GRADED GEOMETRY IN MECHANICS AND FIELD THEORY

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Literature

The talk is based on some ideas of W. M. Tulczyjew and my collaboration with A. Bruce, K. Grabowska, and M. Rotkiewicz:

- Grabowski-Rotkiewicz, *Graded bundles and homogeneity structures*, J. Geom. Phys. **62** (2012), 21–36.
- Bruce-Grabowska-Grabowski, *Higher order mechanics on graded bundles*, J. Phys. A **48** (2015), 205203 (32pp).
- Bruce-Grabowska-Grabowski, *Graded bundles in the category of Lie groupoids*, SIGMA **11** (2015), 090, (25pp).
- Bruce-Grabowska-Grabowski, *Linear duals of graded bundles and higher analogues of (Lie) algebroids*, J. Geom. Phys. **101** (2016), 71–99.
- Bruce-Grabowski-Rotkiewicz, *Polarisation of graded bundles*, SIGMA **12** (2016), 106, (30pp).

Vector bundles as graded bundles

- A vector bundle is a locally trivial fibration $\tau : E \to M$ which, locally over $U \subset M$, reads $\tau^{-1}(U) \simeq U \times \mathbb{R}^n$ and admits an atlas in which local trivializations transform linearly in fibers

$$U \cap V \times \mathbb{R}^n \ni (x, y) \mapsto (x, A(x)y) \in U \cap V \times \mathbb{R}^n\,,$$

  $A(x) \in \mathrm{GL}(n, \mathbb{R})$.
- The latter property can also be expressed in the terms of the gradation in which base coordinates x have degrees 0 and 'linear coordinates' y have degree 1. Linearity in y is now equivalent to the fact that changes of coordinates respect the degrees.
- Morphisms in the category of vector bundles are represented by commutative diagram of smooth maps

$$\begin{array}{ccc} E_1 & \xrightarrow{\ \Phi\ } & E_2 \\ \downarrow{\scriptstyle \tau_1} & & \downarrow{\scriptstyle \tau_2} \\ M_1 & \xrightarrow{\ \varphi\ } & M_2 \end{array}$$

  being linear in fibres (the latter makes sense).

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Graded bundles

- A straightforward generalization is the concept of a graded bundle $\tau : F \to M$ with a local trivialization by $U \times \mathbb{R}^n$ as before, and with the difference that the local coordinates $(y^1, \ldots, y^n)$ in the fibres have now associated positive integer weights $w_1, \ldots, w_n \in \mathbb{N}$, that are preserved by changes of local trivializations:

$$U \cap V \times \mathbb{R}^n \ni (x, y) \mapsto (x, A(x, y)) \in U \cap V \times \mathbb{R}^n ,$$

- One can show that in this case $A(x, y)$ must be polynomial in fiber coordinates, i.e. any graded bundle is a polynomial bundle.
- As these polynomials need not to be linear, graded bundles do not have, in general, vector space structure in fibers. For instance, if $(y, z) \in \mathbb{R}^2$ are coordinates of degrees $1, 2$, respectively, then the map $(y, z) \mapsto (y, z + y^2)$ is a diffeomorphism preserving the degrees, but it is nonlinear.
- If all $w_i \leq r$, we say that the graded bundle is of degree r.

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- Vector bundles are just graded bundles of degree 1.
- Canonical example: $\mathsf{T}^k M \to M$ is a graded bundle of degree k with canonical coordinates $(x, \dot{x}, \ddot{x}, \dddot{x}, \dots)$ of degrees $0, 1, 2, 3,$ etc. For $k = 2$,

$$\begin{aligned} x'^A &= x'^A(x) \\ \dot{x}'^A &= \frac{\partial x'^A}{\partial x^B}(x)\dot{x}^B \\ \ddot{x}'^A &= \frac{\partial x'^A}{\partial x^B}(x)\ddot{x}^B + \frac{\partial^2 x'^A}{\partial x^B \partial x^C}(x)\dot{x}^B\dot{x}^C\,. \end{aligned}$$

- Graded bundles F_k of degree k admit, like jet bundles, a tower of affine fibrations by reductions to coordinates of lower degrees

$$F_k \xrightarrow{\tau^k} F_{k-1} \xrightarrow{\tau^{k-1}} \cdots \xrightarrow{\tau^3} F_2 \xrightarrow{\tau^2} F_1 \xrightarrow{\tau^1} F_0 = M\,.$$

- Note that similar objects has been used in supergeometry by Ševera, Voronov, Roytenberg et al. under the name N-manifolds. However, we will work with classical, purely even manifolds during this talk.

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- With the use of coordinates (x^α, y^a) with degrees 0 for basic coordinates x^α, and degrees $w_a > 0$ for the fibre coordinates y^a, we can define on the graded bundle F a globally defined weight vector field (Euler vector field)

$$\nabla_F = \sum_a w_a y^a \partial_{y^a} .$$

- The flow of the weight vector field extends to a smooth action $\mathbb{R} \ni t \mapsto h_t$ of multiplicative reals on F, $h_t(x^\mu, y^a) = (x^\mu, t^{w_a} y^a)$. Such an action $h : \mathbb{R} \times F \to F$, $h_t \circ h_s = h_{ts}$, we will call a homogeneity structure.

- A function $f : F \to \mathbb{R}$ is called homogeneous of degree (weight) k if $\nabla_F(f) = k\, f$, or equivalently $f(h_t(x)) = t^k f(x)$.

- Note that for graded bundles only non-negative integer degrees of homogeneity are allowed. This is not true for more general 'graded manifolds': for $F = (0, 1)$, with the coordinate x of degree 1, the function x^a is homogeneous of degree a for all $a \in \mathbb{R}$.

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- With the use of coordinates (x^α, y^a) with degrees 0 for basic coordinates x^α, and degrees $w_a > 0$ for the fibre coordinates y^a, we can define on the graded bundle F a globally defined weight vector field (Euler vector field)
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The fundamental fact (cf. [Grabowski-Rotkiewicz]) says that graded bundles and homogeneity structures are in fact equivalent concepts. There is namely a canonical isomorphism of the category of graded bundles and the category of homogeneity structures. This is because any manifold equipped with a homogeneity structure admits an atlas consisting of homogeneous functions.

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- However, thanks to the simple description in terms of a homogeneity structure, the categorial and 'diagrammatic' definition can be substantially simplified.
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Double graded bundles - examples

- Lifts. If $\tau : F \to M$ is a graded bundle of degree k, then $\mathsf{T}F$ and T^*F carry canonical double graded bundle structure: one is the obvious vector bundle, the other is of degree k.
- The above examples are double graded bundle whose one structure is linear. We will call such structures GrL-bundles.
- There are also lifts of graded structures on F to T^rF.
- In particular, if $\tau : E \to M$ is a vector bundle, then $\mathsf{T}E$ and T^*E are double vector bundles. The latter is isomorphic with T^*E^* (Tulczyjew, Mackenzie & Xu), with an isomorphism

$$\mathcal{R}_E : \mathsf{T}^*E^* \to \mathsf{T}^*E\,.$$

- Since a linear Poisson structure on E^* yields a map $\mathsf{T}^*E^* \to \mathsf{T}E^*$, a Lie algebroid structure on E can be encoded as a morphism of double vector bundles (!),

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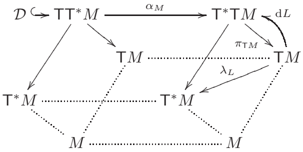
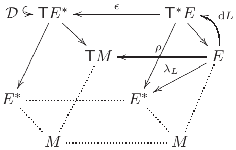
Motivation - higher order mechanics

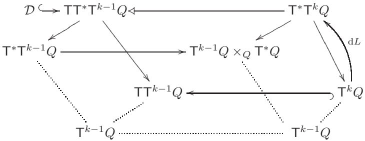
First order Lagrangian mechanics



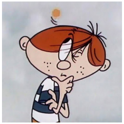
Reduction w.r.t. symmetry



k-th order Lagrangian mechanics



reduced ...



The Tulczyjew triple - Lagrangian side

Any $\mathcal{D} \subset \mathrm{T}N$ can be viewed as implicit dynamics whose solutions are curves $\gamma : \mathbb{R} \to N$ s.t. $\dot{\gamma} \in \mathcal{D}$. For the lagrangian phase equations:

M - positions,
$\mathrm{T}M$ - (kinematic) configurations,
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T^*M - phase space

$$\mathcal{D} = \varepsilon_M(\mathrm{d}L(\mathrm{T}M)) = \mathcal{T}L(\mathrm{T}M)\,,$$

the image of the Tulczyjew differential $\mathcal{T}L$, is the phase dynamics,

$$\mathcal{D} = \left\{ (x, p, \dot{x}, \dot{p}) : \quad p = \frac{\partial L}{\partial \dot{x}}, \quad \dot{p} = \frac{\partial L}{\partial x} \right\},$$

whence the Euler-Lagrange equation: $\frac{\partial L}{\partial x} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right)$. Note that L can be as well singular for the price that $\mathcal{D}$ is an implicit equation.

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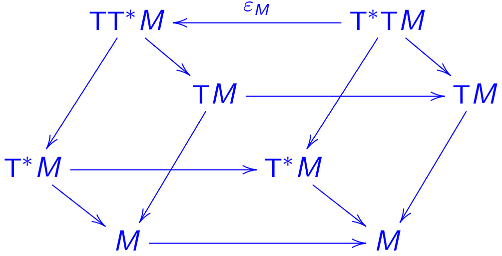

$$\mathcal{D} = \varepsilon_M(\mathrm{d}L(\mathsf{T}M)) = \mathcal{T}L(\mathsf{T}M)\,,$$

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Any $\mathcal{D} \subset \mathsf{T}N$ can be viewed as implicit dynamics whose solutions are curves $\gamma : \mathbb{R} \to N$ s.t. $\dot{\gamma} \in \mathcal{D}$. For the lagrangian phase equations:

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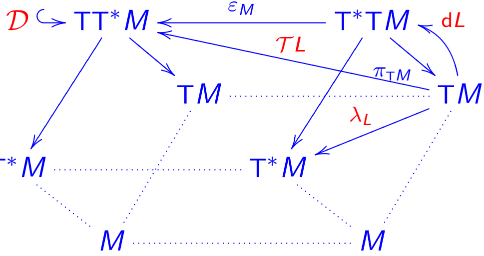

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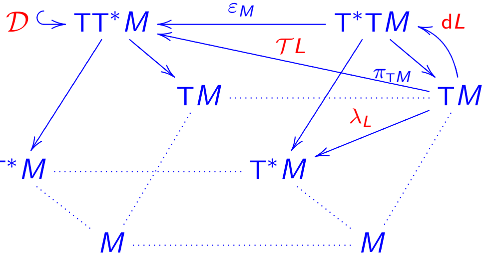

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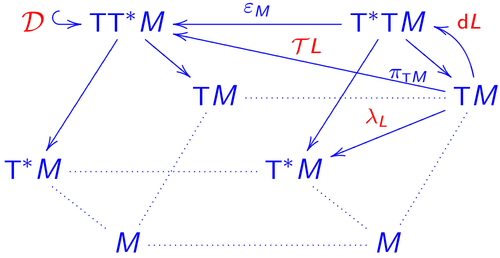

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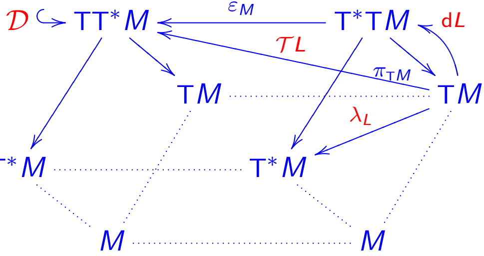

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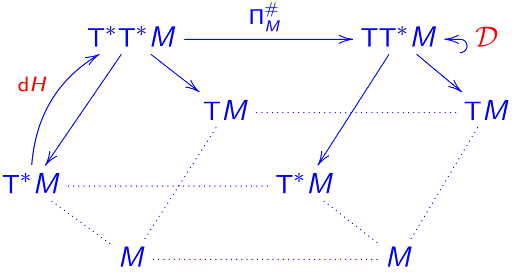
The Tulczyjew triple - Hamiltonian side

$H : \mathsf{T}^*M \to \mathbb{R}$

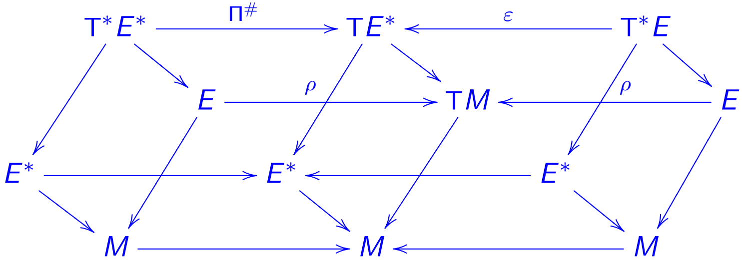


$$\mathcal{D} = \Pi_M^{\#}(\mathrm{d}H(\mathsf{T}^*M))$$

$$\mathcal{D} = \left\{ (x, p, \dot{x}, \dot{p}) : \quad \dot{p} = -\frac{\partial H}{\partial x}, \quad \dot{x} = \frac{\partial H}{\partial p} \right\},$$

whence the Hamilton equations.

Algebroid setting



$H : E^* \longrightarrow \mathbb{R}$ $\qquad$ $\mathcal{D} = \mathcal{T}L(E)$ $\qquad$ $L : E \longrightarrow \mathbb{R}$

$\mathcal{D}_H \subset \mathsf{T}^*E^*$ $\qquad$ $\mathcal{D} = \Pi^\#(\mathrm{d}H(E^*))$ $\qquad$ $\mathcal{D}_L \subset \mathsf{T}^*E$

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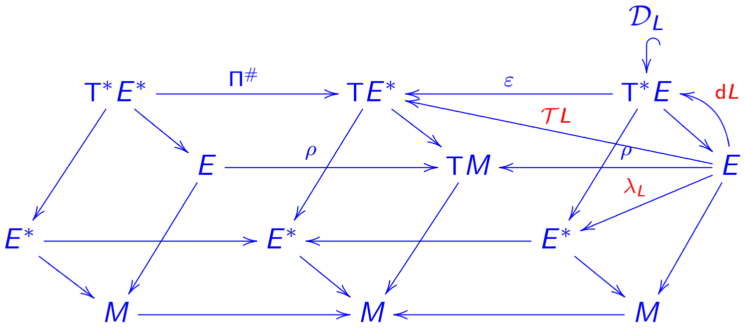

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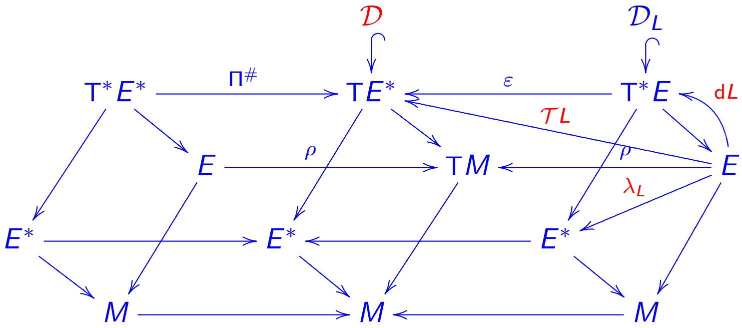

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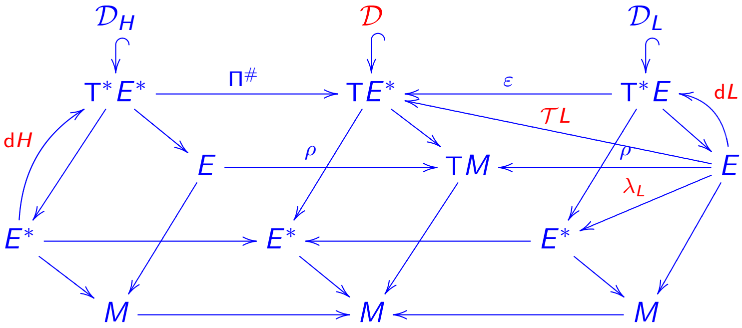

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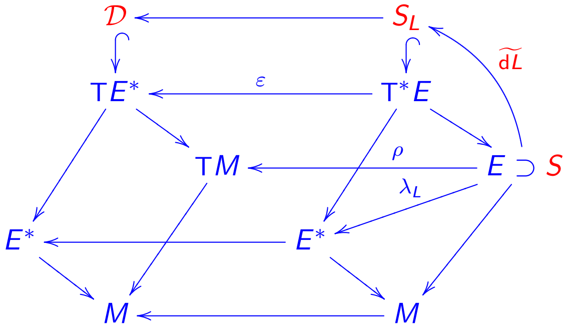
Algebroid setting with vakonomic constraints



where S_L is the lagrangian submanifold in T^*E induced by the Lagrangian on the constraint S, and $\widetilde{\mathrm{d}L} : S \to \mathsf{T}^*E$ is the corresponding relation,

$$S_L = \{\alpha_e \in \mathsf{T}^*_e E : e \in S \text{ and } \langle \alpha_e, v_e \rangle = \mathrm{d}L(v_e) \text{ for every } v_e \in \mathsf{T}_e S\}\,.$$

The vakonomically constrained phase dynamics is just $\mathcal{D} = \varepsilon(S_L) \subset \mathsf{T}E^*$.

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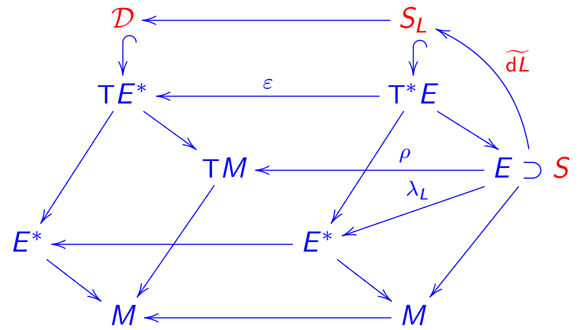

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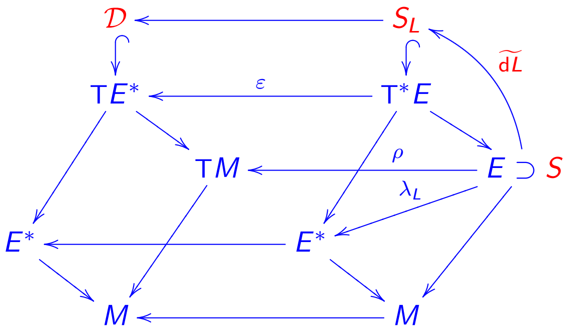

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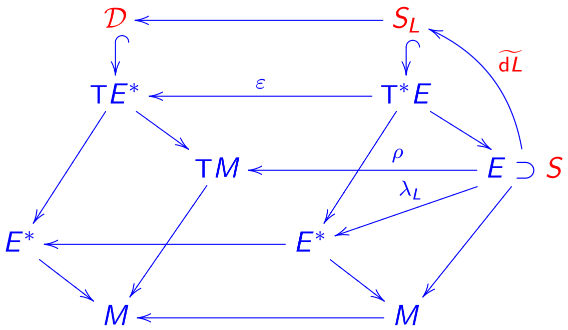

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Higher order Lagrangians

The mechanics with a higher order Lagrangian $L : \mathsf{T}^k Q \to \mathbb{R}$ is traditionally constructed as a vakonomic mechanics, thanks to the canonical embedding of of the higher tangent bundle $\mathsf{T}^k Q$ into the tangent bundle $\mathsf{TT}^{k-1}Q$ as an affine subbundle of holonomic vectors:

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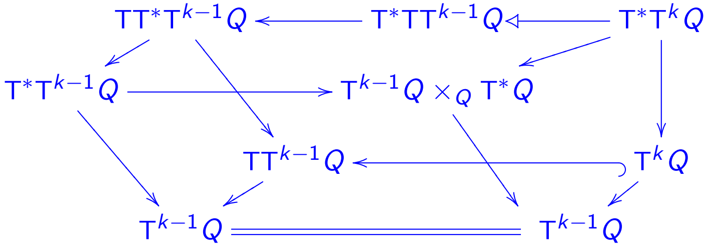

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The linearisation of the graded bundle $A^k(\mathcal{G})$ is given as

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A weighted Lie algebroid on $\mathsf{I}(F_k)$ gives the Tulczyjew triple

Here, the diagram consists of relations, $\hat{\varepsilon} : \mathsf{T}^*F_k \longrightarrow\!\!\triangleright \mathsf{T}^*\,\mathsf{I}(F_k) \to \mathsf{T}\,\mathsf{I}^*(F_k)$, and $\mathrm{Mi}(F_k) = F_{k-1} \times_M \bar{F}_k$ is the so called Mironian of F_k. In the classical case, $\mathrm{Mi}(\mathsf{T}^kM) = \mathsf{T}^{k-1}M \times_M \mathsf{T}^*M$. $\mathcal{T}L$ is the Tulczyjew differential and λ_L the Legendre relation.

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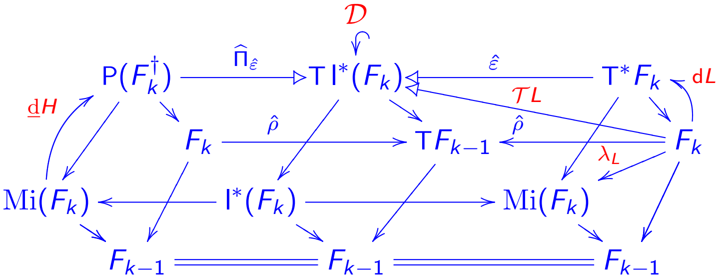

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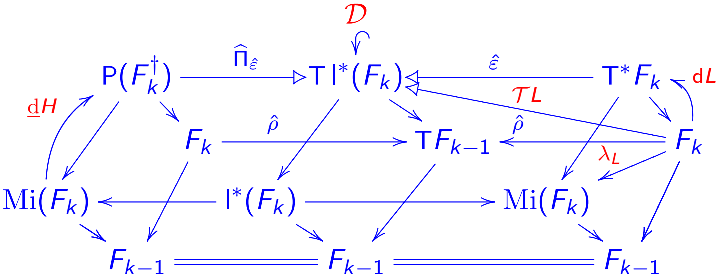

Here, the diagram consists of relations, $\hat\varepsilon : \mathrm{T}^* F_k \longrightarrow\!\!\triangleright\, \mathrm{T}^*\,\mathrm{I}(F_k) \to \mathrm{T}\,\mathrm{I}^*(F_k)$, and $\mathrm{Mi}(F_k) = F_{k-1} \times_M \bar{F}_k$ is the so called Mironian of F_k. In the classical case, $\mathrm{Mi}(\mathrm{T}^k M) = \mathrm{T}^{k-1}M \times_M \mathrm{T}^*M$. $\mathcal{T}L$ is the Tulczyjew differential and λ_L the Legendre relation.

What replaces Lie algebroids in this version of higher Lagrangian theory are linearizations of graded bundles equipped with weighted Lie algebroid structures (weighted Lie algebroids on symmetric GrL-bundles).

Lagrangian framework for graded bundles

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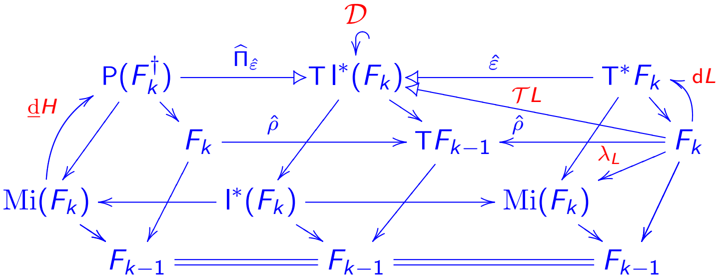

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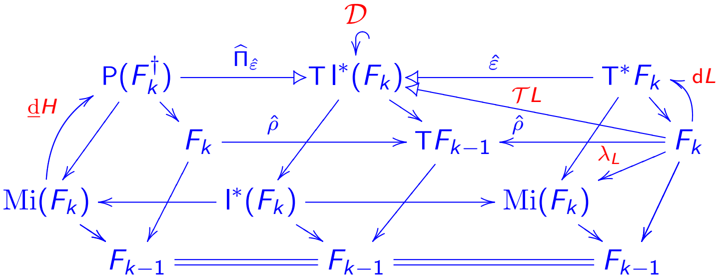

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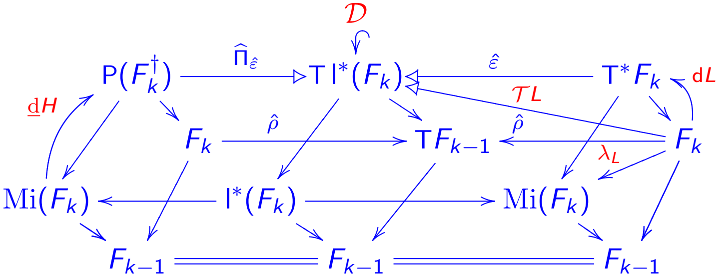

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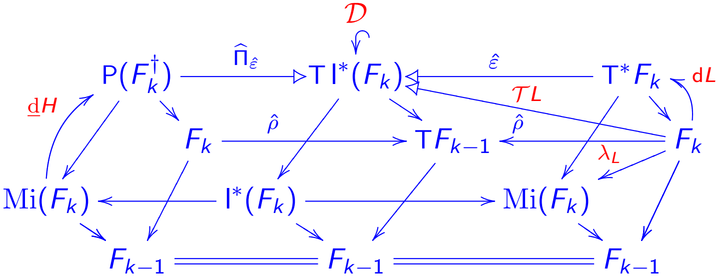

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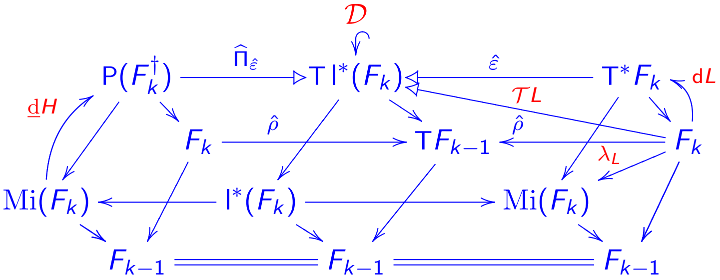

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Example

Let g be a Lie algebra and put $F_2 = g_2 = g[1] \times g[2]$, with coordinates (x^i, z^j) on g_2 and coordinates (x^i, y^j, z^k) on $\mathsf{I}(g_2) = g[1] \times g[1] \times g[2]$. The vector bundle projection is $\tau(x, y, z) = x$ and the corresponding diagram looks like

The embedding $\iota : g_2 \hookrightarrow \mathsf{I}(g_2)$ takes the form $\iota(x, z) = (x, x, z)$. In coordinates $(x, y, z, \alpha, \beta, \gamma)$ on $\mathsf{T}^*\,\mathsf{I}(g_2)$, the phase relation $\mathsf{T}^*\iota : \mathsf{T}^*g_2 \longrightarrow\!\!\triangleright\, \mathsf{T}^*\,\mathsf{I}(g_2)$ relates $(x, z, \alpha + \beta, \gamma)$ with $(x, x, z, \alpha, \beta, \gamma)$.

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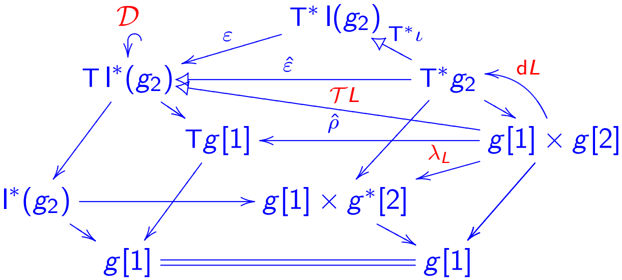

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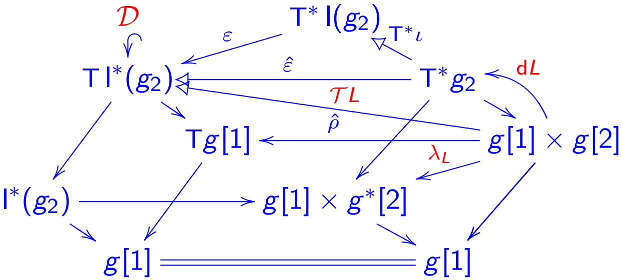

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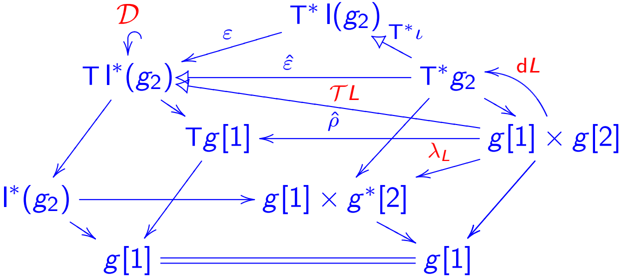

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Hence, for the phase dynamics,

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Higher Euler-Lagrange equations

This leads to the Euler-Lagrange equations on $\mathfrak{g}_2$:

$$\begin{aligned} \dot{x} &= z\,, \\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial x}(x,z) - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial z}(x,z)\right)\right) &= \mathrm{ad}^*_x\left(\frac{\partial L}{\partial x}(x,z) - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial z}(x,z)\right)\right). \end{aligned}$$

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For instance, the 'free' Lagrangian $L(x,z) = \frac{1}{2}\sum_i I_i(z^i)^2$ induces the equations on $\mathfrak{g}$ (c^k_{ij} are structure constants, no summation convention):

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The latter can be viewed as 'higher Euler equations'.

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This leads to the Euler-Lagrange equations on $\mathfrak{g}_2$:

$$\begin{aligned}\dot{x} &= z\,,\\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial x}(x,z)-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial z}(x,z)\right)\right) &= \mathrm{ad}^*_x\left(\frac{\partial L}{\partial x}(x,z)-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial z}(x,z)\right)\right).\end{aligned}$$

These equations are second order and induce the Euler-Lagrange equations on $\mathfrak{g}$ which are of order 3:

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For instance, the 'free' Lagrangian $L(x,z)=\frac{1}{2}\sum_i I_i(z^i)^2$ induces the equations on $\mathfrak{g}$ (c^k_{ij} are structure constants, no summation convention):

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Higher order Lagrangian mechanics on Lie algebroids

Let us consider a general Lie groupoid $\mathcal{G}$ and a Lagrangian $L : A^k \to \mathbb{R}$ on $A^k = A^k(\mathcal{G})$. We will refer to such systems as a k-th order Lagrangian system on the Lie algebroid $A(\mathcal{G})$. The relevant diagram here is

Here, $\mathsf{I}(A^k(\mathcal{G}))$ is the corresponding Lie algebroid prolongation, $\mathcal{D} = \varepsilon \circ r \circ \mathrm{d}L(A^k(\mathcal{G}))$, and λ_L is the Legendre relation.

Note that we deal with reductions: in the case $\mathcal{G}$ is a Lie group,

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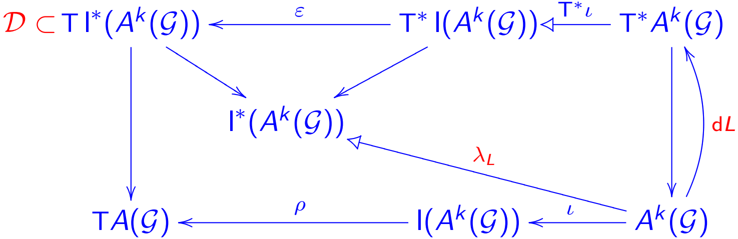

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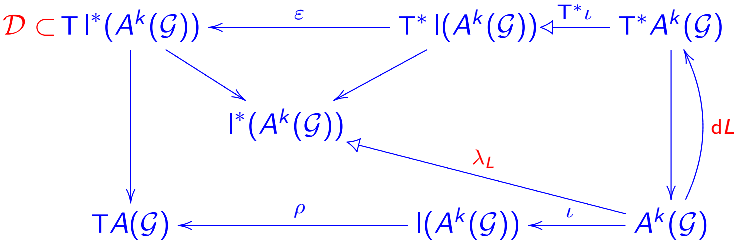

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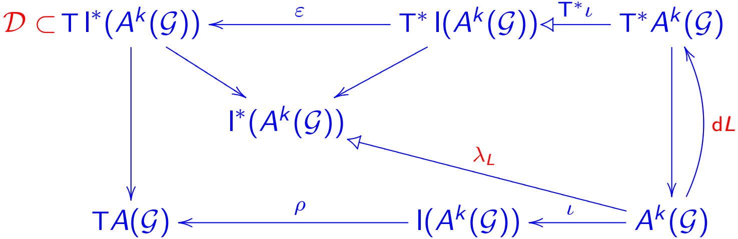

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The remaining equation for the dynamics is

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The above higher order algebroid Euler-Lagrange equations are in complete agrement with the ones obtained by Jóźwikowski & Rotkiewicz, Colombo & de Diego, as well as Martínez. We clearly recover the standard higher Euler–Lagrange equations on $\mathsf{T}^k M$ as a particular example.

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Higher order Lagrangian mechanics on Lie algebroids

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$$\frac{d}{dt}\pi^k_a = \rho^A_a(x)\frac{\partial L}{\partial x^A} + y^b_1 C^c_{ba}(x)\pi^k_c\,,$$

where ρ^A_a and C^c_{ba} are structure functions of the Lie algebroid $A = A(\mathcal{G})$. The above equation can then be rewritten as

$$\rho^A_a(x)\tfrac{\partial L}{\partial x^A} =$$
$$\left(\delta^c_a \tfrac{d}{dt} - y^b_1 C^c_{ba}(x)\right)\left(\tfrac{\partial L}{\partial y^c_1} - \tfrac{1}{2!}\tfrac{d}{dt}\left(\tfrac{\partial L}{\partial y^c_2}\right)\cdots - (-1)^k \tfrac{1}{k!}\tfrac{d^{k-1}}{dt^{k-1}}\left(\tfrac{\partial L}{\partial y^c_k}\right)\right)$$

which we define to be the k-th order Euler–Lagrange equations on $A(\mathcal{G})$.

The above higher order algebroid Euler-Lagrange equations are in complete agrement with the ones obtained by Jóźwikowski & Rotkiewicz, Colombo & de Diego, as well as Martínez. We clearly recover the standard higher Euler–Lagrange equations on $\mathsf{T}^k M$ as a particular example.

The Tulczyjew triple for strings

Using the canonical multisymplectic structure on $\wedge^2 T^* M$, we get the following Tulczyjew triple for multivector bundles, consisting of double graded bundle morphisms:

The way of obtaining the implicit phase dynamics $\mathcal{D}$, as a submanifold of $\wedge^2 T \wedge^2 T^* M$, from a Lagrangian $L : \wedge^2 TM \to \mathbb{R}$ (or from a Hamiltonian $H : \wedge^2 T^* M \to \mathbb{R}$) is now standard: $\mathcal{D} = \mathcal{T}L(\wedge^2 TM)$.

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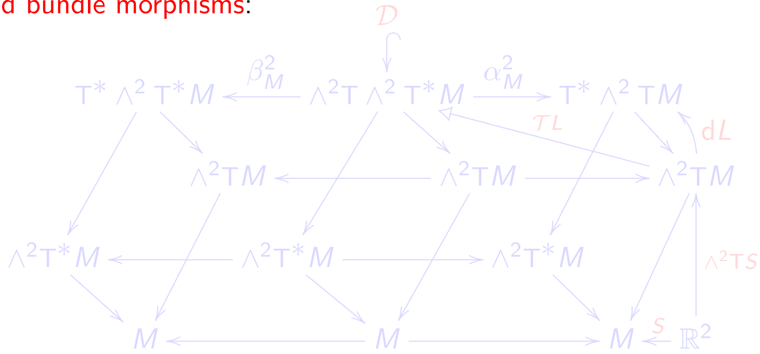

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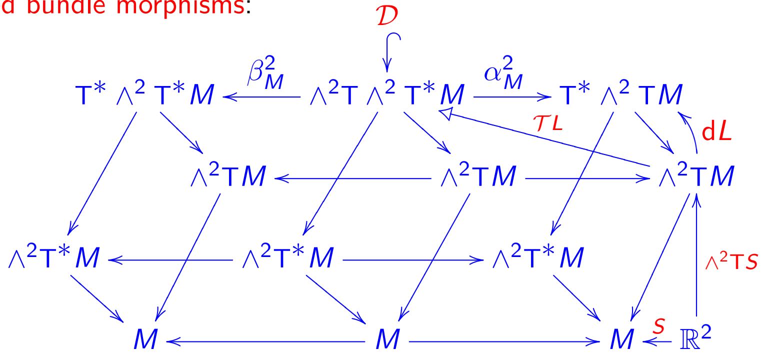

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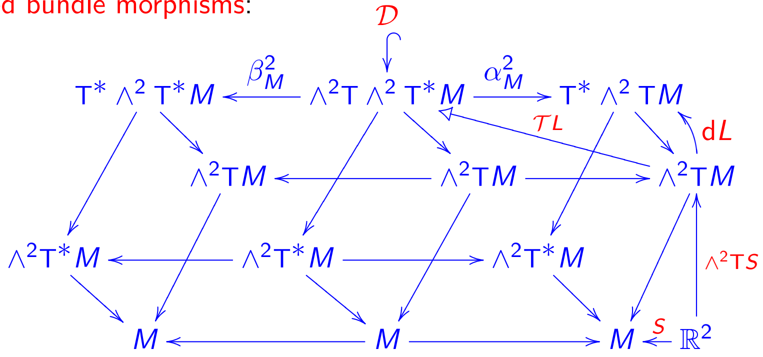

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The Euler-Lagrange equations

A surface $S:(t,s)\mapsto(x^\sigma(t,s))$ in M satisfies the Euler-Lagrange equations if the image by $\mathrm{d}L$ of its prolongation to $\wedge^2\mathrm{T}M$,

$$(t,s)\mapsto\left(x^\sigma(t,s),\dot{x}^{\mu\nu}=\frac{\partial x^\mu}{\partial t}\frac{\partial x^\nu}{\partial s}-\frac{\partial x^\mu}{\partial s}\frac{\partial x^\nu}{\partial t}\right),$$

is α^2_M-related to an admissible surface, i.e. the prolongation of a surface living in the phase space $\wedge^2\mathrm{T}^*M$ to $\wedge^2\mathrm{T}\wedge^2\mathrm{T}^*M$.

In coordinates, the Euler-Lagrange equations read

$$\begin{aligned}\dot{x}^{\mu\nu} &= \frac{\partial x^\mu}{\partial t}\frac{\partial x^\nu}{\partial s}-\frac{\partial x^\mu}{\partial s}\frac{\partial x^\nu}{\partial t},\\ \frac{\partial L}{\partial x^\sigma} &= \frac{\partial x^\mu}{\partial t}\frac{\partial}{\partial s}\left(\frac{\partial L}{\partial \dot{x}^{\mu\sigma}}(t,s)\right)-\frac{\partial x^\mu}{\partial s}\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{x}^{\mu\sigma}}(t,s)\right).\end{aligned}$$

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Plateau problem

In particular, if $M = \mathbb{R}^3 = \{(x^1 = x, x^2 = y, x^3 = z)\}$ with the Euclidean metric, the canonically induced 'free' Lagrangian on $\wedge^2 TM$ reads

$$L(x^\mu, \dot{x}^{\kappa\lambda}) = \sqrt{\sum_{\kappa,\lambda} (\dot{x}^{\kappa\lambda})^2}\,.$$

The Euler-Lagrange equation for surfaces being graphs $(x, y) \mapsto (x, y, z(x, y))$ provides the well-known equation for minimal surfaces, found already by Lagrange :

$$\frac{\partial}{\partial x}\left(\frac{z_x}{\sqrt{1+z_x^2+z_y^2}}\right) + \frac{\partial}{\partial y}\left(\frac{z_y}{\sqrt{1+z_x^2+z_y^2}}\right) = 0\,.$$

In another form:

$$(1+z_x^2)z_{yy} - 2z_x z_y z_{xy} + (1+z_y^2)z_{xx} = 0\,.$$

Starting with a Lorentz metric, we can obtain analogously the Euler-Lagrange equations for the Nambu-Goto Lagrangian.

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THANK YOU FOR YOUR ATTENTION!