac{\partial}{\partial v}$,

$$e_3 = (b \sinh u, -a \sinh v, b \cosh u, -a \cosh v), \quad e_4 = \mathbf{x}$$

then $\{e_1, e_2, e_3, e_4\}$ form an orthonormal frame field on M in $\mathbb{E}^4_2$.

A straightforward computation gives

$$\begin{aligned} &h^3_{11} = -\frac{b}{a},\ h^3_{12} = h^4_{12} = 0,\ h^3_{22} = \frac{a}{b}, \\ &h^4_{11} = h^4_{22} = -1,\ \omega_{12} = \omega_{34} = 0, \omega_{13} = -\frac{b}{a}\omega^1, \\ &\omega_{23} = \frac{a}{b}\omega^2, \omega_{14} = -\omega^1, \omega_{24} = -\omega^2. \end{aligned} \tag{9}$$

- The equation (9) yields $\hat{H} = \frac{a^2-b^2}{ab}e_3$ which gives M is a maximal surface if and only if $a = b = \frac{1}{\sqrt{2}}$
- Therefore, $\mathbb{H}^1(-2) \times \mathbb{H}^1(-2) \subset \mathbb{H}^3_1(-1) \subset \mathbb{E}^4_2$ is a maximal and flat surface. It is obvious that $\mathbb{H}^1(-2) \times \mathbb{H}^1(-2)$ has 1-type pseudo-hyperbolic Gauss map by Theorem 1.

Theorem 2

Let M be a space-like surface in an anti-de Sitter space $\mathbb{H}^{m-1}_1(-1) \subset \mathbb{E}^m_2$. Then, M has 1-type pseudo-hyperbolic Gauss map if and only if M is congruent to an open part of $\mathbb{H}^1(-2) \times \mathbb{H}^1(-2)$ lying in $\mathbb{H}^3_1(-1) \subset \mathbb{H}^{m-1}_1(-1) \subset \mathbb{E}^m_2$ for some $m \geq 5$ or the totally geodesic hyperbolic space $\mathbb{H}^2(-1)$ lying in $\mathbb{H}^{m-1}_1(-1) \subset \mathbb{E}^m_2$ for some $m \geq 5$.

Example 2

- **Space-like surface with flat normal bundle and null mean curvature vector in $\mathbb{H}^4_1(-1)$**
- Let $\mathbf{x} : M \longrightarrow \mathbb{H}^4_1(-1) \subset \mathbb{E}^5_2$ be a space-like isometric immersion from a surface M into an anti-de Sitter space $\mathbb{H}^4_1(-1)$. We consider a surface M in $\mathbb{H}^4_1(-1) \subset \mathbb{E}^5_2$ as follows

$$\mathbf{x}(u, v) = (1, \cosh u \sinh v, \sinh u, \cosh u \cosh v, 1),$$

[7].
- If we put $e_1 = \frac{\partial}{\partial u}, \quad e_2 = \frac{1}{\cosh u}\frac{\partial}{\partial v},$

$$e_3 = (\frac{3}{2}, \cosh u \sinh v, \sinh u, \cosh u \cosh v, \frac{1}{2})$$

and

$$e_4 = (\frac{1}{2}, \cosh u \sinh v, \sinh u, \cosh u \cosh v, -\frac{1}{2}), \; e_5 = \mathbf{x}$$

then $\{e_i\}$ for $i = 1, 2, 3, 4, 5$ form an orthonormal frame field on M.

- A straightforward computation gives

$$\begin{aligned} h^3_{11} = h^3_{22} = h^4_{11} = h^4_{22} = -1, \quad h^3_{12} = h^4_{12} = 0, \\ \omega_{12}(e_1) = 0, \quad \omega_{12}(e_2) = \tanh u, \quad \omega_{34} = 0, \\ \|\hat{h}\|^2 = 0, \quad \hat{H} = e_4 - e_3 = (-1, 0, 0, 0, -1). \end{aligned} \tag{10}$$

- If we use (10), then equation (7) reduces to

$$\triangle \tilde{\nu} = 2\hat{H} \wedge e_1 \wedge e_2 = -2e_3 \wedge e_1 \wedge e_2 + 2e_4 \wedge e_1 \wedge e_2. \tag{11}$$

- If we put

$$\tilde{c} = \tilde{\nu} - e_3 \wedge e_1 \wedge e_2 + e_4 \wedge e_1 \wedge e_2 \tag{12}$$

and

$$\tilde{\nu}_p = e_3 \wedge e_1 \wedge e_2 - e_4 \wedge e_1 \wedge e_2 \tag{13}$$

then we have $\tilde{\nu} = \tilde{c} + \tilde{\nu}_p$.

- It can be shown that $e_i(\tilde{c}) = 0$ for $i = 1, 2$, i.e., $\tilde{c}$ is a constant vector. Using (11), (12) and (13), we get $\Delta \tilde{\nu}_p = -2\tilde{\nu}_p$.
- Thus, M has 1-type pseudo-hyperbolic Gauss map with a nonzero constant component in its spectral decomposition.

Now, we determine space-like surfaces of $\mathbb{H}^4_1(-1) \subset \mathbb{E}^5_2$ with 1-type pseudo-hyperbolic Gauss map containing a nonzero constant component in its spectral decomposition.

Theorem 3 A space-like surface M in the anti-de Sitter space $\mathbb{H}^4_1(-1) \subset \mathbb{E}^5_2$ has 1-type pseudo-hyperbolic Gauss map with a nonzero constant component in its spectral decomposition if and only if M is an open part of the following surfaces:

(1) A non-totally geodesic, totally umbilical space-like surface in a totally geodesic hyperbolic space $\mathbb{H}^3(-1) \subset \mathbb{H}^4_1(-1)$ with mean curvature $|\hat{\alpha}| \neq 1$, that is, M is an open part of a hyperbolic 2-space $\mathbb{H}^2(-c)$ of curvature $-c$ for $0 < c < 1$ in $\mathbb{H}^3(-1) \subset \mathbb{H}^4_1(-1)$ or a 2-sphere $\mathbb{S}(c)$ of curvature c for $c > 0$ in $\mathbb{H}^3(-1) \subset \mathbb{H}^4_1(-1)$ or

(2) A hyperbolic 2-space $\mathbb{H}^2(-c)$ of curvature $-c$ for $c > 1$ in $\mathbb{H}^3_1(-1) \subset \mathbb{H}^4_1(-1)$ or

(3) The surface defined by

$$\mathbf{x}(u, v) = (1, \cosh u \sinh v, \sinh u, \cosh u \cosh v, 1)$$

which is of curvature -1 and totally umbilical with constant lightlike mean curvature vector.

- In this section, we give a characterization of space-like hypersurfaces in an anti-de Sitter space $\mathbb{H}_1^{n+1}(-1)$ with constant mean curvature vector and 2-type pseudo-hyperbolic Gauss map.

Theorem 4 Let M be a space-like, non-totally umbilical hypersurface with nonzero constant mean curvature $\hat{\alpha}$ in an anti-de Sitter space $\mathbb{H}_1^{n+1}(-1) \subset \mathbb{E}_2^{n+2}$. Then, M has 2-type pseudo-hyperbolic Gauss map $\tilde{\nu}$ if and only if it has constant scalar curvature.

- In [1], Zhen-qi and Xian-hua determined space-like, isoparametric hypersurface M^n in $\mathbb{H}_1^{n+1}(-1) \subset \mathbb{E}_2^{n+2}$. They showed that M is congruent to an open subset of a umbilical hypersurface $\mathbb{H}^n(-c)$ of curvature $-c$ with $c > 0$ or the product of two hyperbolic spaces,

$$\begin{aligned} \mathbb{H}^k(-c_1)\times \quad \mathbb{H}^{n-k}(-c_2) = \{(x,y) \in \mathbb{R}_1^{k+1} \times \mathbb{R}_1^{n-k+1} : \\ < x, x >= -{c_1}^{-1}, < y, y >= -{c_2}^{-1}\}, \quad (14) \end{aligned}$$

  where $c_1, c_2 > 0$.
- The product hypersurface $\mathbb{H}^k(-c_1) \times \mathbb{H}^{n-k}(-c_2)$ with $c_1 \neq c_2 > 0$ is a non-umbilical isoparametric hypersurface in an anti-de Sitter space $\mathbb{H}_1^{n+1}(-1)$ having non-zero mean curvature and constant scalar curvature. Therefore, we obtain the following Corollary.

Corollary 3 A product two hyperbolic spaces $\mathbb{H}^k(-c_1) \times \mathbb{H}^{n-k}(-c_2)$ with $c_1 \neq c_2 > 0$ in $\mathbb{H}_1^{n+1}(-1)$ is the only isoparametric hypersurface with 2-type pseudo-hyperbolic Gauss map.

We also classify space-like surfaces with constant mean curvature in an anti-de Sitter space $\mathbb{H}^3_1(-1) \subset \mathbb{E}^4_2$ having 2-type pseudo-hyperbolic Gauss map.

Theorem 4 A space-like surface M in an anti-de Sitter space $\mathbb{H}^3_1(-1) \subset \mathbb{E}^4_2$ with a non-totally umbilical and nonzero constant mean curvature in $\mathbb{H}^3_1(-1)$ has 2-type pseudo-hyperbolic Gauss map it is congruent to open portion of the $\mathbb{H}^1(-a^{-2}) \times \mathbb{H}^1(-b^{-2})$ in $\mathbb{H}^3_1(-1)$ with $a^2 + b^2 = 1$, $a \neq b$.

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THANK YOU ...

- Assume that M is a space-like surface in an anti-de Sitter space $\mathbb{H}_1^4(-1) \subset \mathbb{E}_2^5$ with 1-type pseudo-hyperbolic Gauss map.
- Then, $\Delta\tilde{\nu} = \lambda_p \tilde{\nu}$ for some $\lambda_p \in \mathbb{R}$.
- From equation (7) the pseudo-hyperbolic Gauss map $\tilde{\nu}$ is 1-type if and only if $\hat{H} = R^D = 0$ and $\|\hat{h}\|^2$ is constant.
- Moreover, by (4) it seen that M has constant scalar curvature.

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First, we will calculate $\Delta(e_{n+1} \wedge e_1 \wedge \cdots \wedge e_n)$. We take $\bar{\nu} = e_{n+1} \wedge e_1 \wedge \cdots \wedge e_n$. If we differentiate $\bar{\nu}$, we obtain

$$e_i \bar{\nu} = e_{n+1} \wedge e_1 \wedge \cdots \wedge \underbrace{\mathbf{x}}_{i-th} \wedge \cdots \wedge e_n. \tag{15}$$

Considering (6) and $n\hat{\alpha} = \sum_{i=1}^{n} h_{ii}^{n+1}$ we get

$$\Delta\bar{\nu} = -n\hat{\alpha}\tilde{\nu} - n\bar{\nu} + \sum_{i,j=1}^{n} (\omega_{ij}(e_i) + \omega_{ji}(e_i)) e_{n+1} \wedge e_1 \wedge \cdots \wedge \underbrace{\mathbf{x}}_{j-th} \wedge \cdots \wedge e_n$$
$$= -n(\hat{\alpha}\tilde{\nu} + \bar{\nu}). \tag{16}$$

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