Projective Bivector Parametrization of Isometries

Part I: Rodrigues' Vector

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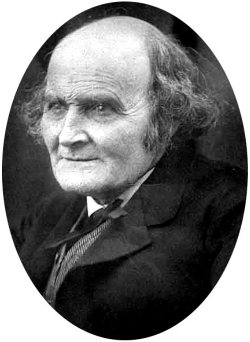
A long time ago in a galaxy far, far away...

O. Rodrigues [1840]

W. Hamilton [1843]



A. Cayley [1846]

Recommended Readings

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- Piña E., *Rotations with Rodrigues' Vector*, Eur. J. Phys. **32** (2011).
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Rotation Vectors and Rodrigues' Formula

The axis-angle representation of rotations in $\mathbb{R}^3$ yields the *rotation vector*

$$\mathbf{s} = \varphi\, \mathbf{n}, \qquad (\mathbf{n}, \varphi) \in \mathbb{S}^2 \times \mathbb{S}^1$$

which the Hodge map $\star$ transforms into a $\mathfrak{so}(3)$ generator as

$$\mathbf{s} \xrightarrow{g[\star]} \mathbf{s}^{\times} = \varphi\, \mathbf{n}^{\times} \in \mathfrak{so}(3).$$

Now, one obtains the group element via the exponential map

$$\mathbf{s}^{\times} \xrightarrow{\exp} \mathcal{R}(\mathbf{n}, \varphi) = \cos\varphi\, \mathcal{I} + (1 - \cos\varphi)\, \mathbf{n}\mathbf{n}^t + \sin\varphi\, \mathbf{n}^{\times}$$

that is the famous *Rodrigues' rotation formula*.

Pros and Cons

On he positive side:

- one has a handy explicit formula for the matrix entries
- the relation to mechanics is straightforward:

$$\dot{\mathbf{s}} = \boldsymbol{\Omega} \qquad (\text{fixed axis}).$$

On the other hand:

- composing spherical vectors is cumbersome, while working with matrices is rather inefficient
- this representation involves transcendent functions, so one needs to work with approximations
- the parametrization is topologically incorrect since

$$\mathrm{SO}(3) \cong \mathbb{RP}^3 \neq \mathbb{S}^2 \times \mathbb{S}^1.$$

Euler's Trigonometric Substitution

The famous Euler trigonometric substitution

$$\mathbf{s} = \varphi \mathbf{n} \quad \longrightarrow \quad \mathbf{c} = \tau \mathbf{n}, \qquad \tau = \tan \frac{\varphi}{2}$$

allows for writing

$$\sin \varphi = \frac{2\tau}{1+\tau^2}, \qquad \cos \varphi = \frac{1-\tau^2}{1+\tau^2}$$

which yields rational expressions for the rotation matrix entries

$$\mathcal{R}(\mathbf{c}) = \frac{(1-\mathbf{c}^2)\,\mathcal{I} + 2\,\mathbf{c}\mathbf{c}^t + 2\,\mathbf{c}^\times}{1+\mathbf{c}^2}.$$

Dealing with Infinities

As $\tau \xrightarrow[\varphi\to\pi]{} \infty$ one applies l'Hôpital's rule to obtain

$$\mathcal{R}(\mathbf{c}) = \frac{(1-\mathbf{c}^2)\mathcal{I} + 2\,\mathbf{c}\mathbf{c}^t + 2\,\mathbf{c}^\times}{1+\mathbf{c}^2} \quad \xrightarrow[\mathbf{c}^2\to\infty]{} \quad 2\mathbf{n}\mathbf{n}^t - \mathcal{I} = \mathcal{O}(\mathbf{n}).$$

Half-turns are mapped on the "plane at infinity" in $\mathbb{RP}^3$. Moreover

$$\langle\, \mathbf{c}_2, \mathbf{c}_1 \rangle \quad \xrightarrow[\mathbf{c}_{1,2}^2\to\infty]{} \frac{\hat{\mathbf{c}}_1 \times \hat{\mathbf{c}}_2}{(\hat{\mathbf{c}}_1, \hat{\mathbf{c}}_2)}$$

where $\hat{\mathbf{c}}_k$ denote the corresponding unit vectors.

The Vector-Parameter

Some major advantages of Rodrigues' construction:

- compact expressions and no excessive parameters whatsoever
- topologically correct parametrization of $\mathrm{SO}(3) \cong \mathbb{RP}^3$, instead of coordinates on $\mathbb{T}^3$ (e.g., Euler angles), which yield singularities
- allows for rational expressions for the rotation's matrix entries

$$\mathcal{R}(\mathbf{c}) = \frac{(1-\mathbf{c}^2)\,\mathcal{I} + 2\,\mathbf{c}\mathbf{c}^t + 2\,\mathbf{c}^\times}{1+\mathbf{c}^2}$$

- an efficient composition to replace the usual matrix multiplication

$$\langle \mathbf{c}_2, \mathbf{c}_1 \rangle = \frac{\mathbf{c}_2 + \mathbf{c}_1 + \mathbf{c}_2 \times \mathbf{c}_1}{1 - (\mathbf{c}_2, \mathbf{c}_1)} \quad \Leftrightarrow \quad \mathcal{R}(\mathbf{c}_2)\,\mathcal{R}(\mathbf{c}_1) = \mathcal{R}(\langle \mathbf{c}_2, \mathbf{c}_1 \rangle)$$

- numerically fast and analytically convenient representation.

Generalized Euler Decomposition

Consider the decomposition

$$\mathcal{R}(\mathbf{c}) = \mathcal{R}(\mathbf{c}_3)\mathcal{R}(\mathbf{c}_2)\mathcal{R}(\mathbf{c}_1)$$

with $\mathbf{c} = \tau\mathbf{n}$ and similarly $\mathbf{c}_k = \tau_k\hat{\mathbf{c}}_k$, $|\hat{\mathbf{c}}_k| = 1$. Denoting

$$g_{ij} = (\hat{\mathbf{c}}_i, \hat{\mathbf{c}}_j), \quad r_{ij} = (\hat{\mathbf{c}}_i, \mathcal{R}(\mathbf{c})\,\hat{\mathbf{c}}_j), \quad \omega = (\hat{\mathbf{c}}_1, \hat{\mathbf{c}}_2 \times \hat{\mathbf{c}}_3)$$

the explicit form of some matrix entries r_{ij} yields a system of QE's

$$(r_{32} + g_{32} - 2g_{12}r_{31})\,\tau_1^2 - 2\tilde{\omega}\tau_1 + r_{32} - g_{32} = 0$$

$$(r_{31} + g_{31} - 2g_{12}g_{23})\,\tau_2^2 + 2\omega\tau_2 + r_{31} - g_{31} = 0$$

$$(r_{21} + g_{21} - 2g_{23}r_{31})\,\tau_3^2 - 2\tilde{\omega}\tau_3 + r_{21} - g_{21} = 0$$

where the two solutions for τ_2 determine the double-valued parameter

$$\tilde{\omega}^{\pm} = \left(\mathcal{R}(\tau_2^{\pm}\hat{\mathbf{c}}_2)\,\hat{\mathbf{c}}_1, \hat{\mathbf{c}}_2, \hat{\mathbf{c}}_3\right), \qquad \tilde{\omega}^- + \tilde{\omega}^+ = 0.$$

The Solutions

NSC for real solutions (Gram determinant for the moving frame):

$$\Delta = \begin{vmatrix} 1 & g_{12} & r_{31} \\ g_{21} & 1 & g_{23} \\ r_{31} & g_{32} & 1 \end{vmatrix} \geq 0.$$

Moreover, with the notation

$$\omega_1 = \left(\hat{\mathbf{c}}_1, \hat{\mathbf{c}}_2, \mathcal{R}^t(\mathbf{c})\,\hat{\mathbf{c}}_3\right), \qquad \omega_2 = \omega, \qquad \omega_3 = \left(\mathcal{R}(\mathbf{c})\,\hat{\mathbf{c}}_1, \hat{\mathbf{c}}_2, \hat{\mathbf{c}}_3\right)$$

the solutions for τ_k can be given in a closed decoupled form as

$$\tau_k^{\pm} = \frac{\sigma^k}{\omega_k \pm \sqrt{\Delta}}, \qquad \sigma^k = \varepsilon^{ijk}(g_{ij} - r_{ij}), \qquad i > j.$$

Specific Cases

Similarly, we have NSC for decomposition in two factors

$$r_{21} = g_{21} \quad \Leftrightarrow \quad \mathcal{R}(\mathbf{c}) = \mathcal{R}(\mathbf{c}_2)\,\mathcal{R}(\mathbf{c}_1)$$

and the solution is given as

$$\tau_1 = \frac{r_{22} - 1}{\mathring{\omega}_1}\,, \qquad \tau_2 = \frac{r_{11} - 1}{\mathring{\omega}_2}\,.$$

where we have denoted

$$\mathring{\omega}_1 = \left(\hat{\mathbf{c}}_1, \hat{\mathbf{c}}_2, \mathcal{R}^t(\mathbf{c})\,\hat{\mathbf{c}}_2\right)\,, \qquad \mathring{\omega}_2 = \left(\mathcal{R}(\mathbf{c})\,\hat{\mathbf{c}}_1, \hat{\mathbf{c}}_1, \hat{\mathbf{c}}_2\right)\,.$$

In the critical points of the map $\mathbb{RP}^3 \cong \mathbb{S}^3/\mathbb{Z}_2 \to \mathbb{T}^3 \cong (\mathbb{RP}^1)^3$ given as

$$\hat{\mathbf{c}}_3 = \pm\mathcal{R}(\mathbf{c})\,\hat{\mathbf{c}}_1$$

the rank drops (gimbal lock) and $\varphi_{1,3}$ become dependent, namely

$$\varphi_2 = 2\arctan\frac{r_{11} - 1}{\mathring{\omega}_2}, \qquad \varphi_1 \pm \varphi_3 = 2\arctan\frac{r_{22} - 1}{\mathring{\omega}_1}.$$

Linear-Fractional Relations

The composition law yields a linear-fractional dependance between each pair of parameters in the decomposition, namely

$$\tau_i(\tau_j) = \frac{\Gamma^1_{ij}\,\tau_j}{\Gamma^2_{ij}\,\tau_j + \Gamma^3_{ij}}.$$

With the notation $c_k = (\mathbf{c}, \hat{\mathbf{c}}_k)$ and $\kappa_{ij} = g_{ij} + r_{ij}$, the matrices Γ^k are given in the generic case as

$$\Gamma^1 = \begin{pmatrix} 1 & g_{32} - r_{32} & r_{32} - g_{32} \\ g_{31} - r_{31} & 1 & g_{31} - r_{31} \\ g_{21} - r_{21} & g_{21} - r_{21} & 1 \end{pmatrix} = -(\Gamma^3)^t$$

$$\Gamma^2 = \begin{pmatrix} 0 & \kappa_{32}c_1 - \kappa_{31}c_2 & \kappa_{21}c_3 - \kappa_{32}c_1 \\ \kappa_{32}c_1 - \kappa_{31}c_2 & 0 & \kappa_{21}c_3 - \kappa_{31}c_2 \\ \kappa_{21}c_3 - \kappa_{32}c_1 & \kappa_{21}c_3 - \kappa_{31}c_2 & 0 \end{pmatrix}.$$

Coordinate Changes in SO(3)

Euler to Bryan angles:

$$\tilde{\tau}_\phi^\pm = \frac{\cos\phi\sin\vartheta}{\cos\vartheta \pm \sqrt{1-\sin^2\phi\sin^2\vartheta}}, \qquad \tilde{\tau}_\vartheta^\pm = -\frac{\sin\phi\sin\vartheta}{1 \pm \sqrt{1-\sin^2\phi\sin^2\vartheta}}$$

$$\tilde{\tau}_\psi^\pm = \frac{\cos\phi\sin\psi + \sin\phi\cos\vartheta\cos\psi}{\cos\phi\cos\psi - \sin\phi\cos\vartheta\sin\psi \pm \sqrt{1-\sin^2\phi\sin^2\vartheta}}.$$

Bryan to Euler angles:

$$\tau_\phi^\pm = -\frac{\sin\tilde{\vartheta}}{\sin\tilde{\phi}\cos\tilde{\vartheta} \pm \sqrt{1-\cos^2\tilde{\phi}\cos^2\tilde{\vartheta}}}, \qquad \tau_\vartheta^\pm = \pm\sqrt{\frac{1-\cos\tilde{\phi}\cos\tilde{\vartheta}}{1+\cos\tilde{\phi}\cos\tilde{\vartheta}}}$$

$$\tau_\psi^\pm = \frac{\cos\tilde{\phi}\sin\tilde{\vartheta}\cos\tilde{\psi} + \sin\tilde{\phi}\sin\tilde{\psi}}{\sin\tilde{\phi}\cos\tilde{\psi} - \cos\tilde{\phi}\sin\tilde{\vartheta}\sin\tilde{\psi} \pm \sqrt{1-\cos^2\tilde{\phi}\cos^2\tilde{\vartheta}}}.$$

Weakening Davenport's Condition

In order to ensure the NSC $\Delta \geq 0$ for all group elements, one needs

$$\hat{\mathbf{c}}_2 \perp \hat{\mathbf{c}}_{1,3} \qquad \text{(Davenport's condition)}.$$

It is possible, however, to use weaker restrictions if we:

- parameterize only regions of SO(3)
- decompose into more factors.

For example, in the case $\hat{\mathbf{c}}_3 \equiv \hat{\mathbf{c}}_1$ one may decompose into three factors if

- for the rotation angle and the one between the axes one has

$$|\varphi| \leq 2\gamma, \qquad \gamma = \min |\measuredangle(\hat{\mathbf{c}}_1, \hat{\mathbf{c}}_2)|$$

- the invariant axis satisfies $\beta = \min |\measuredangle(\mathbf{n}, \hat{\mathbf{c}}_{1,2})| \leq \gamma$.

Similarly, one may always decompose into N factors with

$$N \leq 1 + \left[\frac{\pi}{\gamma}\right]^{+} \qquad \text{(Lowenthal's formula)}.$$

Variations and Lie Derivatives

We consider both left and right shifts of a given rotation

$$u_k^-(t)\,\mathbf{c} = \langle t\,\hat{\mathbf{c}}_k, \mathbf{c}\rangle, \qquad u_k^+(t)\,\mathbf{c} = \langle \mathbf{c}, t\,\hat{\mathbf{c}}_k\rangle$$

Direct differentiation yields

$$\partial_k u_k^\pm \mathbf{c} = (\mathcal{I} + \mathbf{c}\mathbf{c}^t \pm \mathbf{c}^\times)\,\hat{\mathbf{c}}_k$$

and respectively

$$\partial_k \mathcal{R}(u_k^\pm \mathbf{c}) = \frac{2}{1+\mathbf{c}^2}\left[(\mathcal{I} \pm \mathbf{c}^\times)\,\hat{\mathbf{c}}_k \mathbf{c}^t_{sym} + (\hat{\mathbf{c}}_k - \rho_k \mathbf{c} \pm \mathbf{c}\times\hat{\mathbf{c}}_k)^\times - 2\rho_k \mathcal{I}\right].$$

Since we also have

$$\partial_k\sqrt{\Delta} = \Delta^{-\frac{1}{2}}\Delta^{31}\partial_k r_{31}, \qquad \Delta^{31} = g_{32}g_{21} - r_{31}$$

the variations of the decomposition parameters are given as

$$\partial_k \tau_i^\pm = \frac{\partial_k \sigma_i - \tau_i(\partial_k \omega_i \pm \Delta^{-\frac{1}{2}}\Delta^{31}\partial_k r_{31})}{\omega_i \pm \sqrt{\Delta}}\,.$$

The Angular Momentum in Bryan Angles

In a decomposition with respect to the coordinate axes, one has

$$\partial_k \omega_j = 2\begin{pmatrix} -r_{32} & r_{31} & 0 \\ 0 & 0 & 0 \\ 0 & -r_{13} & r_{12} \end{pmatrix}, \qquad \partial_k \sigma_j = 2\begin{pmatrix} r_{33} & 0 & -r_{31} \\ 0 & r_{33} & -r_{32} \\ 0 & -r_{23} & r_{22} \end{pmatrix}$$

which yields the QM angular momentum in these coordinates

$$\begin{aligned} L_1 &= \frac{\partial}{\partial \varphi} \\ L_2 &= \tan\vartheta \sin\varphi \frac{\partial}{\partial \varphi} + \cos\varphi \frac{\partial}{\partial \vartheta} + \sec\vartheta \sin\varphi \frac{\partial}{\partial \psi} \\ L_3 &= \tan\vartheta \cos\varphi \frac{\partial}{\partial \varphi} - \sin\varphi \frac{\partial}{\partial \vartheta} + \sec\vartheta \cos\varphi \frac{\partial}{\partial \psi} \end{aligned}$$

and we easily obtain the hamiltonian (Laplace operator) as

$$\Delta = \vec{L}^2 = \sec^2\vartheta \left(\frac{\partial^2}{\partial \varphi^2} + 2\sin\vartheta \frac{\partial^2}{\partial \varphi\, \partial \psi} + \left(\cos\vartheta \frac{\partial}{\partial \vartheta} \right)^2 + \frac{\partial^2}{\partial \psi^2} \right).$$

Decomposition in a Precessing Frame

For any $\hat{\mathbf{c}}_1$, such that $r_{11} \neq \pm 1$, we may decompose $\mathcal{R} = \mathcal{R}_2\mathcal{R}_1$ choosing

$$\hat{\mathbf{c}}_2 = (1 - r_{11}^2)^{-1/2}\, \hat{\mathbf{c}}_1 \times \mathcal{R}(\mathbf{c})\, \hat{\mathbf{c}}_1$$

and with the notation $\rho_k = (\hat{\mathbf{c}}_k, \mathbf{c})$ the decomposition angles are given as

$$\varphi_1 = 2 \arctan \rho_1, \qquad \varphi_2 = \arccos r_{11}.$$

Denoting κ the precession rate of the coordinate frame $\{K\}$, we have

$$\Delta = \sec^2 \frac{\vartheta}{2} \left[\frac{\partial^2}{\partial \varphi^2} + \frac{\partial}{\partial \vartheta} \left(\cos^2 \frac{\vartheta}{2} \frac{\partial}{\partial \vartheta} \right) \right] + \csc^2 \vartheta \frac{\partial^2}{\partial \kappa^2} \cdot$$

Rigid Body Mechanics

Using the kinematical expressions

$$\boldsymbol{\Omega} = \frac{2}{1+\mathbf{c}^2}\left(\mathcal{I} + \mathbf{c}^{\times}\right)\dot{\mathbf{c}}, \qquad \dot{\mathbf{c}} = \frac{1}{2}\left(\mathcal{I} + \mathbf{c}\mathbf{c}^t - \mathbf{c}^{\times}\right)\boldsymbol{\Omega}$$

one easily obtains the system of ODE's

$$\begin{aligned} \dot{\varphi}_1 &= \Omega_1 - \Omega_3 \tan\frac{\varphi_2}{2} \\ \dot{\varphi}_2 &= \Omega_2 \\ \dot{\kappa} &= \Omega_1 + \Omega_3 \cot\varphi_2 \end{aligned}$$

and in the case of rotational inertial ellipsoid the Euler equations yield

$$\Omega_1(t) = \lambda\omega\cos(\omega t + \varphi_\circ), \qquad \Omega_2(t) = -\lambda\omega\sin(\omega t + \varphi_\circ), \qquad \Omega_3 = \frac{\omega}{\mu}\,.$$

Possible Generalizations

What else do we want?

- Higher-dimensional generalizations;
- Pseudo-Euclidean groups;
- Non-homogeneous isometries.

To do all this, however, we need a shift in our perspective...

Thank You!

THANKS FOR YOUR PATIENCE!