Complex and real hypersurfaces of locally conformal Kähler manifolds

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Topics

1. Preliminaries

Let M^n be a $n = 2m$-dimensional differentiable manifold covered with local coordinate systems (x^i).

Definition 1.

An *almost complex structure* on a manifold M^n is a $(1,1)-$tensor field $J = (J_j^k)$ satisfying the equation

$$J^2 = -E : \quad J_i^k J_j^i = -\delta_j^k$$

where $E = (\delta_j^k)$ is the unit tensor field on M^n. A manifold M^n with such a structure J is called an *almost complex manifold*.

If it is possible to find coordinates so that J takes the canonical form

$$(J_j^i) = \begin{pmatrix} \sqrt{-1}\delta_\beta^\alpha & 0 \\ 0 & -\sqrt{-1}\delta_{\hat{\beta}}^{\hat{\alpha}} \end{pmatrix} \quad \alpha, \beta = \overline{1, m}$$

on an entire neighborhood of any given point p, then J is said to be *integrable*.

A manifold M^n with an integrable structure J is called a *complex manifold*.

A *Hermitian metric* on a manifold M^n is a Riemannian metric g such that

$$g(JX, JY) = g(X, Y): \quad J_i^k g_{kl} J_j^l = g_{ij}$$

for any vector fields X and Y on M^{2m}.

An almost complex manifold with a Hermitian metric is called an *almost Hermitian manifold* and a complex manifold with a Hermitian metric is called a *Hermitian manifold*.

We put $\Omega(X, Y) = g(X, JY)$ and call it a *fundamental 2-form* of an almost complex manifold. A Hermitian metric g on an almost complex manifold M^{2m} is called a *Kähler metric* if the fundamental 2-form Ω is closed:

$$d\Omega = 0.$$

An almost complex manifold M^n with a Kähler metric is called an *almost Kähler manifold*. A complex manifold M^{2m} with a Kähler metric is called a *Kähler manifold*.

Let (M^{2m}, J, g) be a Hermitian manifold of complex dimension m, where J denotes its complex structure, and g its Hermitian metric.

Definition 2.

A Hermitian manifold (M^{2m}, J, g) is called

a *locally conformal Kähler manifold (LCK - manifold)*

if there is an open cover $\mathfrak{U} = \{U_\alpha\}_{\alpha \in A}$ of M^{2m} and a family $\{\sigma_\alpha\}_{\alpha \in A}$ of C^∞ functions $\sigma_\alpha : U_\alpha \to \mathbb{R}$ so that each local metric

$$\hat{g}_\alpha = e^{-2\sigma_\alpha} g|_{U_\alpha}$$

is Kählerian.

An LCK - manifold is endowed with some closed form ω, so called a *Lee form* which can be calculated as

$$\omega = \frac{1}{m-1} \delta\Omega \circ J : \quad \omega_i = -\frac{2}{n-2} J^\alpha_{\beta,\alpha} J^\beta_i .$$

Let $\overline{M}^k$ be an k-dimensional manifold immersed in an $2m$-dimensional Riemannian manifold M^{2m}. Let ∇ and $\overline{\nabla}$ be operators of covariant differentiations on M^{2m} and $\overline{M}^k$, respectively. Then the Gauss and Weingarten formulas are given by [Chen B.Y., p. 2] :

$$\begin{aligned} \nabla_X Y &= \overline{\nabla}_X Y + h(X, Y), \\ \nabla_X \xi &= -A_\xi X + \nabla^\perp_X \xi, \end{aligned}$$

respectively, where X and Y are vector fields tangent to $\overline{M}^k$ and ξ normal to $\overline{M}^k$.
$h(X, Y)$ is the *second fundamental form*, $\nabla^\perp$ the linear connection induced in the normal bundle $E(\Psi)$, called the *normal connection*, and A_ξ the *second fundamental tensor* at ξ.

We call $\overline{M}^k$ *$CR-$ submanifold* of (M^{2m}, J, g) if $\overline{M}^k$ carries a C^∞ distribution $\mathcal{D}$ so that

1. $\mathcal{D}$ is *holomorphic* (i.e. $J_x(\mathcal{D}_x) = \mathcal{D}_x$) for any $x \in \overline{M}^k$,
2. the orthogonal complement $\mathcal{D}^\perp$ with respect to $\overline{g} = \Psi^* g$ of $\mathcal{D}$ in $T(\overline{M}^k)$ is *anti-invariant* (i.e. $J_x({\mathcal{D}^\perp}_x) \subseteq E(\Psi)_x$) for any $x \in \overline{M}^k$)[Dragomir S., Ornea L., p. 153]

Let $(\overline{M}^k, \mathcal{D})$ be a $CR-$submanifold of the Hermitian manifold M_0^{2m}. Set $p = \dim_{\mathbb{C}} \mathcal{D}_x$ and $p = \dim_{\mathbb{R}} \mathcal{D}_x^{\perp}$; for any $x \in \overline{M}^k$ so that $2p + q = k$. If $q = 0$ then $\overline{M}^k$ is a *complex submanifold*, i.e. it is a complex manifold and Ψ is a holomorphic immersion. If $p = 0$ then $\overline{M}^k$ is an *anti-invariant submanifold* (i.e. $J_x(T_x(\overline{M}^k)) \subseteq E(\Psi)_x$ for any $x \in \overline{M}^k$). A $CR-$submanifold $(\overline{M}^k, \mathcal{D})$ is *proper* if $p \neq 0$ and $q \neq 0$. Also $(\overline{M}^k, \mathcal{D})$ is *generic* if $q = 2m - k$ (i.e. $J_x(T_x(\overline{M}^k)) = E(\Psi)_x$ for any $x \in \overline{M}^k$). A submanifold $\overline{M}^k$ of the complex manifold (M^{2m}, J) is *totally real* if

$$T_x(\overline{M}^k) \cap J_x(T_x(\overline{M}^k)) = \{0\}$$

for any $x \in \overline{M}^k$

Let submanifold $\overline{M}^k$ is immersed in LCK-manifold M^{2m}

$$\Psi : \overline{M}^k \longrightarrow M^{2m},$$

so that $k = 2p$ and for any $x \in \overline{M}^{2p}$

$$J_x(T_x(\overline{M}^{2p})) = T_x(\overline{M}^{2p}).$$

Let $\overline{M}^{2p}$ be represented by

$$x^\alpha = x^\alpha(y^1, ..., y^{2p}),$$

where $\alpha = 1, ..., 2m$ and y^i, $i = 1, ..., 2p$ are local coordinate systems respectively on M^{2m} and on $\overline{M}^{2p}$. Then the tangent subspace of $\overline{M}^{2p}$ at each point $x = x(y)$ is spanned by vectors:

$$B_i^\alpha = \partial_i x^\alpha.$$

If a tensor $g_{\alpha\beta}$ is a Riemannian metric of M^{2m} then induced metric of $\overline{M}^{2p}$ take the form:

$$\overline{g}_{ij} = B_i^\alpha g_{\alpha\beta} B_j^\beta.$$

We can define tensors

$$B_\alpha^i = B_j^\beta \overline{g}^{ij} g_{\alpha\beta};$$

and

$$\overline{J}_i^j = J_\beta^\alpha B_\alpha^j B_i^\beta.$$

The late is an almost complex structure induced on $\overline{M}^{2p}$ by the immersion.

We obtained the following Theorem:

Theorem 1.

If a complex submanifold $\overline{M}^{2p}$ is immersed in a LCK-manifold M^{2m} then immersed $\overline{M}^{2p}$ is a LCK-manifold. Moreover if Lee field $B = \omega^{\#}$ defined in M^{2m} is normal to $\overline{M}^{2p}$, then immersed $\overline{M}^{2p}$ is a Kähler one.

Similar results were published in [Ianuş S. Matsumoto K. Ornea L.]

But it is important to explore the immersions also with regard to the position of $\Psi(\overline{M}^{2p})$ with respect to the Lee field of M^{2m}. There are limitations. For instance, we have [Papaghiuc N.]:

Theorem 2.

Let M^k be k-dimensional $(k \geq 2)$ CR-submanifold of a Vaisman manifold M^{2m}. If the anti-Lee field $A = -JB = -J\omega^{\#}$ is normal to M^k then M^k is an anti-invariant submanifold of M^{2m} $(k \leq m)$. Consequently, a Vaisman manifold admits no proper CR-submanifolds so that $\overline{A} = \Psi_* A = 0$. In particular, there are no proper CR-submanifolds of a Vaisman manifold with $B \in \mathcal{D}^{\perp}$. Also, there are no complex submanifolds of a Vaisman manifold normal to the Lee field $B = \omega^{\#}$.

Theorem 3.

LCK-manifold M^{2m} admit immersion of complex hypersurface $\overline{M}^{2m-2}$ so that the Lee field $B = \omega^{\#}$ and the anti-Lee field $A = -JB = -J\omega^{\#}$ are normal to the hypersurface $\overline{M}^{2m-2}$ if and only if the Lee form of M^{2m} satisfies the condition

$$\nabla_X \omega(Y) = \frac{||\omega||^2}{2} g(X, Y).$$

3. Real surfaces of LCK-manifolds

Let M^{2m-1} be a $2m-1$-dimensional manifold and f, ξ, η be a tensor field of type $(1,1)$, a vector field, a 1-form on M^{2m-1} respectively. If f, ξ and η satisfy the conditions

$$\begin{cases} \eta(\xi) = 1; & f\xi = 0; \\ f^2 X = -X + \eta(X); & \eta(fX) = 0, \end{cases}$$

for any vector field $X \in \mathfrak{X}(M^{2m-1})$, then M^{2m-1} is said to have an *almost contact structure* (f, ξ, η) and is called an *almost contact manifold* [Yano, p. 252]. If an almost contact manifold M^{2m-1} admits a Riemannian metric tensor field g such that

$$\begin{aligned} &1) \quad \eta(X) = g(\xi, X); \\ &2) \quad g(fX, fY) = g(X, Y) - \eta(X)\eta(Y), \end{aligned}$$

then M^{2m-1} is said to have an *almost contact metric structure* (f, ξ, η, g) and is called an *almost contact metric manifold*.

An almost contact structure is called *integrable* if $N_{ij}^k = 0$ and *normal* if $N_{ij}^k + \xi^k(\eta_{j,i} - \eta_{i,j}) = 0$. Here

$$N_{ij}^k = f_{j,t}^k f_i^t - f_{j,i}^t f_t^k - f_{i,t}^k f_j^t + f_{i,j}^t f_t^k$$

is so called *Nijenhuis tensor* of the almost contact structure. There are very important theorem [Tashiro Y.]:

Theorem 4.

A hypersurface $\overline{M}^{2m-1}$ in an almost complex manifold M^{2m} has an almost contact structure.

We explore the case when a hypersurface $\overline{M}^{2m-1}$ is the maximal integral submanifold of the distribution defined by the equation

$$\omega = 0,$$

where ω is the Lee form of the LCK-manifold M^{2m}.

As above $\overline{M}^{2m-1}$ be represented by

$$x^{\alpha} = x^{\alpha}(y^1, ..., y^{2m-1}),$$

where $\alpha = 1, ..., 2m$ and y^i, $i = 1, ..., 2m-1$ are local coordinate systems respectively on M^{2m} and on $\overline{M}^{2m-1}$. Then the tangent subspace of $\overline{M}^{2m-1}$ at each point $x = x(y)$ is spanned by vectors:

$$B_i^{\alpha} = \partial_i x^{\alpha}.$$

If a tensor $g_{\alpha\beta}$ is a Riemannian metric of M^{2m} then induced metric of $\overline{M}^{2m-1}$ take the form:

$$\overline{g}_{ij} = B_i^{\alpha} g_{\alpha\beta} B_j^{\beta}.$$

We can define tensors

$$B_{\alpha}^{i} = B_j^{\beta} \overline{g}^{ij} g_{\alpha\beta};$$

and

$$f_i^j = J_{\beta}^{\alpha} B_{\alpha}^{j} B_i^{\beta}.$$

The late is an almost contact structure induced in $\overline{M}^{2p}$ by the immersion.

Theorem 5.

If a hypersurface $\overline{M}^{2m-1}$ of a LCK-manifold M^{2m} is integral manifold of the distribution defined by the equation

$$\omega = 0,$$

where ω is Lee form of the LCK-manifold M^{2m} then induced by the immersion the almost contact structure

$$\begin{aligned} &1) \quad f_i^j = J_\beta^\alpha B_\alpha^j B_i^\beta; \\ &2) \quad \eta_k = \tfrac{1}{||\omega||} B_k^\beta J_\beta^\alpha \omega_\alpha; \\ &3) \quad \xi^k = -\tfrac{1}{||\omega||} B_\beta^k J_\alpha^\beta \omega^\alpha. \end{aligned}$$

is a *normal* one.

To formulate the next two theorems we should recall:
The *cosymplectic structure* is characterized by [Dragomir S., Ornea L., p. 232]

$$1) d\eta = 0, \quad 2) d\overline{\Omega} = 0, \quad 3) N_{ij}^k = 0.$$

The form $\overline{\Omega}$ is defined as

$$\overline{\Omega}(X, Y) = \overline{g}(X, fY).$$

The *c-Sasakian structure* is a normal almost contact metric structure such that [Dragomir S., Ornea L., p. 41-42]

$$d\eta = c\overline{\Omega}.$$

Theorem 6.

If a hypersurface $\overline{M}^{2m-1}$ of a LCK-manifold M^{2m} is integral manifold of the distribution defined by the equation

$$\omega = 0$$

where ω is Lee form of the LCK-manifold M^{2m} that satisfies the condition

$$\nabla_X \omega(Y) = \frac{||\omega||^2}{2} g(X, Y)$$

then induced by the immersion the almost contact structure

$$\begin{aligned} &1) \quad f_i^j = J_\beta^\alpha B_\alpha^j B_i^\beta; \\ &2) \quad \eta_k = \tfrac{1}{||\omega||} B_k^\beta J_\beta^\alpha \omega_\alpha; \\ &3) \quad \xi^k = -\tfrac{1}{||\omega||} B_\beta^k J_\alpha^\beta \omega^\alpha. \end{aligned}$$

is a *cosymplectic structure*. Moreover $\overline{M}^{2m-1}$ is totally umbilical hypersurface of M^{2m}

Theorem 7.

If a hypersurface $\overline{M}^{2m-1}$ of a LCK-manifold M^{2m} is integral manifold of the distribution defined by the equation

$$\omega = 0$$

where ω is Lee form of the LCK-manifold M^{2m} that satisfies the condition

$$\nabla_X \omega(Y) = 0$$

then induced by the immersion the almost contact structure

$$\begin{aligned} &1) \quad f_i^j = J_\beta^\alpha B_\alpha^j B_i^\beta; \\ &2) \quad \eta_k = \frac{1}{||\omega||} B_k^\beta J_\beta^\alpha \omega_\alpha; \\ &3) \quad \xi^k = -\frac{1}{||\omega||} B_\beta^k J_\alpha^\beta \omega^\alpha. \end{aligned}$$

is a *c-Sasakian structure*, $c = ||\omega||$. Moreover $\overline{M}^{2m-1}$ is totally geodesic hypersurface in M^{2m}

Similar results are obtained by [Vaisman I.] and [Kirichenko V.F.]. Moreover Vaisman I. proved that if M^{2m} is conformally flat manifold then $\overline{M}^{2m-1}$ is a constant curvature manifold.
But we have proved normality of almost contact metric structure in $\overline{M}^{2m-1}$ which satisfies the condition $\omega = 0$ in LCK-manifold M^{2m}, for common case.
Taking into account that LCK-manifolds with Lee form which satisfies the condition

$$\nabla_X \omega(Y) = \frac{||\omega||^2}{2} g(X, Y)$$

have very particular properties, we propose call such LCK-manifolds as the *Pseudo-Vaisman manifolds*.

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Thank you for your attention!