Mathematical Models of Classical Electrodynamics

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I. INTRODUCTION

- **The Original Maxwell's Equations**

- In 1865 the great Scottish physicist James C. Maxwell published his spectacular theory that became a basis of Classical Electrodynamics (CED);
- They covered four different physical laws: 1) Faraday's law; 2) Ampere's law (with a correction introduced pure theoretically by Maxwell); 3) Gauss's law of electric field; 4) Gauss's law of magnetic field (no magnetic charges!).
- Originally: 20 equations, then: 12 equations. Ether included?

- **Different EM Models**
- I. The Concept of Vector Fields

- The vector notations (in 3-D space) are listed below. It is curious than even after introduction into CED, the main equations are remained the same in principle.

I. The Concept of Vector Fields

Introduced in CED by the British electrical engineer Oliver Heaviside in 1885. He proposed the following system of partial differential equations in 3-D vector space to express the four laws in Maxwell's theory:

$$\begin{aligned} \operatorname{div} \mathbf{D} &= \rho \,, \\ \operatorname{div} \mathbf{B} &= 0 \,, \\ \operatorname{curl} \mathbf{E} &= -\partial_t \mathbf{B} \,, \\ \operatorname{curl} \mathbf{H} &= \sigma \mathbf{E} + \partial_t \mathbf{D} \,. \end{aligned} \tag{1}$$

∂_t - time-derivative; ρ [C/m^3] - electric charge density; σ [S/m] – conductivity;
E [V/m] - electric field intensity; **B** [Wb/m^2 = T] - magnetic flux density;
D [C/m^2] - electric field displacement; **H** [A/m] - magnetic field intensity.

. I. The Concept's of Vector Fields

- The material equations:
- $\mathbf{D} = \varepsilon\, \mathbf{E}$,
- $\mathbf{B} = \mu\, \mathbf{H}$,
- $\mathbf{J} = \sigma\, \mathbf{E}$. (2)
- ε [F/m] - permittivity of the medium; μ [H/m] - its permeability;
- **J** [A/m^2] - electric current density.
- In 1884 Hertz discovered experimentally the electromagnetic waves Maxwell first understood that the light is such a wave propagating in space with a finite velocity $c = 3 \times\ 10^8\, m/s$.
- In 1892 Lorentz discovered: the Maxwell's equations are invariant to the transforms (v = const, in x – direction):
- $x\ \rightarrow (\,x - vt)/\sqrt{1 -\ v^2/c^2},$
- $y\ \rightarrow y, \quad z\ \rightarrow z.$ (3)

II. The Tensor Notations

- In 1905 Einstein – Minkowski (Special Theory of Relativity).
- 4 – D space-time frame of inertial coordinate systems:

$$[x]=\begin{bmatrix} x \\ y \\ z \\ jct \end{bmatrix},$$

$$[\partial]=\begin{bmatrix} \partial_x \\ \partial_y \\ \partial_z \\ \partial_{jct} \end{bmatrix},$$

$$[J]=\begin{bmatrix} Z_0 J_x \\ Z_0 J_x \\ Z_0 J_x \\ j\rho/\varepsilon_0 \end{bmatrix}, \qquad (4)$$

$$[F]=\begin{bmatrix} 0 & cB_z & -cB_y & -jE_x \\ -cB_z & 0 & cB_x & -jE_y \\ cB_y & -cB_x & 0 & -jE_z \\ jE_x & jE_y & jE_z & 0 \end{bmatrix},$$

II. The Tensor Notations

$x^{\mu} = [x,\ y,\ z,\ jct]^T$ - 4-D coordinates;

$\partial_{\nu} = [\ \partial_x ,\ \ \partial_y ,\ \ \partial_z,\ \ \partial_{jct}]^T$ - 4-D covariant first derivatives;

$J^{\mu} = [Z_0 J_x , Z_0 J_y , Z_0 J_z , j\rho/\epsilon_0]^T$ - 4-tensor 1st rank of the electromagnetic sources;

$\varepsilon_0 \approx 8.85 \times 10^{-12}$F/m; $Z_0 \approx 377\ \Omega$ (const).

$F^{\mu\nu}$ - 4-tensor 2nd rank (anti-simmetric) of the electromagnetic fields; ($F^{\mu\mu} = 0$).

$F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu}$;

$A_{\mu} = [Z_0 A_x , Z_0 A_y , Z_0 A_z , j\varphi/\epsilon_0]^T$ $-$ 4 $-$ tensor 1st rank of the electromagnetic potentials.

II. The Tensor Notations

- $j = i = \sqrt{-1}$ - imaginary unit;
- The Maxwell's equations in tensor notations:
- $$\partial_\nu F^{\mu\nu} = J^\mu ,$$ $$\partial_\rho F_{\sigma\tau} + \partial_\sigma F_{\tau\rho} + \partial_\tau F_{\rho\sigma} = 0 . \qquad (5)$$
- Advantages of the tensor notations:
- a) More concise form;

  b) Obvious covariant

  (automatic fulfillment of the restrictions of the Special Theory of Relativity).

III. The Geometric Algebra Model

- Created by William Clifford in late 19th century. Our presentation follows the recent developments of this theory by D. Hestenes.
- The GA (Geometric Algebra) approach is a vector space approach, based on the theory of multi-vectors.
- In theory of quaternions (developed by W. Hamilton) the result is in the same vector space.
- In GA the result is based on a graded hierarchy of n-vectors.
- The multi-vectors here are written as symbols in ***bold italic*** (***u, v,*** etc).
- Here a combination of different products (1: uni-vector ***u.v***; 2: bi-vector: $\boldsymbol{u} \wedge \boldsymbol{v}$; and 3: three-vector $I = \hat{\mathbf{x}}\,\hat{\boldsymbol{y}}\,\hat{\boldsymbol{z}}$) are used; I is a generalization of imaginary unit *j*.
- The product is defined as:
- $$\mathbf{uv} = \mathbf{u.v} + \mathbf{u} \wedge \boldsymbol{v} = \mathbf{u.v} + I\,\mathbf{u} \times \boldsymbol{v}\,. \qquad (6)$$
- This yields:

III. The geometric algebra model

$$\nabla \mathbf{E} = \nabla \bullet \mathbf{E} + I \nabla \times \mathbf{E}$$

$$= \frac{1}{\varepsilon_0} \rho - \frac{1}{c} \partial_t \left(Ic\mathbf{B} \right)$$

- and (7)

$$\nabla \left(Ic\mathbf{B} \right) = Ic \nabla \bullet \mathbf{B} + I^2 c \nabla \times \mathbf{B}$$

$$= -Z_0 \mathbf{J} - \frac{1}{c} \partial_t \mathbf{E} \,,$$

- We define three new multi-vectors:

- $\boldsymbol{F} = \boldsymbol{E} + Ic\boldsymbol{B}$ - multi – vector of electromagnetic field; (8)

$\boldsymbol{J} = \frac{1}{\epsilon_0} \rho - Z_0 \mathbf{J}$ - multi-vector of electromagnetic source;

$\boldsymbol{G} = \frac{1}{\epsilon_0} \mathbf{D} + IZ_0 \mathbf{H}$ - multi-vector of macroscopic media.

III. The geometric algebra model

- The equations (7) are then presented in the form:
- $$(\nabla + \partial_t)\, F = J, \qquad (9)$$
- The Maxwell's equations can now be expressed in a single equation form:
- $$\nabla F = J\,, \qquad (11)$$
- where: $\nabla = \hat{\mathbf{x}}\,\partial_x + \hat{\mathbf{y}}\,\partial_y + \hat{\mathbf{z}}\,\partial_z - \frac{\mathbf{1}}{\boldsymbol{c}}\hat{\boldsymbol{t}}\,\partial_t$.
- The inverse of the last equations is given by the time-domain Green's function:
- $$F = \nabla^{-1} J. \qquad (12)$$

IV. The Differential Forms Model

- The DF (differential forms) method is developed in the beginning of 20th century by Elie Cartan.
- It is related to the Integral Form of Maxwell's Equations
- There is a correspondence (↔) between the usual vector notations and the DF notations:
- A) for polar vector (1-form):
- $$f = f_x dx + f_y dy + f_z dz \leftrightarrow \mathbf{f} = f_x \hat{\mathbf{x}} + f_y \hat{\mathbf{y}} + f_z \hat{\mathbf{z}} \qquad (13)$$
- B) for axial vector (bi-vector, or 2-form):
- $$U = U_x dydz + U_y dzdx + U_z dxdy \leftrightarrow \mathbf{U} = U_x \hat{\mathbf{x}} + U_y \hat{\mathbf{y}} + U_z \hat{\mathbf{z}} \qquad (14)$$
- $\hat{\boldsymbol{x}} = \hat{\boldsymbol{y}} \times \hat{\mathbf{z}}$, $\hat{\boldsymbol{y}} = \hat{\mathbf{z}} \times \hat{\boldsymbol{x}}$, and $\hat{\boldsymbol{z}} = \hat{\boldsymbol{x}} \times \hat{\boldsymbol{y}}$;
- $\hat{\boldsymbol{x}} \leftrightarrow dydz$, $\hat{\boldsymbol{y}} \leftrightarrow dzdx$, $\hat{\mathbf{z}} \leftrightarrow dxdy$.
- Provided: (f) and (g) are 1-forms, then new "exterior product" (or "wedge product") creates a 2-form, denoted by $(f \wedge g)$.

IV. The Differential Forms Model

- *It is anti-symmetric:* $du \wedge dv = - dv \wedge du$,
- *And:* $du \wedge du = dv \wedge dv = 0$.
- Here we will drop the $\wedge$ sign, writing $dx \wedge dy = dxdy$, etc.
- A key 1-form is *the exterior vector derivative d*, which in 3-D space takes the form
- $$\boldsymbol{d} = \boldsymbol{\partial_x dx} + \boldsymbol{\partial_y dy} + \boldsymbol{\partial_z dz} \ . \qquad (15)$$
- $$dU \equiv d\wedge U \equiv (\partial_x dx + \partial_y dy + \partial_z dz)\wedge U.$$
- However, applying it to a *1-form*, we find:
- $$\begin{aligned} dE &= (\partial_x dx + \partial_y dy + \partial_z dz)\wedge(E_x dx + E_y dy + E_z dz) \\ &= (\partial_x E_y - \partial_y E_x)dxdy + (\partial_y E_z - \partial_z E_y)dydz \\ &\quad + (\partial_z E_x - \partial_x E_z)dzdx \\ &\Leftrightarrow \ dE \ \leftrightarrow \ \nabla \times \mathbf{E}\ . \end{aligned} \qquad (16)$$

IV. The Differential Forms Model

- In the case of a *2-form* by applying the computation rules we find:

$$dD = \left(\partial_x dx + \partial_y dy + \partial_z dz\right) \wedge \left(D_x dydz + D_y dzdx + D_z dxdy\right)$$

$$= \left(\partial_x D_x + \partial_y D_y + \partial_z D_z\right) dxdydz \qquad (17)$$

$$\Leftrightarrow \quad dD \leftrightarrow \nabla \bullet \mathbf{D}\,.$$

In DF model:

0-form: scalar;

1-form: vector;

2-form: bi-vector;

3-form: pseudo-scalar.

$(q = \rho dxdydz)$

See the Table I.

↓

IV. The Differential Forms Model

Table 1. The electromagnetic quantities and source densities represented by different degrees of forms.

Form	Electromagnetic Quantity	Differential Element
0-forms	q, c, ϕ	$(du)^0 \equiv 1$
1-forms	E, H, A	dx, dy, dz
2-forms	D, B, J	$dydz, dzdx, dxdy$
3-forms	ρ	$dxdydz$

IV. The Differential Forms Model

- The DF of Maxwell's equations is:

$$dD = \rho^{free} ,$$

$$dB = 0 ,$$

$$dE = -\partial_t B ,$$

$$dH = J^{free} + \partial_t D .$$

(18)

- The operator (d) transforms the forms as follows: 0-form -> 1-form -> 2-form -> 3-form -> 0-form. The Integral Form of Maxwell's equations is:

$$\int_{\partial V} dD = \int_V \rho^{free} ,$$

$$\int_{\partial V} dB = \int_V 0 = 0 ,$$

$$\int_{\partial A} dE = -\partial_t \int_A B ,$$

$$\int_{\partial A} dH = \int_A \left(J^{free} + \partial_t D \right) .$$

(19)

IV. The Differential Forms Model

- We introduce a more general d-operator in 4-D space-time domain:
- $$\boldsymbol{d} = \left(\boldsymbol{\partial_x dx} + \boldsymbol{\partial_y dy} + \boldsymbol{\partial_z dz} + \boldsymbol{\partial_t dt}\right) \qquad (20)$$
- Here, like in the GA formalism, we can introduce new complete *F* and *G* fields. The system of equations (18) (the Maxwell's equations (1))
- may now be written as a compact DF system of equations:

- $$dF = 0;$$
- $$dG = J, \qquad (21)$$

- where
- $$F = E \wedge dt + B;$$
- $$G = -H \wedge dt + D; \qquad (22)$$
- $$J = -J^{free} \wedge dt + \rho^{free}.$$

IV. The Differential Forms Model

(compare the last system to the GA system (11))!

- Given the "*operator* $*$ " (*so-called a "star-operator"*) that in 3-D *converts* a 1-form into a 2-form (its dual), and *vice versa*.

- This operator allows to write the material equations (2) into the following form:

$$B = \mu * H,$$
$$D = \varepsilon * E, \qquad (23)$$
$$J = \sigma * E.$$

III. CONCLUSION

- In this paper four different models of the Maxwell's Equations (explained the Classical Electrodynamics) are considered and compared:

- I. Vector Approach;
- II. Tensor Approach;
- III. Geometric Algebra Model;
- IV. Differential Forms Model.

- The first one is a classical approach (19^{th} century), introduced by Heaviside. The other ones are modern: they are introduced in the beginning of 20^{th} century. All they have one important advantage: the original 8 scalar equations are converted to only 2 ones.
- This brief notation allows not only to deal with short system, but easy to check the important property called "covariance".

- Thank you for your attention!
- Any questions?