Weyl Manifold, a Quantized Symplectic Manifold

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§1. Deformation quantization

1. Weyl manifold naturally emerges when we consider to glue together quantized canonical coordinates by means of quantized canonical transformations, and is deeplly related to deformation quantization.
2. Actually, from a Weyl manifold we can construct a deformation quantization, and also from a deformation quantization we obtain a Weyl manifold.
3. In this talk we explain an idea of Weyl manifold as a quantized symplectic manifold.

§1.1. The Moyal product

Weyl manifold is deeply related to deformation quantization. We start by giving a very important example of deformation quantization, the Moyal product.

Canonical symplectic strucure. Let us consider $2n$ dimensional euclidean space $\mathbb{R}^{2n}$ with coordinates

$$(x_1, \ldots, x_n, y_1, \ldots, y_n)$$

and the canonical symplectic structure

$$\omega_0 = \sum_{k=1}^{n} dy_k \wedge dx_k.$$

Poisson biderivation.

Its canonical Poisson bracket is given by

$$\{f, g\}_0 = \sum_{k=1}^{n} (\partial_{x_k} f\, \partial_{y_k} g - \partial_{y_k} f\, \partial_{x_k} g), \quad f, g \in C^\infty(\mathbb{R}^{2n})$$

and this can be written as the Poisson biderivation as

$$= \sum_{k=1}^{n} (f \overleftarrow{\partial}_{x_k} \overrightarrow{\partial}_{y_k} g - f \overleftarrow{\partial}_{y_k} \overrightarrow{\partial}_{x_k} g) = f \overleftarrow{\partial}_x \cdot \overrightarrow{\partial}_y g - f \overleftarrow{\partial}_y \cdot \overrightarrow{\partial}_x g$$

$$= f(\overleftarrow{\partial}_x \cdot \overrightarrow{\partial}_y - \overleftarrow{\partial}_y \cdot \overrightarrow{\partial}_x) g = f \overleftarrow{\partial}_x \wedge \overrightarrow{\partial}_y g.$$

The l th power of the Poisson biderivation

is calculated by means of the binomal theorem such as

$$\left(\overleftarrow{\partial}_x \wedge \overrightarrow{\partial}_y\right)^l = \sum_{k=0}^{l} \begin{pmatrix} l \\ k \end{pmatrix} (-1)^k (\overleftarrow{\partial}_x \cdot \overrightarrow{\partial}_y)^{l-k} (\overleftarrow{\partial}_y \cdot \overrightarrow{\partial}_x)^k$$

which defines a bidifferential operator $f \left(\overleftarrow{\partial}_x \wedge \overrightarrow{\partial}_y\right)^l g$.

Moyal product $*_0$

The Moyal product $*_0$ is given by a formal power seris of the Poisson biderivation of the exponential type such that

$$\begin{aligned} f *_0 g &= fg + (\tfrac{\nu}{2}) f(\overleftarrow{\partial}_x \wedge \overrightarrow{\partial}_y) g + \cdots + (\tfrac{\nu}{2})^l \tfrac{1}{l!} f(\overleftarrow{\partial}_x \wedge \overrightarrow{\partial}_y)^l g + \cdots \\ &= f \exp\left(\tfrac{\nu}{2} \overleftarrow{\partial}_x \wedge \overrightarrow{\partial}_y\right) g, \quad f, g \in C^\infty(\mathbb{R}^{2n}), \end{aligned}$$

where ν is a formal parameter.

This is also written in general form such that

$$f *_0 g = fg + \nu C_1(f,g) + \nu^2 C_2(f,g) + \cdots + \nu^l C_l(f,g) + \cdots ,$$

where $C_l(f,g) = f \frac{1}{l!}(\frac{1}{2})^l (\overleftarrow{\partial}_x \wedge \overrightarrow{\partial}_y)^l g$, $l = 1, 2, \cdots$ are bidifferential operators.

Quantized canonical coordinates

- The Moyal product is naturally extended to the space of all formal power series such as $f, g \in C^\infty(\mathbb{R}^{2n})[[\nu]]$.
- Then it is easy to see

Proposition

The Moyal product is an associative product on the space of formal power series $C^\infty(\mathbb{R}^{2n})[[\nu]]$.

- The Moyal product $*_0$ is depending on the canonical coordinates $(x_1, \ldots, x_n, y_1, \ldots, y_n)$. Then the associative algebra $(C^\infty(\mathbb{R}^{2n})[[\nu]], *_0)$ can be regarded as quantized canonical coordinates.

§1.2. Deformation quantization on symplectic manifold

Deformation quantization is defined similary as the Moyal product.

Let (M, ω) be a symplectic manifold. We consider a binary product on the space of formal power series $C^\infty(M)[[\nu]]$ such that

$$f * g = fg + \nu C_1(f,g) + \nu^2 C_2(f,g) + \cdots + \nu^l C_l(f,g) + \cdots ,$$

where $C_l(\cdot, \cdot)$ are bidifferential operators from $C^\infty(M) \times C^\infty(M)$ to $C^\infty(M)$.

$$f * g = fg + \nu C_1(f,g) + \nu^2 C_2(f,g) + \cdots + \nu^l C_l(f,g) + \cdots ,$$

Definition

A product $f * g$ is called a deformation quantization of symplectic manifold (M, ω) if it is associative on the space $C^\infty(M)[[\nu]]$ and $C_1(f,g)$ coincides with the Poisson bracket of ω.

Then for a deformation quantization $*$ of (M, ω), we have an associative algebra $(C^\infty(M)[[\nu]], *)$, called a deformation quantization algebra.

§2. Weyl manifold

Let (M, ω) be a $2n$ dimensional symplectic manifold.

1. A Weyl manifold W_M is a Weyl algebra bundle over (M, ω) with certain properties.
2. Weyl manifold has a deep relationship with deformation quantization of symplectic manifold.
3. This section is based on the joint work with H. Omori, Y. Maeda.

§2.1. Idea

1. By the Darboux theorem, symplectic manifold can be obtained by patching together the canoical coordinates by canonical transoformations.
2. A similar theorem to the Darboux theoerm holds for deformation quantization of symplectic manifolds.

Quantized Darboux theorem

Suppse we have a deformation quantization $*$ of the symplectic manifold (M, ω):

$$f * g = fg + \nu C_1(f,g) + \nu^2 C_2(f,g) + \cdots + \nu^l C_l(f,g) + \cdots .$$

We have a ”quantized Darboux theorem” as follows.

Proposition

On every canonical coordinate neighbourhood U, the star product algebra $(C^\infty(U)[[\nu]], *)$ is isomorphic to the Moyal product algebra $(C^\infty(U)[[\nu]], *_0)$.

Deformation quantization and Weyl manifold

1. Simlarly to a symplectic manifod (M, ω), we consider to construct a deformation quantization of (M, ω) by patching together quantized canonical coordinates by the algebra isomorphisms.
2. But this can be done not directly and not so easily. For this purpose, we first construct a Weyl algebra bundle over (M, ω) from which we can obtain a deformation quantization.
3. The algebra bundle is called a Weyl manifold.

§2.2. Weyl manifold

In order to define a Weyl manifold, we need a formal Weyl algebra.

Formal Weyl algebra

A formal Weyl algebra W is the set of all formal power series of elements $\nu, X_1, \ldots, X_n, Y_1, \ldots, Y_n$,

$$W = \mathbb{C}[[\nu, X_1, \ldots, X_n, Y_1, \ldots, Y_n]]$$

with the product $\hat{*}$ such that

$$\begin{aligned} F \hat{*} G &= F \exp\left(\frac{\nu}{2} \overleftarrow{\partial}_X \wedge \overrightarrow{\partial}_Y \right) G \\ &= FG + (\tfrac{\nu}{2}) F (\overleftarrow{\partial}_X \wedge \overrightarrow{\partial}_Y) G + \cdots + (\tfrac{\nu}{2})^l \tfrac{1}{l!} F (\overleftarrow{\partial}_X \wedge \overrightarrow{\partial}_Y)^l G + \cdots \end{aligned}$$

for $F = \sum_{l\alpha} a_{l\alpha} \nu^l Z^\alpha$, $G = \sum_{m\beta} b_{m\beta} \nu^m Z^\beta \in W$.

Here use notations for simplicity such that

$$z = (z_1, \cdots, z_{2n}) = (x_1, \cdots, x_n, y_1, \cdots, y_n)$$
$$Z = (Z_1, \ldots, Z_{2n}) = (X_1, \ldots, X_n, Y_1, \ldots, Y_n)$$

The formal Weyl algebra W is an associative algebra satisfying the canonical commutation relation

$$[X_j, Y_k]_* = \nu\delta_{jk}, \quad [X_j, X_k]_* = [Y_j, Y_k]_* = 0, \quad j,k = 1,2,\ldots,n.$$

Here the bracket $[\cdot,\cdot]_*$ is the commutator of W ;
$[F,G]_* = F\hat{*}G - G\hat{*}F$, $F,G \in W$.

Weyl function

Let U be an open subset of $\mathbb{R}^{2n}$.

- We consider to embed a function f on U into a formal Weyl algebra W. The embedding is called a Weyl continuation of function f denoted by $f^{\#}$ such that

$$f^{\#}(z) = \sum_{\alpha} \frac{1}{\alpha!} \partial_z^{\alpha} f(z) Z^{\alpha}, \quad z \in U.$$

- The Weyl continuation $f^{\#}(z)$ is called a Weyl function of f and gives a section of the trivial Weyl algebra bundle $U \times W = W_U$.
- We denote the set of all Weyl functions by $\mathcal{F}(W_U)$.
- $\mathcal{F}(W_U)$ is naturally equipped with the multiplication $\hat{*}$ and becomes an associative algebra.

It is direct to see

Proposition

the Weyl continuation gives an algebra isomorphism

$$\# : (C^\infty(U)[[\nu]], *_0) \to (\mathcal{F}(W_U), \hat{*})$$

namely

$$(f *_0 g)^\# = f^\# \hat{*} g^\#, \quad \forall f, g.$$

Weyl diffeomorphism

1. Instead of gluing local quantized canonical coordinates $(C^\infty(U)[[\nu]], *_0)$, we glue the Weyl function algebras $(\mathcal{F}(W_U), \hat{*})$.
2. Since $\mathcal{F}(W_U)$ is a certain class of sections of the trivial budle $W_U = U \times W$, we consider the following bundle isomorphism.

Definition

A bundle isomorphism $\Phi : W_U \to W_{U'}$ with induced map $\phi : U \to U'$ is called a Weyl diffeomorphism when

(i) $\Phi(\nu) = \nu$.

(ii) $\Phi^* \mathcal{F}(W_{U'}) = \mathcal{F}(W_U)$.

(iii) $\Phi^* f^\# = (\phi^* f)^\# + O(\nu^2), \quad f \in C^\infty(U')[[\nu]]$.

As to the induced map of Weyl diffeomorpism we have the following.

Lemma

The induced map $\phi : U \to U'$ *of a Weyl differomorphism* $\Phi : W_U \to W_{U'}$ *is a canonical transformation.*

On the other hand, we have

Theorem

For a canonical transformation $\phi : U \to U'$*, there exists a Weyl diffeomorphism* $\Phi : W_U \to W_{U'}$ *with induced map* ϕ*.*

The Weyl diffeomorphism $\Phi : W_U \to W_{U'}$ is regarded as a quantized canonical transformation.

Existence of Weyl manifold and deformation quantizaiton

1. We take canoical coordinete systems $\{(U_\alpha, z_\alpha)\}_{\alpha\in\Lambda}$ for a symplectic manifold (M, ω). Then (M, ω) is given by patching together $\{(U_\alpha, z_\alpha)\}_{\alpha\in\Lambda}$ by canonical transformations $\phi_{\alpha\beta}$ between U_α and U_β.
2. Then we can take Weyl diffeomorphisms $\Phi_{\alpha\beta}$ between trivial bundles W_{U_α} and W_{U_β} by quantizing the canoical transformations $\phi_{\alpha\beta}$.
3. We glue local trivializations $\{W_{U_\alpha}\}_{\alpha\in\Lambda}$ by the Weyl diffeomorphisms $\Phi_{\alpha\beta}$ and then we obtain

Theorem

For any symplectic manifold (M, ω), there exists a Weyl manifold W_M.

Deformation quantization

From a Weyl manifold we can obtain a deformation quantization of the symplectic manifold in the following way.

- By Weyl diffeomorphisms $\Phi_{\alpha\beta}$, the local Weyl functions $\mathcal{F}(W_{U_\alpha})$ are also glued togher to give global Weyl functions, which are subsets of sections of the Weyl manifold W_M.
- We denote this algebra of the global Weyl functions by $(\mathcal{F}(W_M), \hat{*})$ called a Weyl function algebra on M.
- Then we have

Theorem

We have a $\mathbb{C}[[\nu]]$-linear map $\sigma : C^\infty(M)[[\nu]] \to \mathcal{F}(W_M)$.

- By means of this linear isomorphism we can define an associative product on $C^\infty(M)[[\nu]]$ by

$$f * g = \sigma^{-1}(\sigma(f)\hat{*}\sigma(g)).$$

- By expanding this assciative product in the power of ν we see that the prouct $*$ is a deformation quantization of (M, ω).
- Namely we have

Theorem

For every symplectic manifold (M, ω), there exists a deformation quantization of the symplectic manifold.

Thank you very much for your attention!

Reference

Hideki Omori, Yoshiaki Maeda, Akira Yoshioka,

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