LOW-TYPE SUBMANIFOLDS OF REAL SPACE FORMS

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1. Introduction

The sphere S^m and the real projective and hyperbolic spaces $\mathbb{R}P^m$ and $\mathbb{R}H^m$ allow nice immersions into a (pseudo) Euclidean space by projection operators. We will use the symbol $\mathbb{R}Q^m$ to denote either of the last two space forms. We first recall the construction of an appropriate immersion Φ of these spaces into a (pseudo) Euclidean space of certain matrices by projection operators. In this manner any submanifold of $\mathbb{R}Q^m$ or the unit sphere S^m may be considered as a submanifold of that (pseudo) Euclidean space of matrices.

On the other hand, a submanifold $x : M^n \to E^N_{(K)}$ of Euclidean or pseudo-Euclidean space is said to be of finite type in $E^N_{(K)}$ if the position vector x can be decomposed into a finite sum of vector eigenfunctions of the Laplacian Δ_M on M, viz.

$$x = x_0 + x_1 + \cdots + x_k, \tag{1}$$

where $x_0 = \text{const}$, $x_i \neq \text{const}$, and $\Delta x_i = \lambda_i x_i$, $i = 1, ..., k$. For a compact submanifold, x_0 is its center of mass. If all eigenvalues λ_i are different, the submanifold is said to be of Chen-type k or simply of $k-$type. These kinds of immersions were introduced by B.-Y. Chen in late 1970s. The notion of a $k-$type submanifold is a natural generalization of minimal submanifold of a sphere or the ambient Euclidean space (which are of 1-type).

A k- type immersion satisfies a polynomial equation of degree k in the Laplacian

$$P(\Delta)(x - x_0) = 0, \quad \text{where} \quad P(t) = \prod_{i=1}^{k}(t - \lambda_i)$$

Example 1. The ordinary sphere $S^n(r) \subset E^{n+1}$, any minimal submanifold M of $S^n(r)$ and any minimal submanifold of E^{n+1} are examples of 1-type submanifolds. For example, for a minimal submanifold of a sphere $x : M^k \to S^n$, $\Delta x = kx$, so the position vector is the eigenvector of the Laplacian.

Example 2. Right circular cylinder is of 2-type

$$x(t, \theta) = (\cos\theta, \sin\theta, t) = (0, 0, t) + (\cos\theta, \sin\theta, 0) = x_1 + x_2$$

where $\Delta x_1 = 0 \cdot x_1$, $\Delta x_2 = 1 \cdot x_2$

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Example 3. Any isoparametric hypersurface (constant principal curvatures) of S^n is either of 1-type (if minimal or a small hypersphere) or of 2-type (otherwise).

Conjecture (B.-Y. Chen) The only compact FT hypersurfaces of E^{n+1} are the ordinary spheres. Without restriction on compactness, the only FT hypersurfaces are open portions of minimal hypersurfaces, spheres and circular cylinders.

This notion can be extended to submanifolds $x : M^n \to \bar{M}$ of a general manifold $\bar{M}$ as long as there is a reasonably "nice" embedding $\Phi : \bar{M} \to E^N_{(K)}$ of the ambient manifold $\bar{M}$ into a suitable (pseudo) Euclidean space, in which case M^n is said to be of $k-$type (via Φ) if the composite immersion $\tilde{x} = \Phi \circ x$ is of $k-$type.

The study of submanifolds of the sphere S^m whose position vector in the ambient space E^{m+1} is of finite type has been a fruitful area of research and there are also investigations on low-type submanifolds of $\mathbb{R}H^m$ in E^{m+1}_1. The first paper to treat low-type submanifold in sphere via the second standard immersion Φ of the sphere into the space of symmetric matrices by projection operators is a paper by A. Ros [1984]. This was extended to the study of submanifolds of complex projective space, naturally embedded into an appropriate set of Hermitian matrices. This study produced some new and interesting information on the spectral geometry of submanifolds of these spaces, in particular regarding the eigenvalue estimates and characterization of certain submanifolds by eigenvalue equalities.

We study submanifolds of $\mathbb{R}Q^m$ and sphere which are of low-type via the embedding Φ. More specifically, we show that the only 1-type submanifolds of $\mathbb{R}Q^m$ are the totally geodesic ones (Theorem 1). We extend some earlier results of Ros, Barros-Chen and the author on 2-type submanifolds of sphere to other space forms as well as to the unit sphere. For example, we give a characterization of 2-type submanifolds with parallel mean curvature vector in these spaces in terms of the Ricci tensor and properties of the shape operators (Theorem 3). We classify 2-type hypersurfaces of $\mathbb{R}Q^m$ (Theorems 4 and 5) and give some necessary conditions for minimal hypersurfaces as well as for mass-symmetric hypersurfaces with constant mean curvature to be of Chen 3-type (Theorem 6). The classification of these kinds of 3-type hypersurfaces is carried out for dimmensions $n \leq 5$, save for one case of hypersurfaces of $\mathbb{R}^{n+1}$ which satisfy $\operatorname{tr} A^2 = 2$.

2. The Background and Notation

In the vector space $\mathbb{R}^{m+1}$ we consider the metric

$$g(x, y) = cx_0y_0 + \sum_{k=1}^{m} x_k y_k,$$

where $x = (x_k)$, $y = (y_k) \in \mathbb{R}^{m+1}$ $c = \pm 1$. Define the hyperquadric $\mathcal{Q}^m$ by

$$\mathcal{Q}^m = \{x = (x_0, x_1, \cdots, x_m) \in \mathbb{R}^{m+1} |\, g(x, x) = c\}.$$

When $c = 1$, the ambient space becomes the Euclidean space E^{m+1} and $\mathcal{Q}^m$ is just the unit sphere S^m centered at the origin. When $c = -1$, the metric has the signature $(1, m)$, the ambient space becomes the Lorentz-Minkowski space E^{m+1}_1 and the hyperquadric $\mathcal{Q}^m = H^m$ (which is not connected) consists of two copies of the hyperbolic space. The quotient space (pseudo-Riemannian submersion) defines the real projective space as $\mathbb{R}P^m = S^m/\mathbb{Z}_2$ and the real hyperbolic space as $\mathbb{R}H^m = H^m/\mathbb{Z}_2$. Equivalently, the projective space can be defined also as the set of all lines through the origin of E^{m+1}, equipped with a proper differentiable structure and metric, and the hyperbolic space is the set of time-like lines in E_1^{m+1} through the origin, i.e. those lines on which g is negative-definite. The induced metric on $\mathbb{R}H^m$ is then positive definite and of the sectional curvature -1. We define the embedding Φ of $\mathbb{R}Q^m$ into a suitable (pseudo) Euclidean space by identifying a line $L = [x]$, $g(x,x) = c$, (a time-like line in the hyperbolic case) with the operator of the orthogonal projection P_L with respect to g onto that line. We get the embedding $L \to P_L$ of $\mathbb{R}Q^m$ given by

$$P_L(v) = cg(x,v)x = Mv,$$

where $v \in \mathbb{R}^{m+1}$ and M is the matrix

$$M = \begin{pmatrix} x_0^2 & cx_0x_1 & \cdots & cx_0x_m \\ x_1x_0 & cx_1^2 & \cdots & cx_1x_m \\ \cdots & \cdots & \cdots & \cdots \\ x_mx_0 & cx_mx_1 & \cdots & cx_m^2 \end{pmatrix}$$

It is easy to see that P satisfies the following properties:

$$(i)\ P \text{ is linear}; \quad (ii)\ P^2 = P; \quad (iii)\ g(Pv,w) = g(v,Pw); \quad (iv)\ \operatorname{tr} P = 1.$$

The property (iii) states that P is self-adjoint operator i.e. $P^* = P$, where $P^* = P^t$ in the projective case and $P^* = GP^tG$ in the hyperbolic case, with $G = \operatorname{diag}(-1, I_m)$.

Let $\mathbb{S}^{(1)}(m+1)$ be the set of $g-$symmetric matrices, i.e. $\mathbb{S}^{(1)}(m+1) = \{A \in M_{m+1}(\mathbb{R}) | A^* = A\}$. This space is a linear subspace of the matrix space M_{m+1} of dimension $N = (m+1)(m+2)/2$. Equipped with the trace metric $\langle A, B\rangle = \frac{c}{2}\operatorname{tr}(AB)$ it becomes a (pseudo) Euclidean space $E^N_{(K)}$, where $N = \binom{m+2}{2}$. When $c = -1$, the trace metric is indefinite, of index $K = \binom{m+1}{2} + 1$.

The image of $\mathbb{R}Q^m$ under this embedding is

$$\Phi(\mathbb{R}Q^m) = \{P \in M_{m+1}\,|P^* = P,\ P^2 = P,\ \operatorname{tr} P = 1\}$$

and it lies fully in the hyperplane $\{\operatorname{tr} P = 1\}$ as a space-like submanifold (with a time-like normal bundle in the hyperbolic space). It is easy to see that in the case of $\mathbb{R}P^m$ or S^m this immersion is given by $\Phi([x]) = xx^t$, where x is considered as a column vector with $g(x,x) = 1$. This is the so called first standard embedding of the projective space (in the case of sphere Φ is the second standard immersion of the sphere). The image $\Phi(\mathbb{R}Q^m)$ is a minimal submanifold of the hyperquadric $\mathcal{C}^{N-1}_{I/(m+1)}$ of $E^N_{(K)}$defined by the equation

$$\left\langle M - \frac{I}{m+1},\, M - \frac{I}{m+1} \right\rangle = \frac{cm}{2(m+1)}, \quad M \in \mathbb{S}^{(1)}(m+1), \tag{2}$$

centered at $\frac{I}{m+1}$, which, in the projective case, is the ordinary sphere of radius $\sqrt{\frac{m}{2(m+1)}}$. We recall that a submanifold of a sphere, or, more generaly, of a central hyperquadric, is said to be mass-symmetric in that hyperquadric if the center of mass (the term x_0 in the decomposition (1)) coincides with the center of the hyperquadric. In particular, if $x : M^n \to \mathbb{R}Q^m$ is an isometric immersion then $\tilde{x} := \Phi \circ x$ is mass-symmetric in the hyperquadric $\mathcal{C}^{N-1}_{I/(m+1)}$ if $\tilde{x}_0 = I/(m+1)$.

We shall identify $\mathbb{R}Q^m$ with its image under the embedding Φ. The tangent space and the normal space of $\Phi(\mathbb{R}Q^m)$ at a point (projector) P are given below:

$$T_P(\mathbb{R}Q^m) = \{X \in \mathbb{S}^{(1)}(m+1) | XP + PX = X\}, \tag{3}$$

$$T_P^{\perp}(\mathbb{R}Q^m) = \{Z \in \mathbb{S}^{(1)}(m+1) | ZP = PZ\}. \tag{4}$$

Further, the second fundamental form σ of the embedding Φ is given by

$$\sigma(X,Y) = (XY + YX)(I - 2P), \qquad \bar{\nabla}\sigma = 0, \tag{5}$$

where X, Y are tangent to $\mathbb{R}Q^m$ and Z is normal to it. Also,

$$\langle \sigma(X,Y), I\rangle = 0, \qquad \langle \sigma(X,Y), P\rangle = -\langle X, Y\rangle. \tag{6}$$

We also have

$$\bar{A}_{\sigma(X,Y)}V = c\,[2\langle X,Y\rangle V + \langle X,V\rangle Y + \langle Y,V\rangle X]. \tag{7}$$

These formulas were first obtained by Ros for the second standard immersion of sphere

In this section we compute the iterated Laplacians $\Delta^k\tilde{x},\ k = 1,2,3$, for submanifolds of $\mathbb{R}Q^m$ of dimension ≥ 2 satisfying various conditions.

Let $\bar{\nabla}, \bar{A}, \bar{D}$, denote respectively the Levi-Civita connection, the Weingarten endomorphism, and the metric connection in the normal bundle, related to $\mathbb{R}Q^m$ and the embedding Φ. Let the same letters without bar denote the respective objects for a submanifold M and the immersion x, whereas the same symbols with tilde will denote the corresponding objects related to the composite immersion $\tilde{x} = \Phi \circ x$ of M into the (pseudo) Euclidean space $\mathbb{S}^{(1)}(m+1)$. As usual, we use σ for the second fundamental form of $\mathbb{R}Q^m$ in $E^N_{(K)}$ via Φ and the symbol h for the second fundamental form of a submanifold in $\mathbb{R}Q^m$. An orthonormal basis of the tangent space T_pM at a general point will be denoted by $\{e_i\}$, $i = 1, 2, \cdots, n$, and a basis of the normal space $T_p^{\perp}M$ of M in $\mathbb{R}Q^m$ will be represented by $\{e_r\}, r = n+1, \cdots, m$. In general, indices i, j will range from 1 to n and indices r, s from $n+1$ to m. The mean curvature vector H is defined by $H := \frac{1}{n}\sum_i h(e_i, e_i) = \frac{1}{n}\sum_r (\operatorname{tr} A_r)e_r$.

We give first some important formulas which will be repeatedly used throughout this paper. For a general submanifold M, local tangent fields $X, Y \in \Gamma(TM)$ and a local normal field $\xi \in \Gamma(T^{\perp}M)$, the formulas of Gauss and Weingarten are

$$\bar{\nabla}_X Y = \nabla_X Y + h(X,Y), \qquad \bar{\nabla}_X \xi = -A_\xi X + D_X \xi. \tag{8}$$

The Laplacian of f is defined by

$$\Delta f = \sum_i [(\nabla_{e_i} e_i) f - e_i(e_i f)].$$

The Laplace operator can be extended to act on a vector field V along $\tilde{x}(M)$ by $\Delta V = \sum_i [\tilde{\nabla}_{\nabla_{e_i} e_i} V - \tilde{\nabla}_{e_i} \tilde{\nabla}_{e_i} V]$.

From the Beltrami's formula $\Delta \tilde{x} = -n\tilde{H}$, we get

$$\Delta \tilde{x} = -nH - \sum_{i=1}^{n} \sigma(e_i, e_i), \tag{9}$$

where we understand the Laplacian to be applied to vector fields along M (viewed as $E^N_{(K)}$−valued functions, i.e. matrices) componentwise, and thus

$$\Delta^2 \tilde{x} := \Delta(\Delta \tilde{x}) = -n\Delta H - \sum_{i=1}^{n} \Delta[\sigma(e_i, e_i)]. \tag{10}$$

The Laplacian of the mean curvature vector equals

$$\Delta H = \sum_i [\tilde{\nabla}_{\nabla_{e_i} e_i} H - \tilde{\nabla}_{e_i} \tilde{\nabla}_{e_i} H].$$

We get

$$\begin{aligned} \Delta H = {} & \operatorname{tr}(\nabla A_H) + \operatorname{tr} A_{DH} + \Delta^{\perp} H + cnH + \sum_i h(e_i, A_H e_i) \\ & - n\sigma(H, H) + 2\sum_i \sigma(e_i, A_H e_i) - 2\sum_i \sigma(e_i, D_{e_i} H), \end{aligned} \tag{11}$$

where

$$\operatorname{tr}(\nabla A_H) := \sum_i (\nabla_{e_i} A_H) e_i, \qquad \operatorname{tr} A_{DH} := \sum_i A_{D_{e_i} H} e_i,$$

and $\Delta^{\perp}$ stands for the Laplacian in the normal bundle $T^{\perp}M$ of M in $\mathbb{R}Q^m$. In a similar manner one computes

$$\begin{aligned} \sum_i \Delta[\sigma(e_i, e_i)] = {} & 2cn(n+2)H + 2c(n+1)\sum_i \sigma(e_i, e_i) - 2n\sum_i \sigma(D_{e_i} H, e_i) \\ & + 2\sum_{i,r} \sigma(A_r e_i, A_r e_i) - 2\sum_{r,s} \operatorname{tr}(A_r A_s)\sigma(e_r, e_s). \end{aligned} \tag{12}$$

Combining these formulas we obtain

$$\begin{aligned} \Delta^2 \tilde{x} = {} & -2n\operatorname{tr} A_{DH} - \frac{n^2}{2}\nabla\alpha^2 - n\Delta^{\perp} H - cn(3n+4)H - n\sum_i h(e_i, A_H e_i) \\ & + n^2\sigma(H, H) - 2c(n+1)\sum_i \sigma(e_i, e_i) - 2n\sum_i \sigma(e_i, A_H e_i) \\ & + 4n\sum_i \sigma(e_i, D_{e_i} H) - 2\sum_{i,r} \sigma(A_r e_i, A_r e_i) + 2\sum_{r,s} \operatorname{tr}(A_r A_s)\sigma(e_r, e_s), \end{aligned} \tag{13}$$

which holds for any $n-$dimensional submanifold of S^m or $\mathbb{R}Q^m$.

In particular, when $DH = 0$ this simplifies to

$$\begin{aligned}\Delta^2\tilde{x} = &-cn(3n+4)H - n\sum_i h(e_i, A_H e_i) + n^2\sigma(H,H)\\ &- 2c(n+1)\sum_i \sigma(e_i,e_i) - 2n\sum_i \sigma(e_i, A_H e_i)\\ &- 2\sum_{i,r}\sigma(A_r e_i, A_r e_i) + 2\sum_{r,s}\operatorname{tr}(A_r A_s)\sigma(e_r,e_s).\end{aligned} \tag{14}$$

In the case of a hypersurface of $\mathbb{R}Q^m$ or a sphere we have $H = \alpha\xi$, where ξ is the unit normal and α is the mean curvature. Letting $A := A_\xi$, $f := \operatorname{tr} A = n\alpha$, and $f_2 := \operatorname{tr} A^2$ we get

$$\begin{aligned}\Delta^2\tilde{x} = &-2A(\nabla f) - f\nabla f - [\,\Delta f + c(3n+4)f + f f_2\,]\,\xi\\ &+ (f^2 + 2f_2)\,\sigma(\xi,\xi) + 4\sigma(\nabla f,\xi) - 2c(n+1)\sum_i \sigma(e_i,e_i)\\ &- 2f\sum_i \sigma(Ae_i,e_i) - 2\sum_i \sigma(Ae_i,Ae_i).\end{aligned} \tag{15}$$

Continuing further, we can compute the third iterated Laplacian of $\tilde{x}$, this time for hypersurfaces with constant mean curvature. First, let $f_k := \operatorname{tr} A^k$ and as before $f = f_1 = \operatorname{tr} A$. Then if $f = \text{const}$, we further compute

$$\Delta[\sigma(\xi,\xi)] = 2f_2\,\sigma(\xi,\xi) + 2c\sum_i \sigma(e_i,e_i) - 2\sum_i \sigma(Ae_i,Ae_i), \tag{16}$$

Then we obtain

$$\begin{aligned}\Delta^3\tilde{x} = &-\{\,8cf_3 + f[\Delta f_2 + 4cf^2 + f_2(f_2 + 4c(n+4)) + 7n^2 + 16n + 8]\}\,\xi\\ &- 4c\,\nabla f_2 - 2fA(\nabla f_2) + 6f\,\sigma(\xi,\nabla f_2) + 12\,\sigma(\xi, A(\nabla f_2)) + \frac{8}{3}\,\sigma(\xi,\nabla f_3)\\ &+ \{2\Delta f_2 + 4f_4 + 4ff_3 + f_2\,[\,4f_2 + 3f^2 + 4c(n+1)] + c(3n+4)f^2\}\,\sigma(\xi,\xi)\\ &- 4\,[cf^2 + (n+1)^2]\sum_i \sigma(e_i,e_i) - 4f\,[f_2 + c(n+4)]\sum_i \sigma(Ae_i,e_i)\\ &- 8(f_2+c)\sum_i \sigma(Ae_i,Ae_i) - 4\sum_i \sigma(A^2e_i, A^2e_i)\\ &+ 4\sum_{i,j}\sigma((\nabla_{e_j}A)e_i, (\nabla_{e_j}A)e_i),\end{aligned} \tag{17}$$

where A is the shape operator, ξ the unit normal and $f_k = \operatorname{tr} A^k$, $f = f_1 = \operatorname{tr} A$.

3. Submanifolds of $\mathbb{R}Q^m$ whose Chen-type is 1 or 2

We start with submanifolds of the lowest type, namely submanifolds of $\mathbb{R}Q^m$ which are of Chen-type 1 via the embedding Φ. It is not difficult to classify these: they turn out to be open portions of lower dimensional canonically embedded (totally geodesic) $\mathbb{R}Q^n \subset \mathbb{R}Q^m$. Indeed, let us assume that $\tilde{x} : M^n \to E^N_{(K)}$ is of 1-type, where, as usual, $\tilde{x} = \Phi \circ x$. This means

$$\tilde{x} = \tilde{x}_0 + \tilde{x}_1, \quad \text{with } \tilde{x}_0 = \text{const}, \ \tilde{x}_1 \neq \text{const}, \ \text{and } \Delta\tilde{x}_1 = \lambda\tilde{x}_1.$$

Then $\Delta\tilde{x} = \lambda(\tilde{x} - \tilde{x}_0)$, i.e. according to (10)

$$\lambda(\tilde{x} - \tilde{x}_0) + nH + \sum_{i=1}^{n} \sigma(e_i, e_i) = 0. \tag{18}$$

Differentiate this formula with respect to a tangent vector X to get

$$\lambda X + n\sigma(H, X) - nA_H X + nD_X H - \sum_i \bar{A}_{\sigma(e_i,e_i)}X + \sum_i \bar{D}_X\sigma(e_i, e_i) = 0.$$

Using (7) and the parallelism of σ we eventually get $2A_\xi = -(\operatorname{tr} A_\xi)\, I$. Taking trace of this we conclude $A_\xi = 0$, i.e. the submanifold is totally geodesic.

It is well known that M^n is then an open portion of a canonically embedded (totally geodesic) $\mathbb{R}Q^n$. Conversely, these submanifolds are of 1-type since the restriction of Φ to them produces standard embeddings of $\mathbb{R}Q^n$.

Therefore, we have

Theorem 1. (*i*) *A submanifold $M^n \subset \mathbb{R}P^m$ is of 1-type in E^N via Φ if and only if it is an open portion of a canonical totally geodesic $\mathbb{R}P^n \subset \mathbb{R}P^m$, $n \leq m$.* (*ii*) *A submanifold $M^n \subset \mathbb{R}H^m$ is of 1-type in E^N_K via Φ if and only if it is an open portion of a canonical totally geodesic $\mathbb{R}H^n \subset \mathbb{R}H^m$, $n \leq m$.*

Next, for 2-type submanifolds, starting with those that are mass-symmetric in the hyperquadric $\mathcal{C}^{N-1}_{I/(m+1)}$ we have the following

Theorem 2. *Let $x : M^n \to \mathbb{R}Q^m$ be an isometric immersion with the mean curvature vector H and the mean curvature α. If M^n is mass-symmetric and of 2-type via Φ then the following conditions hold:*

(*i*) $DH = 0$, *i.e. the mean curvature vector is parallel,*
(*ii*) $\mathfrak{a}(H) = 0$, *i.e. the allied mean curvature vector vanishes,*
(*iii*) *The Ricci tensor S of M satisfies $S = 2nA_H + kI$ for some constant k,*
(*iv*) $tr\,(A_\xi A_\eta) = \rho\,\langle \xi, \eta\rangle - \frac{1}{2}(tr\,A_\xi)(tr\,A_\eta)$, *for every $\xi, \eta \in \Gamma(T^\perp M)$, where ρ is a constant,*
(*v*) $tr\,A_H^2 = \alpha^2[\frac{2m\rho}{n} + n\alpha^2 - c(n+2)]$, *thus also a constant.*

Conversely, if (*i*) $-$ (*v*) *hold then $\tilde{x}$ is mass-symmetric and of Chen-type 1 or 2.*

Therefore we recover the result of Barros-Chen, proved for submanifolds of a sphere immersed by the second standard immersion of the sphere also for submanifolds of $\mathbb{R}Q^m$.

With respect to the trace metric $\ll A, B \gg := \operatorname{tr}(AB)$ on the vector space $\mathbb{S}(TM)$ of symmetric endomorphisms of the tangent space of M, property (v) can be rephrased by saying that the Weingarten map is homothetic on the subspace $(Span\{H\})^{\perp}$.

As a corollary we get the following result, which is an extension of the same result proved by Ros for submanifolds of the sphere.

Corollary 1. *Let $x : M^n \to \mathbb{R}Q^m$ be a full minimal immersion of an n-dimensional Riemannian manifold into a non-flat real space form. Then $\tilde{x}$ is of 2-type if and only if $(i)\, M$ is an Einstein submanifold and $(ii) \ll A_\xi, A_\eta \gg = k < \xi, \eta >$ for every pair of normal vectors ξ, η where k is a constant.*

Remark 1. Theorem 2 and Corollary 1 extend respectivelly the results of [Barros Chen] and [Ros] from sphere to $\mathbb{R}P^m$ and $\mathbb{R}H^m$ but also dispose of the compactness assumption for M.

For submanifolds with parallel mean curvature we obtain the following

Theorem 3. *Let $x : M^n \to \mathbb{R}Q^m$ be a submanifold of a non-flat space form $\mathbb{R}P^m$, $\mathbb{R}H^m$ or S^m with parallel mean curvature vector. If $\tilde{x}$ is of 2-type then*

(i) The mean curvature α, the scalar curvature τ, the squared norm of the second fundamental form $||h||^2$, and $tr\, A_H^2$ are all constant.
(ii) $D_X \mathfrak{a}(H) = 0$, for every $X \in \Gamma(TM)$, i.e. the allied mean curvature vector is parallel.
(iii) The Ricci tensor has the form $S = aI + bA_H + \frac{nc}{4} A_{\mathfrak{a}(H)}$, where a and b are constants.
(iv) $\nabla_X S = 2n \nabla_X A_H$, for every $X \in \Gamma(TM)$.
(v) $S \circ A_H = A_{\mathfrak{a}(H)} + 2nA_H^2 + dA_H + eI$, where d and e are constants. Thus $S \circ A_H$ is a symmetric endomorphism and A_H commutes with S and $A_{\mathfrak{a}(H)}$.
(vi) $S \circ A_\xi = A_{\hat{\mathfrak{a}}(\xi)} + 2nA_H A_\xi + (2cn - p/2)A_\xi + \frac{n}{4} tr\,(A_H A_\xi)I$, for every normal vector $\xi \perp H$, where p is a constant.
(vii) $D_X \hat{\mathfrak{a}}(\xi) = \hat{\mathfrak{a}}(D_X \xi)$, for every $\xi \in \Gamma(T^\perp M)$, i.e. $\hat{\mathfrak{a}} : \Gamma(T^\perp M) \to \Gamma(T^\perp M)$ is a parallel operator.

4. Hypersurfaces in non-flat space forms whose type is two or three

Assume $x : M^n \to \mathbb{R}Q^{n+1}$ is an isometric immersion of a Riemannian $n-$manifold as a hypersurface of a non-flat space form (including also the unit sphere) for which the associated immersion $\tilde{x} : M^n \to \mathbb{S}^{(1)}(n+2)$ is of 2-type. Then for some constants $p = \lambda_1 + \lambda_2$, $q = \lambda_1 \lambda_2$ the equation following equation holds:

$$\Delta^2 \tilde{x} - p\Delta\tilde{x} + q(\tilde{x} - \tilde{x}_0) = 0 \tag{19}$$

Taking the metric product of this equation with $\tilde{x}$ we get

$$A(\nabla f) = -\frac{3}{2} f \nabla f. \tag{20}$$

and taking the metric product of (19) with $\sigma(\xi, \xi)$ we get

$$2\nabla f_2 + 2A(\nabla f) - f\nabla f = 0. \tag{21}$$

Combining these two formulas we have $\nabla f_2 = 2f\nabla f = \nabla f^2$, i.e. $f_2 - f^2 = \text{const}$.

Lemma 1. *If M^n is a 2-type hypersurface of $\mathbb{R}Q^{n+1}$ which is of 2-type via Φ then the mean curvature $\alpha = f/n$, the scalar curvature τ, and the squared norm of the second fundamental form f_2 are all constant. Moreover, M^n has at most two principal curvatures, both of which are constant.*

This Lemma will be crucial in the classification of 2-type hypersurfaces of $\mathbb{R}Q^m$. Before we proceed with such classification, however, let us try to understand the situation in the projective space vis à vis that in the sphere. Recall that we have the canonical projection (2-fold covering) $\pi : S^m \to \mathbb{R}P^m$. For any connected submanifold M of $\mathbb{R}P^m$ there is the associated (not necessarily connected) submanifold $\hat{M} = \pi^{-1}(M)$ of S^m, which is invariant under the antipodal map. If g denotes the metric on $\mathbb{R}P^m$ as well as the induced metric on M and $\hat{g} = \pi^* g$ denotes the metric on S^m as well as the induced metric on $\hat{M}$, then the restriction of π to $\hat{M}$ gives the covering $\pi : (\hat{M}, \hat{g}) \to (M, g)$, which is a Riemannian submersion with totally geodesic (two-point) fibers. For such submersions there is a relation between the Laplacians $\Delta_{\hat{M}}(f \circ \pi) = (\Delta_M f) \circ \pi$ for any function $f \in C^2(M)$. If Φ is the first standard embedding of the projective space then $\Phi \circ \pi$ is the second standard immersion of the sphere. Let $M^n \subset \mathbb{R}P^m$ be a submanifold of a projective space, with x being the inclusion. We consider $\hat{M} = \pi^{-1}(M)$ and define the immersion $\hat{x} : \hat{M} \to \mathbb{S}(m+1)$ to be the lift of $\tilde{x}$ i.e. $\hat{x} = \tilde{x} \circ \pi$. If $\tilde{x} = \tilde{x}_0 + \tilde{x}_1 + ... + \tilde{x}_k$ is a $k-$type decomposition then $\hat{x} = \tilde{x} \circ \pi = \tilde{x}_0 \circ \pi + \tilde{x}_1 \circ \pi + \cdots \tilde{x}_k \circ \pi$ gives a decomposition of $k-$type since

$$\Delta_{\hat{M}} \hat{x}_i = \Delta_{\hat{M}}(\tilde{x}_i \circ \pi) = (\Delta_M \tilde{x}_i) \circ \pi = (\lambda_i \tilde{x}_i) \circ \pi = \lambda_i(\tilde{x}_i \circ \pi) = \lambda_i \hat{x}_i.$$

Conversely, if $\hat{x} = \tilde{x} \circ \pi = \hat{x}_0 + \hat{x}_1 + \cdots + \hat{x}_k$ is a $k-$type decomposition of $\hat{x}$, with eigenfunctions $\hat{x}_i$ which are constant on fibers (i.e. $\mathbb{Z}_2-$invariant) then each $\hat{x}_i$ factors through π, $\hat{x}_i = \tilde{x}_i \circ \pi$ and $\tilde{x} = \tilde{x}_0 + \tilde{x}_1 + \cdots + \tilde{x}_k$ is a k-type decomposition of $\tilde{x}$. In other words $M \subset \mathbb{R}P^m$ is of $k-$type via Φ if and only if $\hat{M} \subset S^m$ is of $k - type$ with $\mathbb{Z}_2-$invariant eigenfunctions in the $k-$type decomposition.

Theorem 4. (i) *A complete hypersurface $M^n \subset \mathbb{R}P^{n+1}$ is of 2-type via Φ if and only if M is either a geodesic hypersphere of any radius $\rho \in (0, \pi/2)$ or the canonical projection of the product of spheres $\pi(S^p(r_1) \times S^{n-p}(r_2))$, with the following three possibilities for the radii: $r_1^2 = \frac{p+1}{n+2}$, $r_1^2 = \frac{p+2}{n+2}$, $r_1^2 = \frac{p}{n+2}$, and $r_2^2 = 1 - r_1^2$ in each of the three cases.*
(ii) *A complete hypersurface M^n of $\mathbb{R}H^{n+1}$ is of 2-type via Φ if and only if M^n is a geodesic sphere of arbitrary radius or an equidistant hypersurface to a totally geodesic hyperbolic space $\mathbb{R}H^n \subset \mathbb{R}H^{n+1}$ with an arbitrary (nonzero) distance to it.*

Proof. In the projective case, from the preceding discussion and Lemma 1 it follows that $\hat{M} = \pi^{-1}(M)$ is a hypersurface of S^{n+1} which is of 2-type via the second standard immersion of the sphere and has at most two (constant) principal curvatures.

Complete isoparametric hypersurfaces in a sphere with one or two principal curvatures are compact since they are the geodesic hyperspheres (the umbilical case) or standard products of spheres $S^p(r_1) \times S^{n-p}(r_2)$ (two principal curvatures).

Let us now examine the hyperbolic case. As is well known by the result of E. Cartan, the number of principal curvatures of an isoparametric hypersurface in $\mathbb{R}H^{n+1}$ is one or two. If a 2-type hypersurface $M^n \subset \mathbb{R}Q^{n+1}$ has only one (constant) principal curvature then it is an umbilical hypersurface and therefore one of the following: a totally geodesic $\mathbb{R}H^n$, a geodesic sphere, a horosphere, or an equidistant hypersurface to a canonical totally geodesic $\mathbb{R}H^n$ in $\mathbb{R}H^{n+1}$, but it can be shown that no hypersurface with two constant principal curvatures is of 2-type.

We state now the characterization of a horosphere.

Theorem 5. *A complete hypersurface $M^n \subset \mathbb{R}H^{n+1}$ is a horosphere if and only if $\Delta\tilde{H}$, the Laplacian of the mean curvature vector of M in $\mathbb{S}^{(1)}(m+1)$, is a constant matrix.*

We turn our attention next to hypersurfaces of real space forms S^{n+1}, $\mathbb{R}Q^{n+1}$ with constant mean curvature (CMC) which are of Chen-type 3 in $\mathbb{S}^{(1)}(n+2)$. These hypersurfaces satisfy the condition

$$\Delta^3\tilde{x} + p\Delta^2\tilde{x} + q\Delta\tilde{x} + r(\tilde{x} - \tilde{x}_0) = 0, \tag{22}$$

where p, q, r are constants given by the (signed) elementary symmetric functions of the eigenvalues λ_1, λ_2, and λ_3 associated with a 3-type decomposition $\tilde{x} = \tilde{x}_0 + \tilde{x}_1 + \tilde{x}_2 + \tilde{x}_3$. Differentiating with respect to an arbitrary tangent vector X leads to the relation

$$\tilde{\nabla}_X(\Delta^3\tilde{x}) + p\tilde{\nabla}_X(\Delta^2\tilde{x}) + q\tilde{\nabla}_X(\Delta\tilde{x}) + rX = 0. \tag{23}$$

We consider first a minimal 3-type hypersurface M^n in S^{n+1} or $\mathbb{R}Q^{n+1}$. For such hypersurface the iterated Laplacians reduce to $\Delta\tilde{x} = -\sum_i \sigma(e_i, e_i)$,

$$\Delta^2\tilde{x} = 2f_2\sigma(\xi,\xi) - 2c(n+1)\sum_i \sigma(e_i, e_i) - 2\sum_i \sigma(Ae_i, Ae_i), \tag{24}$$

$$\begin{aligned} \Delta^3\tilde{x} = &- 8cf_3\,\xi + \frac{8}{3}\sigma(\xi, \nabla f_3) + 4[f_4 + f_2^2 + c(n+1)f_2]\}\,\sigma(\xi,\xi) \\ &- 4(n+1)^2\sum_i \sigma(e_i, e_i) - 8(f_2 + c)\sum_i \sigma(Ae_i, Ae_i) \\ &- 4\sum_i \sigma(A^2e_i, A^2e_i) + 4\sum_{i,j} \sigma((\nabla_{e_j}A)e_i, (\nabla_{e_j}A)e_i). \end{aligned} \tag{25}$$

Theorem 6. *Let M^n be a complete minimal hypersurface of dimension $2 \le n \le 5$ in a non-flat real space form which is of 3-type via Φ. Then (i) The only such submanifolds in S^{n+1} are the Cartan minimal hypersurface $SO(3)/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ in S^4 and the Clifford minimal hypersurfaces $M_{p,n-p} = S^p(\sqrt{\frac{p}{n}}) \times S^{n-p}(\sqrt{\frac{n-p}{n}})$ in S^4, S^5 and S^6 for which $n \neq 2p$. (ii) Minimal 3-type hypersurfaces in $\mathbb{R}P^{n+1}$ are the projections $\pi(M_{p,n-p})$ of the Clifford hypersurface mentioned above. (iii) There are no minimal 3-type hypersurfaces in $\mathbb{R}H^{n+1}$ in these dimensions.*

The Cartan isoparametric family of hypersurfaces with three (constant) principal curvatures in S^4, is an algebraic family defined by the equation

$$x_5^3 + \frac{3}{2}(x_1^2 + x_2^2)x_5 - 3(x_3^2 + x_4^2)x_5 + \frac{3\sqrt{3}}{2}(x_1^2 - x_2^2)x_4 + 3\sqrt{3}\, x_1 x_2 x_3 = \cos(3t),$$

where $t \in (0, \pi/3)$ and the minimal one is obtained when $t = \pi/6$. The values $t = 0$ and $t = \pi/3$ correspond to focal submanifolds (in this case two projective planes $\mathbb{R}P^2$). These cubic isoparametric hypersurfaces are of the homogeneous type $SO(3)/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ i.e. obtained as orbits of an action of $SO(3)$

With $\theta = \pi/2 - t$, the principal curvatures are

$$\mu_1 = \tan\left(\theta - \frac{\pi}{3}\right), \quad \mu_2 = \tan\theta, \quad \mu_3 = \tan\left(\theta + \frac{\pi}{3}\right).$$

This theorem is a generalization of the author's earlier result from in that a classification is achieved without assuming mass-symmetry, and such classification is also considered for $\mathbb{R}Q^m$. It also surpases a result of J. T. Lu who only showed nonexistence of 3-type minimal surfaces in S^3. We investigate now 3-type CMC hypersurfaces M^n of real space forms which are mass-symmetric in the hyperquadric $\mathcal{C}^{N+1}_{I/(n+2)}$. This means that they satisfy the equation

$$\Delta^3\tilde{x} + p\Delta^2\tilde{x} + q\Delta\tilde{x} + r(\tilde{x} - \frac{I}{n+2}) = 0, \tag{26}$$

and we conclude that for dimensions $n \le 5$ the hypersurface is isoparametric.

Theorem 7. *Let M^n be a complete hypersurface of a real space form of constant mean curvature and of dimension $2 \le n \le 5$. Then*

(i) The only CMC hypersurface $M^n \subset S^{n+1}$ which is mass-symmetric and of 3-type in $\mathbb{S}(n+2)$ via Φ is the Cartan's minimal hypersurface $M^3 = SO(3)/(\mathbb{Z}_2 \times \mathbb{Z}_2)$.

(ii) The only CMC hypersurface $M^n \subset \mathbb{R}P^{n+1}$ which is mass-symmetric and of 3-type via Φ is the projection of the Cartan minimal hypersurface above, $\pi(SO(3)/(\mathbb{Z}_2\times \mathbb{Z}_2))$.

(iii) Assuming that $\mathrm{tr}\, A \neq \pm 2$, the only CMC hypersurfaces in $\mathbb{R}H^{n+1}$ which are mass-symmetric and of 3-type via Φ are the product hypersurfaces

$$M^n = \mathbb{R}H^k\left(-\frac{k-l}{k}\right) \times S^l\left(\frac{k-l}{l}\right), \qquad k + l = n,\ k > l,$$

with the sectional curvatures of the hyperbolic and spherical factors as indicated.