Classification theorem for the static and asymptotically flat Einstein-Maxwell-dilaton spacetimes possessing a photon sphere

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Varna, 2017

Outline

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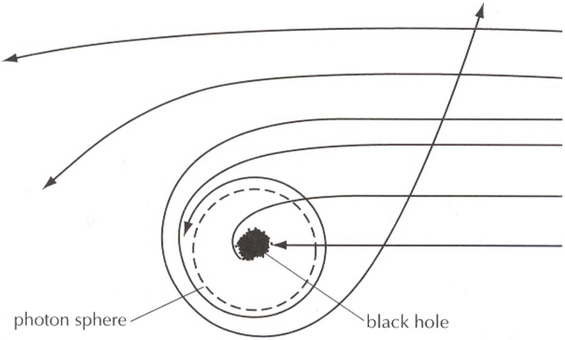
What is a photon sphere?

Any null geodesic initially tangent to the photon sphere remains tangent to it.



What is a photon sphere?

- Region where light can be confined to closed orbits.
- Characteristic of non-rotating black holes and compact objects with radii smaller than $3M$.
- Closely connected to gravitational lensing. Presence of a photon sphere leads to relativistic images.
- Astrophysically – timelike hypersurface on which the light bending angle is unboundedly large.

Mathematical properties

- In static spacetimes the lapse function is constant on the photon sphere.
- Photon spheres are totally umbilic hypersurfaces with constant mean and scalar curvatures as well as constant surface gravity.
- Resemblance to event horizons, which leads to the question of classification of solutions using photon spheres.

The problem

- Richer class of solutions possessing a photon sphere, compared to the one with an event horizon (neutron stars with radii smaller than $3M$ have a photon sphere, but don't have an event horizon).
- Uniqueness theorems have been proven for the static asymptotically flat solutions to the Einstein equations in vacuum, the Einstein-scalar field equations and the Einstein-Maxwell equations possessing a photon sphere.
- Naturally the next question is the classification of the static solutions to the Einstein-Maxwell-dilaton field equations.

Outline

Equations and staticity

We start with the standard Einstein-Maxwell-dilaton equations,

$$\begin{aligned}
&\mathfrak{R}_{\mu\nu} = 2\,{}^{\mathfrak{g}}\nabla_\mu\varphi\,{}^{\mathfrak{g}}\nabla_\nu\varphi + 2e^{-2\alpha\varphi}\left(F_{\mu\beta}F_\nu{}^\beta - \frac{\mathfrak{g}_{\mu\nu}}{4}F_{\beta\gamma}F^{\beta\gamma}\right),\\
&{}^{\mathfrak{g}}\nabla_{[\beta}F_{\mu\nu]} = 0,\\
&{}^{\mathfrak{g}}\nabla_\beta\left(e^{-2\alpha\varphi}F^{\beta\mu}\right) = 0,\\
&{}^{\mathfrak{g}}\nabla_\beta\,{}^{\mathfrak{g}}\nabla^\beta\varphi = -\frac{\alpha}{2}e^{-2\alpha\varphi}F_{\mu\nu}F^{\mu\nu},
\end{aligned}$$

and define staticity of the Maxwell and dilaton field in the usual way,

$$\begin{aligned}
\mathcal{L}_\xi F &= 0,\\
\mathcal{L}_\xi\varphi &= 0.
\end{aligned}$$

For static spacetimes we can write the spacetime and the metric in the form

$$\mathfrak{L}^4 = \mathbb{R}\times M^3,\ \mathfrak{g} = -N^2\mathrm{d}t^2 + g.$$

Photon sphere and electric potential Φ

The photon sphere P^3 is defined to be the photon surface of constant N.

We modify this with additional properties: the one-forms $\iota_\xi F$ and $\mathrm{d}\varphi$ are normal to P^3.

We will use later the electric field one-form E and the electric potential Φ, defined in the usual way:

$$E = -\iota_\xi F, \quad \mathrm{d}\Phi = E.$$

With this and considering the purely electric case, where $\iota_\xi \star F = 0$, F is given explicitly by

$$F = -N^{-2}\xi \wedge \mathrm{d}\Phi.$$

Asymptotic flatness

For asymtotically flat spacetimes the following expansions for the spatial metric and lapse function hold:

$$g = \delta + O(r^{-1}), \quad N = 1 - \frac{M}{r} + O(r^{-2}).$$

Next, the asymptotic expansions for the dilaton field φ and the electric potential Φ are given by the following:

$$\varphi = \varphi_\infty - \frac{q}{r} + O(r^{-2}),$$
$$\Phi = \Phi_\infty + \frac{Q}{r} + O(r^{-2}),$$

where we set $\varphi_\infty = 0$ and $\Phi_\infty = 0$.

Geometric picture

P^3 is the outermost photon sphere.

M^3 is a time slice.

M^3_{ext} is the spatial part of the spacetime $\mathfrak{L}^4$ outside of the photon sphere. We assume that N regularly foliates M^3_{ext}.

Σ is the intersection of P^3 and M^3, and is the inner boundary of M^3_{ext}. It is a level set of N by definition. All level sets of N are topological spheres as a consequence of our assumption.

Reduced field equations

The reduced Einstein-Maxwell-dilaton equations become:

$$
\begin{aligned}
{}^g\Delta N &= N^{-1} e^{-2\alpha\varphi}\, {}^g\nabla_i \Phi\, {}^g\nabla^i \Phi, \\
{}^g R_{ij} &= 2\, {}^g\nabla_i \varphi\, {}^g\nabla_j \varphi + N^{-1}\, {}^g\nabla_i\, {}^g\nabla_j N \\
&\quad + N^{-2} e^{-2\alpha\varphi} (g_{ij}\, {}^g\nabla_k \Phi\, {}^g\nabla^k \Phi - 2\, {}^g\nabla_i \Phi\, {}^g\nabla_j \Phi), \\
{}^g\nabla_i (N^{-1} e^{-2\alpha\varphi}\, {}^g\nabla^i \Phi) &= 0, \\
{}^g\nabla_i (N\, {}^g\nabla^i \varphi) &= \alpha N^{-1} e^{-2\alpha\varphi}\, {}^g\nabla_i \Phi\, {}^g\nabla^i \Phi.
\end{aligned}
$$

Mass, scalar charge and electric charge

Using the asymptotic expansions of the potentials, the electric charge is given by

$$Q = -\frac{1}{4\pi}\int_{\Sigma} N^{-1} e^{-2\alpha\varphi}\, {}^{g}\nabla^{i}\Phi \mathrm{d}\Sigma_{i}.$$

The mass and the dilaton charge are given by the following:

$$M = M_0 + \Phi_0 Q,$$
$$q = q_0 + \alpha\Phi_0 Q.$$

On the photon sphere we have:

$$M_0 = \frac{1}{4\pi}\int_{\Sigma} {}^{g}\nabla^{i} N \mathrm{d}\Sigma_{i},$$
$$q_0 = \frac{1}{4\pi}\int_{\Sigma} N\, {}^{g}\nabla^{i}\varphi \mathrm{d}\Sigma_{i}.$$

Outline

Dependence between the potentials N, φ and Φ

Using a new metric $\gamma_{ij} = N^2 g_{ij}$ on M^3_{ext}, and introducing new functions u, U, Ψ and $\hat{\Phi}$, such that

$$N^2 = e^{2u}, \quad U = u + \alpha\varphi, \quad \Psi = \varphi - \alpha u, \quad \hat{\Phi} = \sqrt{1+\alpha^2}\Phi,$$

we can rewrite the field equations in the following form:

$$
\begin{aligned}
&{}^{\gamma}R_{ij} = \frac{1}{1+\alpha^2}\left(2D_iUD_jU - 2e^{-2U}D_i\hat{\Phi}D_j\hat{\Phi} + 2D_i\Psi D_j\Psi\right),\\
&D_iD^iU = e^{-2U}D_i\hat{\Phi}D^i\hat{\Phi},\\
&D_iD^i\Psi = 0,\\
&D_i(e^{-2U}D^i\hat{\Phi}) = 0.
\end{aligned}
$$

Dependence between the potentials N, φ and Φ

From those we can obtain a functional dependence between U and $\hat{\Phi}$,

$$e^{2U} - 1 - \hat{\Phi}^2 + \frac{2(M + \alpha q)}{Q_\alpha}\hat{\Phi} = 0,$$

where $Q_\alpha = \sqrt{1+\alpha^2}Q$. Introducing yet another potential by

$$\mathrm{d}\zeta = e^{-2U}\mathrm{d}\hat{\Phi}, \quad \zeta_\infty = 0$$

we get another functional dependence,

$$(q_0 - \alpha M_0)\zeta - Q_\alpha \Psi = 0,$$

which finally reduces the EMD equations to

$$^{\gamma}R_{ij} = \frac{2}{1+\alpha^2}\left(\frac{M^2+q^2}{Q^2} - 1\right) D_i\zeta D_j\zeta,$$
$$D_i D^i \zeta = 0.$$

CMC and CSC; some useful relations

As a photon sphere, P^3 is totally umbilic, i. e. its second fundamental form is pure trace.

Codazzi for $P^3 \hookrightarrow \mathfrak{L}^4$ and EMD $\Longrightarrow P^3$ has CMC.

Contracted Gauss for $P^3 \hookrightarrow \mathfrak{L}^4$, EMD, Codazzi for $\Sigma \hookrightarrow M^3$ and dependence between N, φ and $\Phi \Longrightarrow P^3$ has CSC.

We also obtain the following equalities to be used later in the proof:

$$N_0 = \frac{1}{4\pi} e^{-2\alpha\varphi_0} N_0^{-1} E_\nu^2 A_\Sigma - \frac{1}{4\pi} N_0 ({}^{\mathfrak{g}}\nabla_\nu \varphi)^2 A_\Sigma + \frac{3}{8\pi} H [\nu(N)]_0 A_\Sigma,$$
$$2[\nu(N)]_0 = N_0 H.$$

Outline

Theorem 1

Let $(\mathfrak{L}^4_{ext}, \mathfrak{g}, F, \varphi)$ be a static and asymptotically flat spacetime with given mass M, electric charge Q and dilaton charge q, satisfying the Einstein-Maxwell-dilaton equations and possessing a non-extremal photon sphere (i. e. $M^2 + q^2 - Q^2 \neq 0$) as an inner boundary of $\mathfrak{L}^4_{ext}$. Assume that the lapse function regularly foliates $\mathfrak{L}^4_{ext}$. Then $(\mathfrak{L}^4_{ext}, \mathfrak{g}, F, \varphi)$ is spherically symmetric.

Outline of the proof – case 1

Here $M^2 + q^2 > Q^2$ and we consider the potential $\lambda = \sqrt{\left(\frac{M^2+q^2}{Q^2} - 1\right)\frac{1}{1+\alpha^2}\zeta}$ and the following inequalities:

$$\int_{M^3_{ext}} D^i \left[\Omega^{-1}\left(\Gamma D_i\chi - \chi D_i\Gamma\right)\right]\sqrt{\gamma}\mathrm{d}^3x \geq 0$$

and

$$\int_{M^3_{ext}} D^i \left(\Omega^{-1} D_i\chi\right)\sqrt{\gamma}\mathrm{d}^3x \geq \int_{M^3_{ext}} D^i \left[\Omega^{-1}\left(\Gamma D_i\chi - \chi D_i\Gamma\right)\right]\sqrt{\gamma}\mathrm{d}^3x,$$

where $\chi = \left(\gamma^{ij}D_i\Gamma D_j\Gamma\right)^{\frac{1}{4}}$, $\Gamma = -\tanh(\lambda)$ and $\Omega = \frac{1}{\cosh^2(\lambda)}$.

We can show that these transform into an equality,

$$\left(\frac{d\ln(N)}{d\lambda}\right)_0 = -\frac{1}{2}\coth(\lambda_0).$$

Outline of the proof – case 1

This leads to a vanishing Bach tensor, i. e. $R(\gamma)_{ijk} = 0$, which means that $R(g)_{ijk} = 0$.

Finally, considering the surfaces of constant N in M^3, $(\Sigma_N, \sigma) \hookrightarrow (M^3, g)$ we can write the spacetime metric in the form:

$$\mathfrak{g} = -N^2 \mathrm{d}t^2 + \rho^2 \mathrm{d}N^2 + \sigma_{AB} \mathrm{d}x^A \mathrm{d}x^B,$$

since N foliates M^3_{ext} regularly. Calculating $R(g)_{ijk} R(g)^{ijk} = 0$ explicitly we can conclude that

$$h^{\Sigma_N}_{AB} = \frac{1}{2} H^{\Sigma_N} \sigma_{AB}, \quad \partial_A \rho = 0,$$

i. e. the space geometry is spherically symmetric.

Outline of the proof – case 2

Here $M^2 + q^2 < Q^2$ and we consider the potential

$\lambda = \sqrt{\left(1 - \frac{M^2+q^2}{Q^2}\right)\frac{1}{1+\alpha^2}}\zeta.$

We use the same inequalities but with different functions Γ and Ω,

$$\Gamma = -\tan(\lambda), \quad \Omega = \cos^{-2}(\lambda).$$

The two inequalities lead to an equality,

$$\left(\frac{d\ln(N)}{d\lambda}\right)_0 = -\frac{1}{2}\cot(\lambda_0).$$

The same arguments as in case 1 hold and the spacetime is again spherically symmetric.

Theorem 2

The second theorem gives an explicit classification of the static and asymptotically flat Einstein-Maxwell-dilaton spacetimes possessing a photon sphere. We can introduce a new parameter $M_\alpha = M + \alpha q$ and the following formula:

$$M^2 + q^2 - Q^2 = \frac{1}{1+\alpha^2}\left[M_\alpha^2 - Q_\alpha^2 + (q - \alpha M)^2\right].$$

We now consider 2 cases.

Theorem 2 – case 1

Here $M^2 + q^2 > Q^2$. The dimensionally reduced field equations are

$$
\begin{aligned}
{}^{\gamma}R_{ij} &= 2D_i\lambda D_j\lambda, \\
D_i D^i \lambda &= 0.
\end{aligned}
$$

These can be solved for spherically symmetric space,

$$
\begin{aligned}
e^{2\lambda} &= 1 - \frac{2\sqrt{M^2 + q^2 - Q^2}}{r}, \\
\gamma_{ij}dx^i dx^j &= dr^2 + e^{2\lambda}r^2(d\theta^2 + \sin^2\theta d\phi^2).
\end{aligned}
$$

The spacetime metric is thus

$$
ds^2 = -N^2dt^2 + N^{-2}\left[dr^2 + e^{2\lambda}r^2(d\theta^2 + \sin^2\theta d\phi^2)\right]
$$

and we can obtain explicit expressions (3 classes depending on $\frac{Q_\alpha^2}{M_\alpha^2}$) for N, φ and Φ, since we know λ.

Theorem 2 – case 2

Here $M^2 + q^2 < Q^2$. The dimensionally reduced field equations are

$$
\begin{aligned}
{}^{\gamma}R_{ij} &= -2D_i\lambda D_j\lambda, \\
D_iD^i\lambda &= 0.
\end{aligned}
$$

These can be solved for spherically symmetric space,

$$
\begin{aligned}
\lambda &= \arctan\left(\frac{\sqrt{Q^2 - M^2 - q^2}}{r}\right), \\
\gamma_{ij} &= dr^2 + (r^2 + Q^2 - M^2 - q^2)(d\theta^2 + \sin^2\theta d\phi^2).
\end{aligned}
$$

The spacetime metric is thus

$$
ds^2 = -N^2dt^2 + N^{-2}\left[dr^2 + (r^2 + Q^2 - M^2 - q^2)(d\theta^2 + \sin^2\theta d\phi^2)\right]
$$

and we can again obtain explicit expressions for N, φ and Φ.

Outline

Discussion

Assumptions:

The lapse function regularly foliates M^3_{ext}. In general this assumption cannot be easily dropped.

Some of what remains is:

Higher dimensional EMD gravitation and/or similar results for stationary spacetimes.

What else has been done:

Static spacetimes with conformal scalar field. Perturbative approach for the static vacuum case.

Thank you for your attention!

The support by Sofia University Research Grant 80.10-30
is gratefully acknowledged.