Introduction to the theory of Clifford algebras

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Outline

Lecture 1 **Clifford Algebras and Related Structures.**
Definition of Clifford algebras. Examples in small dimensions: complex numbers, double numbers, quaternions, Pauli's matrices, Dirac's matrices. Grassmann algebra. Z_2-grading, grade involution, reversion, Clifford conjugation. Center of Clifford algebra.

Lecture 2 **Unitary Spaces on Clifford Algebras.**
Hermitian scalar product in Clifford algebras. Operation of Hermitian conjugation and unitary groups in Clifford algebras.

Lecture 3 **Matrix Representations of Clifford Algebras.**
Cartan's periodicity of 8 for Clifford algebras. Faithful and irreducible representations. Primitive idempotents and minimal left ideals.

Lecture 4 **Lie Groups and Lie Algebras in Clifford Algebras.**
Spin groups as subgroups of Clifford and Lipschitz groups. Double covers of the orthogonal groups. Cartan-Dieudonne theorem. Spin groups in small dimensions. Lie groups in Clifford algebras and corresponding Lie algebras.

Lecture 5 **Dirac Equation.** Dirac equation in Clifford algebras. Dirac-Hestenes equation. Spinors in n dimensions.

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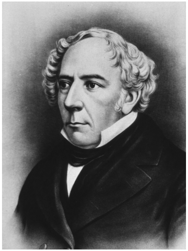
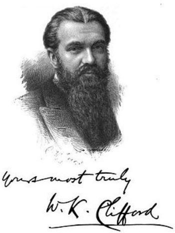
Lecture 1

Clifford Algebras and Related Structures

Definition of Clifford algebras. Examples in small dimensions: complex numbers, double numbers, quaternions, Pauli's matrices, Dirac's matrices. Grassmann algebra. Z_2-grading, grade involution, reversion, Clifford conjugation. Center of Clifford algebra.





William Rowan Hamilton (1805 - 1865)
Hermann Günther Grassmann (1809 - 1877)
William Kingdon Clifford (1845 - 1879)

W. R. Hamilton, On quaternions, or on a new system of imaginaries in algebra, Philosophical Magazine, 1844. (letter to John T. Graves, dated October 17, 1843)

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W. K. Clifford, Application of Grassmann's Extensive Algebra, American Journal of Mathematics, I: 350-358, 1878.

Quaternions

Associative division algebra $\mathbb{H}$:

$$q = a1 + bi + cj + dk \in \mathbb{H}, \qquad a, b, c, d \in \mathbb{R},$$
$$1 \text{ is identity element},$$
$$i^2 = j^2 = k^2 = -1,$$
$$ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j.$$

$$(a_1 + b_1 i + c_1 j + d_1 k)(a_2 + b_2 i + c_2 j + d_2 k) =$$
$$(a_1 a_2 - b_1 b_2 - c_1 c_2 - d_1 d_2) + (a_1 b_2 + b_1 a_2 + c_1 d_2 - d_1 c_2)i$$
$$+(a_1 c_2 + c_1 a_2 - b_1 d_2 + d_1 b_2)j + (a_1 d_2 + d_1 a_2 + b_1 c_2 - c_1 b_2)k.$$

$$\bar{q} := a - bi - cj - dk, \qquad ||q|| := \sqrt{q\bar{q}} = \sqrt{a^2 + b^2 + c^2 + d^2};$$
$$q \neq 0 \Rightarrow \exists q^{-1} = \frac{1}{||q||^2}\bar{q}.$$

Real Clifford algebra $\mathcal{C}\!\ell_{p,q,r}$ (with fixed basis)

Linear space E over $\mathbb{R}$, $n \in \mathbb{N}$, $\dim E = 2^n$ with the basis

$$\{e, e_{a_1}, e_{a_1 a_2}, \ldots, e_{1\ldots n}\}, \qquad 1 \leq a_1 < \cdots < a_k \leq n,$$

and an operation of multiplication $U, V \to UV$ with the following properties:

1. distributivity

$$U(\alpha V + \beta W) = \alpha UV + \beta UW, \quad (\alpha U + \beta V)W = \alpha UW + \beta VW,$$
$$\forall U, V, W \in E, \qquad \forall \alpha, \beta \in \mathbb{R};$$

2. associativity

$$U(VW) = (UV)W, \qquad \forall U, V, W \in E;$$

3. e is the identity element

$$Ue = eU = U, \qquad \forall U \in E;$$

4. $\{e_a,\ a = 1, \ldots, n\}$ are generators

$$e_{a_1} \ldots e_{a_k} = e_{a_1 \ldots a_k}, \qquad 1 \leq a_1 < \cdots < a_k \leq n;$$

5. the main anticommutative property

$$\boxed{e_a e_b + e_b e_a = 2\eta_{ab} e}, \ \eta = ||\eta_{ab}|| = \mathrm{diag}(\underbrace{1, \ldots, 1}_{p}, \underbrace{-1, \ldots, -1}_{q}, \underbrace{0, \ldots, 0}_{r}),\ p+q+r = n.$$

Alternative definitions of Clifford algebra

5 definitions:

Lounesto P., Clifford Algebras and Spinors, Cambridge Univ. Press (1997).

Clifford algebra as a quotient of the tensor algebra:
We consider a vector space V of arbitrary finite dimension n over the field $\mathbb{R}$. We have a quadratic form $Q : V \to \mathbb{R}$. Consider the tensor algebra

$$T(V) = \bigoplus_{k=0}^{\infty} \bigotimes^{k} V$$

and the two-sided ideal $I(V, Q)$ generated by all elements of the form $x \otimes x - Q(x)e$ for $x \in V$. Then we called the Clifford algebra $C\ell(V, Q)$ the following quotient algebra

$$C\ell(V, Q) = T(V)/I(V, Q).$$

Chevalley C., The algebraic theory of Spinors and Clifford algebras, Springer (1996).

Particular cases:

- Nondegenerate Clifford algebra $\mathcal{C}\ell_{p,q} := \mathcal{C}\ell_{p,q,0}$ (Q is nondegenerate);

$$(e_a)^2 = \pm e, \qquad e_a e_b = -e_b e_a, \qquad a \neq b;$$

- Clifford algebra of the Euclidian space $\mathbb{R}^n$: $\mathcal{C}\ell_n := \mathcal{C}\ell_{n,0,0}$ (Q is positive definite);

$$(e_a)^2 = e, \qquad e_a e_b = -e_b e_a, \qquad a \neq b;$$

- Grassmann algebra $\Lambda_r := \mathcal{C}\ell_{0,0,r}$ ($Q \equiv 0$);

$$e_a \wedge e_a = 0, \qquad e_a \wedge e_b = -e_b \wedge e_a.$$

Related structures:

- Complex Clifford algebra $\mathcal{C}\ell(\mathbb{C}^n)$ (linear space V is over $\mathbb{C}$).
- Complexified Clifford algebra $\mathbb{C} \otimes \mathcal{C}\ell_{p,q}$.

$$\mathcal{C}\ell_{p,q,r} \ni U = ue + \sum_{a} u_a e_a + \sum_{a<b} u_{ab} e_{ab} + \cdots + u_{1\ldots n} e_{1\ldots n} = \sum_{A} u_A e_A, \quad u_A \in \mathbb{R},$$

$$A = a_1 \ldots a_k, \qquad |A| = k.$$

Subspaces of grade k**:**

$$\mathcal{C}\ell_{p,q,r} = \bigoplus_{k=0}^{n} \mathcal{C}\ell^k_{p,q,r}, \qquad \mathcal{C}\ell^k_{p,q,r} = \{\sum_{|A|=k} u_A e_A\}, \qquad \dim \mathcal{C}\ell^k_{p,q,r} = C_n^k = \frac{n!}{k!(n-k)!}$$

Z_2**-grading:** Clifford algebra $\mathcal{C}\ell_{p,q,r}$ is the direct sum of even and odd subspaces:

$$\mathcal{C}\ell_{p,q,r} = \mathcal{C}\ell^{(0)}_{p,q,r} \oplus \mathcal{C}\ell^{(1)}_{p,q,r}, \quad \mathcal{C}\ell^{(0)}_{p,q,r} = \bigoplus_{k=0 \bmod 2} \mathcal{C}\ell^k_{p,q,r}, \quad \mathcal{C}\ell^{(1)}_{p,q,r} = \bigoplus_{k=1 \bmod 2} \mathcal{C}\ell^k_{p,q,r},$$

$$\boxed{\mathcal{C}\ell^{(i)}_{p,q,r} \mathcal{C}\ell^{(j)}_{p,q,r} \subset \mathcal{C}\ell^{(i+j) \bmod 2}_{p,q,r}, \quad i = 0, 1}; \quad \dim \mathcal{C}\ell^{(0)}_{p,q,r} = \dim \mathcal{C}\ell^{(1)}_{p,q,r} = 2^{n-1};$$

$\mathcal{C}\ell^{(0)}_{p,q,r}$ is subalgebra of $\mathcal{C}\ell_{p,q,r}$.

Examples in small dimensions

$$Cl_0 \cong \mathbb{R} \quad U = ue, \quad e^2 = e; \qquad \text{(real numbers)}$$

$$Cl_1 \cong \mathbb{R} \oplus \mathbb{R} \quad U = ue + u_1 e_1, \qquad (e_1)^2 = e; \qquad \text{(double numbers)}$$

$$Cl_{0,1} \cong \mathbb{C} \quad U = ue + u_1 e_1, \qquad (e_1)^2 = -e; \qquad \text{(complex numbers)}$$

$$Cl_{0,2} \cong \mathbb{H} \quad U = ue + u_1 e_1 + u_2 e_2 + u_{12} e_{12}; \qquad \text{(quaternions)}$$

$$(e_1)^2 = (e_2)^2 = -e,$$
$$(e_{12})^2 = e_1 e_2 e_1 e_2 = -e_1 e_1 e_2 e_2 = -e,$$
$$e_1 e_2 = -e_2 e_1 = e_{12}, \qquad e_2 e_{12} = -e_{12} e_2 = e_1,$$
$$e_{12} e_1 = -e_1 e_{12} = e_2,$$
$$e_1 \to i, \qquad e_2 \to j, \qquad e_{12} \to k,$$

$Cl_2 \cong Cl_{1,1} \cong \mathrm{Mat}(2, \mathbb{R}) \not\cong Cl_{0,2}$ (see Lecture 3).

Pauli matrices

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\begin{aligned} &\sigma_1\sigma_2 = i\sigma_3, \quad \sigma_2\sigma_3 = i\sigma_1, \quad \sigma_3\sigma_1 = i\sigma_2, \\ &(\sigma_a)^\dagger = \sigma_a, \quad \mathrm{tr}(\sigma_a) = 0, \quad (\sigma_a)^2 = \sigma_0, \qquad a = 1, 2, 3, \\ &\sigma_a\sigma_b = -\sigma_b\sigma_a, \qquad a \neq b, \qquad a, b = 1, 2, 3. \end{aligned}$$

$$\boxed{C\ell_3 \cong \mathrm{Mat}(2, \mathbb{C})},$$

$$e \to \sigma_0, \quad e_a \to \sigma_a, a = 1, 2, 3, \quad e_{ab} \to \sigma_a\sigma_b, a < b, \quad e_{123} \to \sigma_1\sigma_2\sigma_3,$$

$$\{\sigma_0, \quad \sigma_1, \quad \sigma_2, \quad \sigma_3, \quad i\sigma_1, \quad i\sigma_2, \quad i\sigma_3, \quad i\sigma_0\}.$$

W. Pauli, 1927. [Pauli's matrices were introduced by W. Pauli to describe spin of an electron]

Dirac gamma matrices

$$\gamma_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \gamma_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix},$$

$$\gamma_2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}, \quad \gamma_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

$$\gamma_a\gamma_b + \gamma_b\gamma_a = 2\eta_{ab}\mathbf{1}, \quad a, b = 0, 1, 2, 3, \quad \eta = \|\eta_{ab}\| = \mathrm{diag}(1, -1, -1, -1),$$

$$\mathrm{tr}\gamma_a = 0, \qquad \gamma_a^\dagger = \gamma_0\gamma_a\gamma_0, \qquad a = 0, 1, 2, 3.$$

$$\boxed{\mathbb{C} \otimes C\!\ell_{1,3} \cong \mathrm{Mat}(4, \mathbb{C})}, \qquad e_a \to \gamma_a, \quad a = 0, 1, 2, 3.$$

Dirac P.A.M., Proc. Roy. Soc. Lond. A117 (1928).

Dirac P.A.M., Proc. Roy. Soc. Lond. A118 (1928).

Operations of conjugation

- grade involution $\widehat{U} := U|_{e_a \to -e_a}$,

$$\widehat{U} = \widehat{\sum_{k=0}^{n} \overset{k}{U}} = \sum_{k=0}^{n} (-1)^k \overset{k}{U}, \qquad \overset{k}{U} \in C\ell^k_{p,q};$$

$$\widehat{\widehat{U}} = U, \quad \widehat{UV} = \widehat{U}\widehat{V}, \quad \lambda \widehat{U + \mu} V = \lambda \widehat{U} + \mu \widehat{V}, \quad \forall U, V \in C\ell_{p,q}, \forall \lambda, \mu \in \mathbb{R};$$

- reversion (anti-involution) $\widetilde{U} := U|_{e_{a_1 \ldots a_k} \to e_{a_k} \ldots e_{a_1}}$,

$$\widetilde{U} = \widetilde{\sum_{k=0}^{n} \overset{k}{U}} = \sum_{k=0}^{n} (-1)^{\frac{k(k-1)}{2}} \overset{k}{U}, \qquad \overset{k}{U} \in C\ell^k_{p,q};$$

$$\widetilde{\widetilde{U}} = U, \quad \widetilde{UV} = \widetilde{V}\widetilde{U}, \quad \lambda \widetilde{U + \mu} V = \lambda \widetilde{U} + \mu \widetilde{V}, \quad \forall U, V \in C\ell_{p,q}, \forall \lambda, \mu \in \mathbb{R};$$

- Clifford conjugation (anti-involution) = superposition of reversion and grade involution;
- complex conjugation in $\mathbb{C} \otimes C\ell_{p,q}$: $\overline{U} := U|_{u_{a_1 \ldots a_k} \to \bar{u}_{a_1 \ldots a_k}}$;

$$\overline{\overline{U}} = U, \quad \overline{UV} = \overline{U}\,\overline{V}, \quad \overline{\lambda U + \mu V} = \bar{\lambda} \overline{U} + \bar{\mu} \overline{V}, \quad \forall U, V \in \mathbb{C} \otimes C\ell_{p,q}, \forall \lambda, \mu \in \mathbb{C};$$

- hermitian conjugation in $\mathbb{C} \otimes C\ell_{p,q}$ (see Lecture 2).

Quaternion types $\overline{0}, \overline{1}, \overline{2}, \overline{3}$

$$C\ell^{(j)}_{p,q} := \bigoplus_{k=j \bmod 2} C\ell^k_{p,q} = \{U \in C\ell_{p,q} \mid \widehat{U} = (-1)^j U\},\ j = 0, 1;$$ (even and odd subspaces)

$$C\ell^{\overline{j}}_{p,q} := \bigoplus_{k=j \bmod 4} C\ell^k_{p,q} = \{U \in C\ell_{p,q} \mid \widehat{U} = (-1)^j U, \widetilde{U} = (-1)^{\frac{j(j-1)}{2}} U\}, \quad j = 0, 1, 2, 3.$$

$\overline{\mathbf{j}} := C\ell^{\overline{j}}_{p,q}$ is called subspace of quaternion type $j = 0, 1, 2, 3$.

$$C\ell_{p,q} = \overline{\mathbf{0}} \oplus \overline{\mathbf{1}} \oplus \overline{\mathbf{2}} \oplus \overline{\mathbf{3}}, \qquad \mathbb{C} \otimes C\ell_{p,q} = \overline{\mathbf{0}} \oplus \overline{\mathbf{1}} \oplus \overline{\mathbf{2}} \oplus \overline{\mathbf{3}} \oplus i\overline{\mathbf{0}} \oplus i\overline{\mathbf{1}} \oplus i\overline{\mathbf{2}} \oplus i\overline{\mathbf{3}}.$$

$C\ell_{p,q}$	$\overline{\mathbf{0}}$	$\overline{\mathbf{1}}$	$\overline{\mathbf{2}}$	$\overline{\mathbf{3}}$
$\widehat{U} = ?U$	+	-	+	-
$\widetilde{U} = ?U$	+	+	-	-

$\mathbb{C} \otimes C\ell_{p,q}$	$\overline{\mathbf{0}}$	$\overline{\mathbf{1}}$	$\overline{\mathbf{2}}$	$\overline{\mathbf{3}}$	$i\overline{\mathbf{0}}$	$i\overline{\mathbf{1}}$	$i\overline{\mathbf{2}}$	$i\overline{\mathbf{3}}$
$\widehat{U} = ?U$	+	-	+	-	+	-	+	-
$\widetilde{U} = ?U$	+	+	-	-	+	+	-	-
$\bar{U} = ?U$	+	+	+	+	-	-	-	-

$$\dim \overline{\mathbf{0}} = 2^{n-2} + 2^{\frac{n-2}{2}} \cos \frac{\pi n}{4}, \qquad \dim \overline{\mathbf{1}} = 2^{n-2} + 2^{\frac{n-2}{2}} \sin \frac{\pi n}{4},$$

$$\dim \overline{\mathbf{2}} = 2^{n-2} - 2^{\frac{n-2}{2}} \cos \frac{\pi n}{4}, \qquad \dim \overline{\mathbf{3}} = 2^{n-2} - 2^{\frac{n-2}{2}} \sin \frac{\pi n}{4}.$$

Commutator $[U, V] := UV - VU$, anticommutator $\{U, V\} := UV + VU$.

Theorem

We have the following properties:

$$[\bar{\boldsymbol{j}}, \bar{\boldsymbol{j}}] \subset \bar{\boldsymbol{2}}, \quad [\bar{\boldsymbol{j}}, \bar{\boldsymbol{2}}] \subset \bar{\boldsymbol{j}}, \quad j = 0, 1, 2, 3, \quad [\bar{\boldsymbol{0}}, \bar{\boldsymbol{1}}] \subset \bar{\boldsymbol{3}}, \quad [\bar{\boldsymbol{0}}, \bar{\boldsymbol{3}}] \subset \bar{\boldsymbol{1}}, \quad [\bar{\boldsymbol{1}}, \bar{\boldsymbol{3}}] \subset \bar{\boldsymbol{0}},$$
$$\{\bar{\boldsymbol{j}}, \bar{\boldsymbol{j}}\} \subset \bar{\boldsymbol{0}}, \quad \{\bar{\boldsymbol{j}}, \bar{\boldsymbol{0}}\} \subset \bar{\boldsymbol{j}}, \quad j = 0, 1, 2, 3, \quad \{\bar{\boldsymbol{1}}, \bar{\boldsymbol{2}}\} \subset \bar{\boldsymbol{3}}, \quad \{\bar{\boldsymbol{2}}, \bar{\boldsymbol{3}}\} \subset \bar{\boldsymbol{1}}, \quad \{\bar{\boldsymbol{3}}, \bar{\boldsymbol{1}}\} \subset \bar{\boldsymbol{2}}.$$

- D.Sh., "Classification of elements of Clifford algebras according to quaternionic types", Dokl. Math., 80:1, (2009), 610–612
- D.Sh., "Quaternion types of Clifford algebra elements, basis-free approach", Proceedings of ICCA9 (Weimar, 2011), arXiv: 1109.2322
- D.Sh., "Quaternion typification of Clifford algebra elements", Adv. Appl. Clifford Algebr., 22:1, (2012), 243–256 , arXiv: 0806.4299
- D.Sh., "Development of the method of quaternion typification of Clifford algebra elements", Adv. Appl. Clifford Algebr., 22:2, (2012), 483–497, arXiv: 0903.3494

Applications: see Lecture 4.

Theorem

The *center of Clifford algebra* $\mathrm{Cen}(C\ell_{p,q}) := \{U \in C\ell_{p,q} \mid UV = VU,\ \forall V \in C\ell_{p,q}\}$ *is*

$$\mathrm{Cen}(C\ell_{p,q}) = \begin{cases} C\ell^0_{p,q} = \{ue \mid u \in \mathbb{R}\}, & \text{if } n \text{ is even;} \\ C\ell^0_{p,q} \oplus C\ell^n_{p,q} = \{ue + u_{1\ldots n}e_{1\ldots n} \mid u, u_{1\ldots n} \in \mathbb{R}\}, & \text{if } n \text{ is odd.} \end{cases}$$

Proof $U = U^{(0)} + U^{(1)},\ U^{(i)} \in C\ell^{(i)}_{p,q},\ i = 0, 1;$

$$UV = VU,\ \forall V \in C\ell_{p,q} \Leftrightarrow U^{(i)}e_k = e_k U^{(i)},\ k = 1, \ldots, n,\ i = 0, 1.$$

- We represent $U^{(0)}$ in the form $U^{(0)} = A^{(0)} + e_1 B^{(1)}$, where $A^{(0)} \in C\ell^{(0)}_{p,q}$ and $B^{(1)} \in C\ell^{(1)}_{p,q}$ do not contain e_1. For $k = 1$ we obtain
$$(A^{(0)} + e_1 B^{(1)})e_1 = e_1(A^{(0)} + e_1 B^{(1)}).$$
Using $A^{(0)}e_1 = e_1 A^{(0)}$ and $e_1 B^{(1)} e_1 = -e_1 e_1 B^{(1)}$, we obtain $B^{(1)} = 0$. Acting similarly for $e_2, \ldots, e_n$, we obtain $U^{(0)} = ue$.
- We represent $U^{(1)}$ in the form $U^{(1)} = A^{(1)} + e_1 B^{(0)}$, where $A^{(1)} \in C\ell^{(1)}_{p,q}$ and $B^{(0)} \in C\ell^{(0)}_{p,q}$ do not contain e_1. For $k = 1$ we obtain
$$(A^{(1)} + e_1 B^{(0)})e_1 = e_1(A^{(1)} + e_1 B^{(0)}).$$
Using $A^{(1)}e_1 = -e_1 A^{(1)}$ and $e_1 B^{(0)} e_1 = e_1 e_1 B^{(0)}$, we obtain $A^{(1)} = 0$. Acting similarly for $e_2, \ldots, e_k$, we obtain $U^{(1)} = u_{1\ldots n}e_{1\ldots n}$ in the case of odd n and $U^{(1)} = 0$ in the case of even n. ■

Lecture 2

Unitary Spaces on Clifford Algebras

Hermitian scalar product in Clifford algebras. Operation of Hermitian conjugation and unitary groups in Clifford algebras.

Trace of Clifford algebra element (projection onto $\mathbb{C}\otimes C\!\ell^0_{p,q}$ with $e\to 1$)

$$\mathrm{Tr}(U):=u,\qquad U=ue+\sum_a u_a e_a+\cdots+u_{1\ldots n}e_{1\ldots n}\in\mathbb{C}\otimes C\!\ell_{p,q}.$$

Properties:

$$\mathrm{Tr}(U+V)=\mathrm{Tr}(U)+\mathrm{Tr}(V),\quad \mathrm{Tr}(\lambda U)=\lambda\mathrm{Tr}(U),\quad \mathrm{Tr}(UV)=\mathrm{Tr}(VU),$$
$$\mathrm{Tr}(UVW)=\mathrm{Tr}(VWU)=\mathrm{Tr}(WUV),\quad \forall U,V,W\in\mathbb{C}\otimes C\!\ell_{p,q},\forall\lambda\in\mathbb{C},$$
$$\mathrm{Tr}(U^{-1}VU)=\mathrm{Tr}(V),\quad \mathrm{Tr}(U)=\mathrm{Tr}(\hat{U})=\mathrm{Tr}(\tilde{U})=\overline{\mathrm{Tr}\bar{U}}.$$

Theorem

$$\mathrm{Tr}(U)=\frac{1}{2^{[\frac{n+1}{2}]}}\mathrm{tr}(\gamma(U)),$$

where

$$\gamma:\mathbb{C}\otimes C\!\ell_{p,q}\to\begin{cases}\mathrm{Mat}(2^{\frac{n}{2}},\mathbb{C}), & \text{if } n \text{ is even;}\\ \mathrm{Mat}(2^{\frac{n-1}{2}},\mathbb{C})\oplus\mathrm{Mat}(2^{\frac{n-1}{2}},\mathbb{C}), & \text{if } n \text{ is odd.}\end{cases}$$

is faithful matrix representation of $\mathbb{C}\otimes C\!\ell_{p,q}$ *(of minimal dimension).*

D. Sh., Concepts of trace, determinant and inverse of Clifford algebra elements, Proceedings of the 8th congress ISAAC (2011), arXiv: 1108.5447

Theorem

The operation $U, V \in \mathbb{C}\otimes C\!\ell_n \to (U,V) := \mathrm{Tr}(\bar{\tilde{U}} V)$ *is Hermitian (or Euclidian) scalar product on* $\mathbb{C}\otimes C\!\ell_n$ *(or* $C\!\ell_n$ *respectively).* $(q=0)$

Proof We must verify

$$(U,V) = \overline{(V,U)}, \quad (U,\lambda V) = \lambda (U,V), \quad (U, V+W) = (U,V) + (U,W),$$
$$(U,U) \geq 0, \qquad (U,U) = 0 \Leftrightarrow U = 0 \tag{1}$$

for all $U, V, W \in \mathbb{C}\otimes C\!\ell_{p,q}$, $\lambda \in \mathbb{C}$. To prove (1) it is sufficient to prove that basis of $\mathbb{C}\otimes C\!\ell_{p,q}$ is orthonormal:

$$(e_{i_1\ldots i_k}, e_{j_1\ldots j_l}) = \mathrm{Tr}(e_{i_k}\cdots e_{i_1} e_{j_1}\cdots e_{j_l}) = \begin{cases} 1, & \text{if } (i_1,\ldots,i_k) = (j_1,\ldots,j_l); \\ 0, & \text{if } (i_1,\ldots,i_k) \neq (j_1,\ldots,j_l). \end{cases}$$

We have

$$(U,U) = \sum_A u_A \overline{u_A} = \sum_A |u_A|^2 \geq 0. \qquad \blacksquare$$

Hermitian conjugation of Clifford algebra elements:

$$U^\dagger := U|_{e_{a_1 \dots a_k} \to e^{-1}_{a_1 \dots a_k},\, u_{a_1 \dots a_k} \to \bar{u}_{a_1 \dots a_k}}, \qquad U \in \mathbb{C} \otimes C\!\ell_{p,q}.$$

Properties:

$$U^{\dagger\dagger} = U, \qquad (UV)^\dagger = V^\dagger U^\dagger, \qquad (\lambda U + \mu V)^\dagger = \bar{\lambda} U^\dagger + \bar{\mu} V^\dagger,$$
$$\forall U, V \in \mathbb{C} \otimes C\!\ell_{p,q}, \qquad \forall \lambda, \mu \in \mathbb{C}.$$

Theorem

The operation $U, V \in \mathbb{C} \otimes C\!\ell_n \to (U, V) := \mathrm{Tr}(U^\dagger V)$ is Hermitian (or Euclidian) scalar product on $\mathbb{C} \otimes C\!\ell_{p,q}$ (or $C\!\ell_{p,q}$ respectively).

Proof ... $(e_{i_1} \cdots e_{i_k}, e_{i_1} \cdots e_{i_k}) = \mathrm{Tr}(e^{-1}_{i_k} \dots e^{-1}_{i_1} e_{i_1} \cdots e_{i_k}) = \mathrm{Tr}(e) = 1.$ ■

N. Marchuk, D. Sh., Unitary spaces on Clifford algebras, Adv. Appl. Clifford Algebr., 18:2 (2008), 237–254, arXiv: 0705.1641

Real case: the transposition anti-involution in $C\!\ell_{p,q}$.

R. Ablamowicz, B. Fauser, On the transposition anti-involution in real Clifford algebras I, II, III; Linear and Multilinear Algebra (2011, 2011, 2012).

Theorem

We have $\gamma(U^\dagger)=(\gamma(U))^\dagger$, *where*

$$\gamma:\mathbb{C}\otimes C\!\ell_{p,q}\to\begin{cases}\mathrm{Mat}(2^{\frac{n}{2}},\mathbb{C}), & \text{if } n \text{ is even;}\\ \mathrm{Mat}(2^{\frac{n-1}{2}},\mathbb{C})\oplus\mathrm{Mat}(2^{\frac{n-1}{2}},\mathbb{C}), & \text{if } n \text{ is odd.}\end{cases}$$

is faithful matrix representation of $\mathbb{C}\otimes C\!\ell_{p,q}$ *such that (not arbitrary!)* $(\gamma(e_a))^{-1}=(\gamma(e_a))^\dagger$.

Unitary group in Clifford algebra (Lie group):

$$\mathrm{U}C\!\ell_{p,q}:=\{U\in\mathbb{C}\otimes C\!\ell_{p,q}\,|\,U^\dagger U=e\}\cong\begin{cases}\mathrm{U}(2^{\frac{n}{2}}), & \text{if } n \text{ is even;}\\ \mathrm{U}(2^{\frac{n-1}{2}})\oplus\mathrm{U}(2^{\frac{n-1}{2}}), & \text{if } n \text{ is odd.}\end{cases}$$

Example: all basis elements $e_{a_1\ldots a_k}\in\mathrm{U}C\!\ell_{p,q}$.
Unitary Lie algebra in Clifford algebra $\mathfrak{u}C\!\ell_{p,q}:=\{U\in\mathbb{C}\otimes C\!\ell_{p,q}\,|\,U^\dagger=-U\}$.

Theorem

We have the following formulas:

$$U^\dagger = \begin{cases} (e_{1\ldots p})^{-1}\overline{\widetilde{U}}e_{1\ldots p}, & \text{if } p \text{ is odd;} \\ (e_{1\ldots p})^{-1}\overline{\widehat{\widetilde{U}}}e_{1\ldots p}, & \text{if } p \text{ is even.} \end{cases} \qquad U^\dagger = \begin{cases} (e_{p+1\ldots n})^{-1}\overline{\widetilde{U}}e_{p+1\ldots n}, & \text{if } q \text{ is even;} \\ (e_{p+1\ldots n})^{-1}\overline{\widehat{\widetilde{U}}}e_{p+1\ldots n}, & \text{if } q \text{ is odd.} \end{cases}$$

Example: $\gamma_a^\dagger = \gamma_0\gamma_a\gamma_0$ for Dirac gamma-matrices.

Proof It is sufficient to prove the following formulas:

$$e_{i_1\ldots i_k}{}^\dagger = (-1)^{(p+1)k}e_{1\ldots p}^{-1}\widetilde{e_{i_1\ldots i_k}}e_{1\ldots p}, \quad e_{i_1\ldots i_k}{}^\dagger = (-1)^{qk}e_{p+1\ldots n}^{-1}\widetilde{e_{i_1\ldots i_k}}e_{p+1\ldots n}$$

Let s be the number of common indices of $\{i_1,\ldots,i_k\}$ and $\{1,\ldots,p\}$. Then

$$(-1)^{(p+1)k}e_{1\ldots p}^{-1}\widetilde{e_{i_1\ldots i_k}}e_{1\ldots p} = (-1)^{(p+1)k}e_p\cdots e_1 e_{i_k}\cdots e_{i_1}e_1\cdots e_p =$$
$$= (-1)^{(p+1)k}(-1)^{kp-s}e_{i_k}\cdots e_{i_1} = (-1)^{k-s}e_{i_k}\cdots e_{i_1} = e_{i_1\ldots i_k}^{-1}.$$
$$(-1)^{qk}e_{p+1\ldots n}^{-1}\widetilde{e_{i_1\ldots i_k}}e_{p+1\ldots n} = (-1)^{qk}(-1)^q e_n\cdots e_{p+1}e_{i_k}\cdots e_{i_1}e_{p+1}\cdots e_n =$$
$$= (-1)^{qk+q}(-1)^{qk-(k-s)}(-1)^q e_{i_k}\cdots e_{i_1} = (-1)^{k-s}e_{i_k}\cdots e_{i_1} = e_{i_1\ldots i_k}^{-1}. \qquad \blacksquare$$

Lecture 3

Matrix Representations of Clifford Algebras

Cartan's periodicity of 8 for Clifford algebras. Faithful and irreducible representations. Primitive idempotents and minimal left ideals.

Cartan's periodicity of 8.

Theorem (Cartan 1908)

We have the following isomorphism of algebras

$$\mathcal{C}\!\ell_{p,q} \cong \begin{cases} \mathrm{Mat}(2^{\frac{n}{2}}, \mathbb{R}), & \text{if } p-q \equiv 0;2 \mod 8; \\ \mathrm{Mat}(2^{\frac{n-1}{2}}, \mathbb{R}) \oplus \mathrm{Mat}(2^{\frac{n-1}{2}}, \mathbb{R}), & \text{if } p-q \equiv 1 \mod 8; \\ \mathrm{Mat}(2^{\frac{n-1}{2}}, \mathbb{C}), & \text{if } p-q \equiv 3;7 \mod 8; \\ \mathrm{Mat}(2^{\frac{n-2}{2}}, \mathbb{H}), & \text{if } p-q \equiv 4;6 \mod 8; \\ \mathrm{Mat}(2^{\frac{n-3}{2}}, \mathbb{H}) \oplus \mathrm{Mat}(2^{\frac{n-3}{2}}, \mathbb{H}), & \text{if } p-q \equiv 5 \mod 8. \end{cases}$$

Proof see next slides.

Theorem

We have the following isomorphism of algebras

$$\mathcal{C}\!\ell(\mathbb{C}^n) \cong \begin{cases} \mathrm{Mat}(2^{\frac{n}{2}}, \mathbb{C}), & \text{if } n \text{ is even}; \\ \mathrm{Mat}(2^{\frac{n-1}{2}}, \mathbb{C}) \oplus \mathrm{Mat}(2^{\frac{n-1}{2}}, \mathbb{C}), & \text{if } n \text{ is odd}. \end{cases}$$

Lemma

We have the following isomorphisms of associative algebras:

$$1)\, C\!\ell_{p+1,q+1} \cong \mathrm{Mat}(2, C\!\ell_{p,q}), \qquad 2)\, C\!\ell_{p+1,q+1} \cong C\!\ell_{p,q} \otimes C\!\ell_{1,1},$$
$$3)\, C\!\ell_{p,q} \cong C\!\ell_{q+1,p-1}, \quad p \geq 1, \qquad 4)\, C\!\ell_{p,q} \cong C\!\ell_{p-4,q+4}, \quad p \geq 4.$$

Proof Let $e_1, \ldots, e_n$ be generators of $C\!\ell_{p,q}$ and $(e_+)^2 = e$, $(e_-)^2 = -e$ (all generators anticommute).

1. We obtain generators of $\mathrm{Mat}(2, C\!\ell_{p,q})$ in the following way:

$$e_i \to \begin{pmatrix} e_i & 0 \\ 0 & -e_i \end{pmatrix}, \quad i = 1, \ldots, n, \quad e_+ \to \begin{pmatrix} 0 & e \\ e & 0 \end{pmatrix}, \quad e_- \to \begin{pmatrix} 0 & -e \\ e & 0 \end{pmatrix}.$$

2. $e_i e_+ e_-$, $i = 1, \ldots, n$ are generators of $C\!\ell_{p,q}$ and e_+, e_- are generators of $C\!\ell_{1,1}$. Each generator of $C\!\ell_{p,q}$ commutes with each generator of $C\!\ell_{1,1}$.
3. e_1, $e_i e_1$, $i = 2, \ldots, n$ are generators of $C\!\ell_{q+1,p-1}$.
4. $e_i e_1 e_2 e_3 e_4$, $i = 1, 2, 3, 4$ and e_j, $j = 5, \ldots, n$ are generators of $C\!\ell_{p-4,q+4}$. ■

Lounesto P., Clifford Algebras and Spinors, Cambridge Univ. Press (1997).

The table of Clifford algebras

Notations: ${}^2\mathbb{R} := \mathbb{R} \oplus \mathbb{R}$, $\mathbb{R}(2) := \mathrm{Mat}(2, \mathbb{R}), \ldots$

$n \backslash p-q$	-5	-4	-3	-2	-1	0	1	2	3	4	5
0	–	–	–	–	–	$\mathbb{R}$	–	–	–	–	–
1	–	–	–	–	$\mathbb{C}$	–	${}^2\mathbb{R}$	–	–	–	–
2	–	–	–	$\mathbb{H}$	–	$\mathbb{R}(2)$	–	$\mathbb{R}(2)$	–	–	–
3	–	–	${}^2\mathbb{H}$	–	$\mathbb{C}(2)$	–	${}^2\mathbb{R}(2)$	–	$\mathbb{C}(2)$	–	–
4	–	$\mathbb{H}(2)$	–	$\mathbb{H}(2)$	–	$\mathbb{R}(4)$	–	$\mathbb{R}(4)$	–	$\mathbb{H}(2)$	–
5	$\mathbb{C}(4)$	–	${}^2\mathbb{H}(2)$	–	$\mathbb{C}(4)$	–	${}^2\mathbb{R}(4)$	–	$\mathbb{C}(4)$	–	${}^2\mathbb{H}(2)$

- $C\ell_{0,0} \cong \mathbb{R}$, $C\ell_{0,1} \cong \mathbb{C}$, $C\ell_{1,0} \cong \mathbb{R} \oplus \mathbb{R}$, $C\ell_{0,2} \cong \mathbb{H}$, (Lecture 1)
  $C\ell_{0,3} \cong \mathbb{H} \oplus \mathbb{H}$: $e \to (1,1)$, $e_1 \to (i, -i)$, $e_2 \to (j, -j)$, $e_3 \to (k, -k)$.
- Lemma ($C\ell_{p+1,q+1} \cong \mathrm{Mat}(2, C\ell_{p,q})$) $\Rightarrow C\ell_{1,1} \cong \mathrm{Mat}(2, \mathbb{R})$;
- Lemma ($C\ell_{p+1,q+1} \cong C\ell_{p,q} \otimes C\ell_{1,1}$): $n \to n+2 \Rightarrow \mathrm{Mat}(k, \ldots) \to \mathrm{Mat}(2k, \ldots)$;
- Lemma ($C\ell_{p,q} \cong C\ell_{q+1,p-1}$) $\Rightarrow$ symmetry w.r.t. the column"$p - q = 1$";
- Lemma ($C\ell_{p,q} \cong C\ell_{p-4,q+4}$) $\Rightarrow$ symmetry $p - q \leftrightarrow p - q - 8$.

Even subalgebras

Let us remind:

$$\mathcal{C}\ell^{(0)}_{p,q} := \bigoplus_{k=0 \bmod 2} \mathcal{C}\ell^{k}_{p,q} = \{U \in \mathcal{C}\ell_{p,q} \mid \widehat{U} = U\};$$

Theorem

We have the following isomorphism of algebras

$$1)\mathcal{C}\ell^{(0)}_{p,q} \cong \mathcal{C}\ell_{p,q-1}, \quad q \geq 1; \qquad 2)\mathcal{C}\ell^{(0)}_{p,q} \cong \mathcal{C}\ell_{q,p-1}, \quad p \geq 1; \qquad 3)\mathcal{C}\ell^{(0)}_{p,q} \cong \mathcal{C}\ell^{(0)}_{q,p}.$$

Proof Let $e_1, \ldots, e_n$ be generators of $\mathcal{C}\ell_{p,q}$.

1. Then $e_i e_n$, $i = 1, \ldots, n-1$ are generators of $\mathcal{C}\ell^{(0)}_{p,q}$.
2. Then $\begin{cases} e_{p+i} e_p, & i = 1, \ldots, q; \\ e_{j-q} e_p, & j = q+1, \ldots, n-1, \end{cases}$ are generators of $\mathcal{C}\ell^{(0)}_{p,q}$.
3. 1), 2) $\Rightarrow$ 3). ∎

An algebra is simple if it contains no non-trivial two-sided ideals and the multiplication operation is not zero.
A central simple algebra over a field $\mathbb{F}$ is a finite-dimensional associative algebra, which is simple, and for which the center is exactly $\mathbb{F}$.

- If n is even, then $C\ell(V, Q)$ is a central simple algebra.
- If n is odd and $\mathbb{F} = \mathbb{C}$, then $C\ell(V, Q)$ is a direct sum of two isomorphic complex central simple algebras.
- If n is odd, $\mathbb{F} = \mathbb{R}$, and $(e_{1\ldots n})^2 = e$, then $C\ell(V, Q)$ is a direct sum of two isomorphic simple algebras.
- If n is odd, $\mathbb{F} = \mathbb{R}$, and $(e_{1\ldots n})^2 = -e$, then $C\ell(V, Q)$ is simple with center $\cong \mathbb{C}$.

$$(e_{1\ldots n})^2 = (-1)^{q+\frac{n(n-1)}{2}} e = \begin{cases} e, & \text{if } p-q = 0, 1 \mod 4; \\ -e, & \text{if } p-q = 2, 3 \mod 4. \end{cases}$$

Chevalley C., The algebraic theory of Spinors and Clifford algebras, Springer (1996).

Primitive idempotents and minimal left ideals

- $t \in \mathbb{C} \otimes \mathcal{C}\ell_{p,q}$ is said to be Hermitian idempotent if $t^2 = t$, $t^\dagger = t$.
- $\mathrm{I}(t) = \{U \in \mathbb{C} \otimes \mathcal{C}\ell_{p,q} \mid U = Ut\}$ is the left ideal generated by t.
- A left ideal that does not contain other left ideals except itself and the trivial ideal (generated by $t = 0$), is called a minimal left ideal. The corresponding idempotent is called primitive.
- If $V \in \mathrm{I}(t)$ and $U \in \mathbb{C} \otimes \mathcal{C}\ell_{p,q}$, then $UV \in \mathrm{I}(t)$.
- The left ideal $\mathrm{I}(t)$ is a complex vector space with orthonormal basis $\tau_1, \ldots, \tau_d$, $d := \dim \mathrm{I}(t)$. We have Hermitian scalar product $(U, V) = \mathrm{Tr}(U^\dagger V)$ on $\mathrm{I}(t)$, $\tau_k = \tau^k$, $(\tau_k, \tau^l) = \delta^l_k$, $k, l = 1, \ldots, n$.
- We may define linear map $\gamma : \mathbb{C} \otimes \mathcal{C}\ell_{p,q} \to \mathrm{Mat}(d, \mathbb{C})$ as

$$U\tau_k = \gamma(U)^l_k \tau_l, \qquad \gamma(U) = ||\gamma(U)^l_k|| \in \mathrm{Mat}(d, \mathbb{C}). \tag{3}$$

  We have $\gamma(U)^k_l = (\tau^k, U\tau_l)$.
- Linear map γ is representation of Clifford algebra of dimension d: $\gamma(UV) = \gamma(U)\gamma(V)$. **Proof:**

$$\gamma(UV)^m_k \tau_m = (UV)\tau_k = U(V\tau_k) = U\tau_l \gamma(V)^l_k = \gamma(U)^m_l \gamma(V)^l_k \tau_m.$$

- We have $\gamma(U^\dagger)=(\gamma(U))^\dagger$.
  Proof: Using $(A,UB)=(AU^\dagger,B)$ and $(A,B)=\overline{(B,A)}$ for $(A,B)=\mathrm{Tr}(A^\dagger B)$, we obtain $\gamma(U)^k_l=(U^\dagger\tau^k,\tau_l)$, $\overline{\gamma(U)}^k_l=(\tau_l,U^\dagger\tau^k)$. Transposing, we get $(\gamma(U)^k_l)^\dagger=(\tau^k,U^\dagger\tau_l)$, which coincides with $\gamma(U^\dagger)^k_l=(\tau^k,U^\dagger\tau_l)$.
- The choice of t and basis of $\mathrm{I}(t)$.

$$t=\frac{1}{2}(e+i^a e_1)\prod_{k=1}^{[n/2]-1}\frac{1}{2}(e+i^{b_k}e_{2k}e_{2k+1})\in\mathbb{C}\otimes C\!\ell_{p,q},\quad t^2=t^\dagger=t,$$

$$a=\begin{cases}0, & \text{if } p\neq 0;\\ 1, & \text{if } p=0.\end{cases}\qquad b_k=\begin{cases}0, & 2k=p;\\ 1, & 2k\neq p.\end{cases}$$

Details:

N. Marchuk, D. Sh., Unitary spaces on Clifford algebras, Adv. Appl. Clifford Algebr., 18:2 (2008), 237–254, arXiv: 0705.1641

R. Abłamowicz, Spinor representations of Clifford algebras: A symbolic approach, Computer Physics Communications, 115(11), 1998. (Real case)

Pauli's fundamental theorem

Theorem (Pauli)

Consider 2 sets of square complex matrices

$$\gamma_a, \qquad \beta_a, \qquad a = 1, 2, 3, 4.$$

of size 4. *Let these 2 sets satisfy the following conditions*

$$\begin{aligned} \gamma_a\gamma_b + \gamma_b\gamma_a &= 2\eta_{ab}\mathbf{1}, \qquad \eta = \mathrm{diag}(1, -1, -1, -1), \\ \beta_a\beta_b + \beta_b\beta_a &= 2\eta_{ab}\mathbf{1}. \end{aligned}$$

Then there exists a unique (up to multiplication by a complex constant) complex matrix T such that

$$\gamma_a = T^{-1}\beta_a T, \qquad a = 1, 2, 3, 4.$$

W.Pauli, *Contributions mathematiques a la theorie des matrices de Dirac*, Ann. Inst. Henri Poincare 6, (1936).

Faithful and irreducible representations

- in the case of even $n = p + q$ $\mathbb{C} \otimes \mathcal{C}\ell_{p,q}$ has 1 faithful and irreducible representation of dimension $2^{\frac{n}{2}}$; $(\mathbb{C} \otimes \mathcal{C}\ell_{p,q} \cong \mathrm{Mat}(2^{\frac{n}{2}}, \mathbb{C})$, n is even)
- in the case of odd $n = p + q$ $\mathbb{C} \otimes \mathcal{C}\ell_{p,q}$ has 2 irreducible representations of dimension $2^{\frac{n-1}{2}}$;
- in the case of odd $n = p + q$ $\mathbb{C} \otimes \mathcal{C}\ell_{p,q}$ has 2 faithful reducible representation of dimension $2^{\frac{n-1}{2}} + 2^{\frac{n-1}{2}} = 2^{\frac{n+1}{2}}$.
  $(\mathbb{C} \otimes \mathcal{C}\ell_{p,q} \cong \mathrm{Mat}(2^{\frac{n-1}{2}}, \mathbb{C}) \oplus \mathrm{Mat}(2^{\frac{n-1}{2}}, \mathbb{C})$, n is odd)
- Similarly for the real Clifford algebra $\mathcal{C}\ell_{p,q}$ (results depend on $n \mod 2$ and $p - q \mod 8$): see next slides

Generalization of Pauli's theorem

Let the set of Clifford algebra elements satisfies the conditions

$$\beta_a \in C\!\ell_{p,q}, \qquad \beta_a\beta_b + \beta_b\beta_a = 2\eta_{ab}e.$$

Then the set

$$\gamma_a = T^{-1}\beta_a T, \qquad \forall\,\text{invertible } T \in C\!\ell_{p,q}$$

satisfies the conditions

$$\gamma_a\gamma_b + \gamma_b\gamma_a = 2\eta_{ab}e.$$

Really,

$$\gamma_a\gamma_b + \gamma_b\gamma_a = T^{-1}\beta_a TT^{-1}\beta_b T + T^{-1}\beta_b TT^{-1}\beta_a T =$$
$$= T^{-1}(\beta_a\beta_b + \beta_b\beta_a)T = T^{-1}2\eta_{ab}eT = 2\eta_{ab}e.$$

Our question: if we have γ_a and $\beta_a \Rightarrow \exists T$? algorithm to compute T?

D. Sh., Extension of Pauli's theorem to Clifford algebras, Dokl. Math., 84(2), 2011.

D. Sh., Pauli theorem in the description of n-dimensional spinors in the Clifford algebra formalism,Theoret. and Math. Phys., 175:1, 2013.

D. Sh., Calculations of elements of spin groups using generalized Pauli's theorem, AACA, 25(1), 2015; arXiv:1409.2449

Theorem (The case of even n)

Consider real $C\ell_{p,q}$ (or complexified $\mathbb{C}\otimes C\ell_{p,q}$) Clifford algebra with even $n=p+q$. Let the following 2 sets of Clifford algebra elements γ_a, β_a, $a=1,2,\ldots,n$ satisfy conditions

$$\gamma_a\gamma_b+\gamma_b\gamma_a=2\eta_{ab}e,\quad \beta_a\beta_b+\beta_b\beta_a=2\eta_{ab}e.$$

Then both sets of elements generate bases of Clifford algebra and there exists a unique (up to multiplication by a real (complex) constant) Clifford algebra element T such that

$$\gamma_a=T^{-1}\beta_a T,\qquad \forall a=1,\ldots,n.$$

Moreover, we can always find this element T in the form

$$T=\sum_A \beta_A F(\gamma_A)^{-1}$$

where F is any element of a set

$$1)\ \{\gamma_A, |A| \text{is even}\}\quad \text{if}\, \beta_{1\ldots n}\neq-\gamma_{1\ldots n};\qquad 2)\ \{\gamma_A, |A| \text{is odd}\}\quad \text{if}\, \beta_{1\ldots n}\neq\gamma_{1\ldots n}$$

such that corresponding T is nonzero $T\neq 0$.

The case of even n in matrix formalism

Theorem

Let n is even and 2 sets of square matrices $\gamma_a, \beta_a, a = 1, 2, \ldots, n$ satisfy conditions

$$\gamma_a\gamma_b + \gamma_b\gamma_a = 2\eta_{ab}\mathbf{1}, \qquad \beta_a\beta_b + \beta_b\beta_a = 2\eta_{ab}\mathbf{1}.$$

- *If matrices are complex of the order $2^{\frac{n}{2}}$, then there exists a unique (up to a complex constant) matrix T such that*
- *If signature is $p - q \equiv 0, 2 \mod 8$ and matrices are real of the order $2^{\frac{n}{2}}$, then there exists a unique (up to a real constant) matrix T such that*
- *If signature is $p - q \equiv 4, 6 \mod 8$ and matrices are over the quaternions of the order $2^{\frac{n-2}{2}}$, then there exists a unique (up to a real constant) matrix T such that*

$$\gamma_a = T^{-1}\beta_a T, \qquad a = 1, \ldots n.$$

The case of odd n

Example 1: $C\ell_{2,1} \simeq \mathrm{Mat}(2,\mathbb{R}) \oplus \mathrm{Mat}(2,\mathbb{R})$ with generators e_1, e_2, e_3. We can take

$$\gamma_1 = e_1, \qquad \gamma_2 = e_2, \qquad \gamma_3 = e_1 e_2.$$

Then $\gamma_a\gamma_b + \gamma_b\gamma_a = 2\eta_{ab}\mathbf{1}$. Elements $\gamma_1, \gamma_2, \gamma_3$ generate not $C\ell_{2,1}$, but generate $C\ell_{2,0} \simeq \mathrm{Mat}(2,\mathbb{R})$.

Example 2: $C\ell_{3,0} \simeq \mathrm{Mat}(2,\mathbb{C})$ with generators e_1, e_2, e_3.

$$\beta_1 = \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \beta_2 = \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \beta_3 = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

$$\gamma_a = -\sigma_a, \qquad a = 1, 2, 3.$$

Then $\gamma_{123} = -\beta_{123}$. Suppose, that we have $T \in \mathrm{GL}(2,\mathbb{C})$ such that $\gamma_a = T^{-1}\beta_a T$. Then

$$\gamma_{123} = T^{-1}\beta_1 T T^{-1}\beta_2 T T^{-1}\beta_3 T = T^{-1}\beta_1\beta_2\beta_3 T = \beta_{123}$$

and we obtain a contradiction (we use that $\beta_{123} = \sigma_{123} = i\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = i\mathbf{1}$).

But we have element $T = \mathbf{1}$ such that $\gamma_a = -T^{-1}\beta_a T$.

Theorem (The case of odd n)

Consider real $C\ell_{p,q}$ (or complexified $\mathbb{C}\otimes C\ell_{p,q}$) Clifford algebra with odd $n=p+q$. Let the following 2 sets of Clifford algebra elements γ_a, β_a, $a=1,2,\ldots,n$ satisfy conditions

$$\gamma_a\gamma_b+\gamma_b\gamma_a=2\eta_{ab}e,\quad \beta_a\beta_b+\beta_b\beta_a=2\eta_{ab}e.$$

Then in the case of Clifford algebra of signature $p-q\equiv 1 \mod 4$ elements $\gamma_{1\ldots n}$ and $\beta_{1\ldots n}$ equals $\pm e_{1\ldots n}$ and then corresponding sets generate bases of Clifford algebra or equals $\pm e$ and then corresponding sets don't generate bases.
In the case of Clifford algebra of signature $p-q\equiv 3 \mod 4$ elements $\gamma_{1\ldots n}$ and $\beta_{1\ldots n}$ equals $\pm e_{1\ldots n}$, and then corresponding sets generate bases of Clifford algebra or (only for $\mathbb{C}\otimes C\ell_{p,q}$) equals $\pm ie$ and then corresponding sets don't generate bases.

continue →

There exists a unique (up to a invertible element of Clifford algebra center) element T such that

$$
\begin{array}{llll}
1) & \gamma_a = T^{-1}\beta_a T, & \forall a = 1,\ldots,n & \Leftrightarrow \quad \beta_{1\ldots n} = \gamma_{1\ldots n},\\
2) & \gamma_a = -T^{-1}\beta_a T, & \forall a = 1,\ldots,n & \Leftrightarrow \quad \beta_{1\ldots n} = -\gamma_{1\ldots n},\\
3) & \gamma_a = e_{1\ldots n}T^{-1}\beta_a T, & \forall a = 1,\ldots,n & \Leftrightarrow \quad \beta_{1\ldots n} = e_{1\ldots n}\gamma_{1\ldots n},\\
4) & \gamma_a = -e_{1\ldots n}T^{-1}\beta_a T, & \forall a = 1,\ldots,n & \Leftrightarrow \quad \beta_{1\ldots n} = -e_{1\ldots n}\gamma_{1\ldots n},\\
5) & \gamma_a = ie_{1\ldots n}T^{-1}\beta_a T, & \forall a = 1,\ldots,n & \Leftrightarrow \quad \beta_{1\ldots n} = ie_{1\ldots n}\gamma_{1\ldots n},\\
6) & \gamma_a = -ie_{1\ldots n}T^{-1}\beta_a T, & \forall a = 1,\ldots,n & \Leftrightarrow \quad \beta_{1\ldots n} = -ie_{1\ldots n}\gamma_{1\ldots n}.
\end{array}
$$

Note, that all 6 cases can be written in the form $\gamma_a = (\beta_{1\ldots n}(\gamma_{1\ldots n})^{-1})T^{-1}\beta_a T$.
Moreover, in the case of real Clifford algebra $C\ell_{p,q}$ of signature $p-q \equiv 3 \bmod 4$ we can always find this element T in the form

$$\sum_{|A|\text{is even}} \beta_A F(\gamma_A)^{-1}, \tag{4}$$

where F is any element of the set $\{\gamma_A \,|\, |A| \text{is even}\}$ such that corresponding T is nonzero $T \neq 0$.
In the case of real Clifford algebra $C\ell_{p,q}$ of signature $p-q \equiv 1 \bmod 4$ and complexified Clifford algebra $\mathbb{C}\otimes C\ell_{p,q}$ we can always find this element T in the form (4), where F is one of the elements of the set $\{\gamma_A + \gamma_B \,|\, |A|, |B| \text{are even}\}$.

Lecture 4

Lie Groups and Lie Algebras in Clifford Algebras

Spin groups as subgroups of Clifford and Lipschitz groups. Double covers of the orthogonal groups. Cartan-Dieudonne theorem. Spin groups in small dimensions. Lie groups in Clifford algebras and corresponding Lie algebras.

Orthogonal groups

$$\mathrm{O}(p,q) := \{A \in \mathrm{Mat}(n,\mathbb{R}) \mid A^{\mathrm{T}}\eta A = \eta\},\ p+q=n,\ \eta = \mathrm{diag}(\overbrace{1,\ldots,1}^{p},\overbrace{-1,\ldots,-1}^{q}),$$

$$A \in \mathrm{O}(p,q) \Rightarrow \det A = \pm 1,\ |A^{1\ldots p}_{1\ldots p}| \geq 1,\ |A^{p+1\ldots,n}_{p+1\ldots n}| \geq 1,\ A^{1\ldots p}_{1\ldots p} = \frac{A^{p+1\ldots n}_{p+1\ldots n}}{\det A},$$

$$\mathrm{SO}(p,q) := \{A \in \mathrm{O}(p,q) \mid \det A = 1\}, \quad \mathrm{SO}_{\uparrow\downarrow}(p,q) := \{A \in \mathrm{SO}(p,q) \mid A^{1\ldots p}_{1\ldots p} \geq 1\}$$

$$= \{A \in \mathrm{SO}(p,q) \mid A^{p+1\ldots n}_{p+1\ldots n} \geq 1\} = \{A \in \mathrm{O}(p,q) \mid A^{1\ldots p}_{1\ldots p} \geq 1, A^{p+1\ldots n}_{p+1\ldots n} \geq 1\},$$

$$\mathrm{O}_{\uparrow}(p,q) := \{A \in \mathrm{O}(p,q) \mid A^{1\ldots p}_{1\ldots p} \geq 1\}, \quad \mathrm{O}_{\downarrow}(p,q) := \{A \in \mathrm{O}(p,q) \mid A^{p+1\ldots n}_{p+1\ldots n} \geq 1\},$$

$$\mathrm{O}(p,q) = \mathrm{SO}_{\uparrow\downarrow}(p,q) \sqcup \mathrm{O}_{\uparrow}(p,q)' \sqcup \mathrm{O}_{\downarrow}(p,q)' \sqcup \mathrm{SO}(p,q)', \text{(4 connected components)}$$

$$\mathrm{O}_{\uparrow}(p,q) = \mathrm{SO}_{\uparrow\downarrow}(p,q) \sqcup \mathrm{O}_{\uparrow}(p,q)', \quad \mathrm{O}_{\downarrow}(p,q) = \mathrm{SO}_{\uparrow\downarrow}(p,q) \sqcup \mathrm{O}_{\downarrow}(p,q)',$$

$$\mathrm{SO}(p,q) = \mathrm{SO}_{\uparrow\downarrow}(p,q) \sqcup \mathrm{SO}(p,q)'.$$

Examples:

- Orthogonal group $\mathrm{O}(n) := \mathrm{O}(n,0) \cong \mathrm{O}(0,n)$, special orthogonal group $\mathrm{SO}(n) := \mathrm{SO}(n,0) \cong \mathrm{SO}(0,n)$; $\mathrm{O}(n) = \mathrm{SO}(n) \sqcup \mathrm{SO}(n)'$ (2 connected components).
- Lorentz group $\mathrm{O}(1,3)$, special (or proper) $\mathrm{SO}(1,3)$, orthochronous $\mathrm{O}_{\uparrow}(1,3)$, orthochorous (or parity preserving) $\mathrm{O}_{\downarrow}(1,3)$, proper orthochronous $\mathrm{SO}_{\uparrow\downarrow}(1,3)$.

- Subgroup $H \subset G$ of group G is normal ($H \triangleleft G$) if $gHg^{-1} \subseteq H \ \forall g \in G$.
- Quotient group (or factor group) $\frac{G}{H} := \{gH \,|\, g \in G\}$ is the set of all left cosets ($\equiv$ right cosets, because H is normal).

$$\mathrm{SO}_{\uparrow\downarrow}(p,q) \triangleleft \mathrm{O}(p,q), \quad \mathrm{SO}(p,q) \triangleleft \mathrm{O}(p,q), \quad \ldots \text{(all subgroups are normal)}$$

$$\frac{\mathrm{O}(p,q)}{\mathrm{SO}_{\uparrow\downarrow}(p,q)} = \mathbb{Z}_2 \times \mathbb{Z}_2, \qquad \text{(Klein four-group)}$$

$$\frac{\mathrm{O}(p,q)}{\mathrm{SO}(p,q)} = \frac{\mathrm{O}(p,q)}{\mathrm{O}_\downarrow(p,q)} = \frac{\mathrm{O}(p,q)}{\mathrm{O}_\uparrow(p,q)} = \frac{\mathrm{SO}(p,q)}{\mathrm{SO}_{\uparrow\downarrow}(p,q)}$$

$$= \frac{\mathrm{O}_\downarrow(p,q)}{\mathrm{SO}_{\uparrow\downarrow}(p,q)} = \frac{\mathrm{O}_\uparrow(p,q)}{\mathrm{SO}_{\uparrow\downarrow}(p,q)} = \mathbb{Z}_2 = \{\pm 1\}, \qquad \frac{\mathrm{O}(n)}{\mathrm{SO}(n)} = \mathbb{Z}_2.$$

Example: $\mathrm{O}(1,1)$. Four components: $\mathrm{O}'_\uparrow(1,1)$, $\mathrm{O}'_\downarrow(1,1)$, $\mathrm{SO}'(1,1)$, $\mathrm{SO}_{\uparrow\downarrow}(1,1)$

$$\begin{pmatrix} \cosh\psi & \sinh\psi \\ -\sinh\psi & -\cosh\psi \end{pmatrix}, \begin{pmatrix} -\cosh\psi & -\sinh\psi \\ \sinh\psi & \cosh\psi \end{pmatrix}, \begin{pmatrix} -\cosh\psi & -\sinh\psi \\ -\sinh\psi & -\cosh\psi \end{pmatrix}, \begin{pmatrix} \cosh\psi & \sinh\psi \\ \sinh\psi & \cosh\psi \end{pmatrix},$$

$$\psi \in \mathbb{R}, \qquad \cosh^2\psi = 1 + \sinh^2\psi, \qquad \cosh\psi \geq 1.$$

Twisted adjoint representation

- the group of all invertible elements
  $C\ell^{\times}_{p,q} := \{U \in C\ell_{p,q} \,|\, \exists V \in C\ell_{p,q} : UV = VU = e\}$, $\dim C\ell^{\times}_{p,q} = 2^n$,
  Lie algebra: $C\ell_{p,q}$ with Lie bracket $[U, V] = UV - VU$;
- adjoint representation

$$\mathrm{Ad} : C\ell^{\times}_{p,q} \to \mathrm{Aut}\, C\ell_{p,q},\ T \to \mathrm{Ad}_T,\ \mathrm{Ad}_T U = TUT^{-1},\ U \in C\ell_{p,q}.$$

- kernel of Ad: $\ker(\mathrm{Ad}) = \{T \in C\ell^{\times}_{p,q} \,|\, \mathrm{Ad}_T(U) = U \quad \forall U \in C\ell_{p,q}\} =$

$$\begin{cases} C\ell^{0\times}_{p,q}, & \text{if } n \text{ is even;} \\ (C\ell^{0}_{p,q} \oplus C\ell^{n}_{p,q})^{\times}, & \text{if } n \text{ is odd.} \end{cases}$$

(see Theorem about the center of $C\ell_{p,q}$)

- twisted adjoint representation

$$\widetilde{\mathrm{Ad}} : C\ell^{\times}_{p,q} \to \mathrm{End}\, C\ell_{p,q},\ T \to \widetilde{\mathrm{Ad}}_T,\ \widetilde{\mathrm{Ad}}_T U = \widehat{T} UT^{-1},\ U \in C\ell_{p,q}.$$

- kernel of $\widetilde{\mathrm{Ad}}$: $\ker(\widetilde{\mathrm{Ad}}) = \{T \in C\ell^{\times}_{p,q} \,|\, \widetilde{\mathrm{Ad}}_T(U) = U \quad \forall U \in C\ell_{p,q}\} = C\ell^{0\times}_{p,q}$.

- vector space $V = C\ell^1_{p,q}$; quadratic form $Q(x) = g(x,x) \leftrightarrow$ symmetric bilinear form

$$g(x,y) = \frac{1}{2}(Q(x+y) - Q(x) - Q(y)) = \frac{1}{2}(xy+yx)|_{e\to 1},\ x,y \in C\ell^1_{p,q};$$

- **Statement:** $\widetilde{\mathrm{Ad}} : C\ell^{1\times}_{p,q} \to \mathrm{O}(p,q)$ on V. **Proof:** For $v \in C\ell^{1\times}_{p,q}$, $x \in C\ell^1_{p,q}$:

$$Q(\widetilde{\mathrm{Ad}}_v x) = (\hat{v}xv^{-1})^2 = \hat{v}xv^{-1}\hat{v}xv^{-1} = x^2 = Q(x), \text{because } x^2 \in C\ell^0_{p,q}. \quad \blacksquare$$

- $\widetilde{\mathrm{Ad}}_v$ acts on V as a reflection along v (in the hyperplane orthogonal to v):

$$\widetilde{\mathrm{Ad}}_v x = \hat{v}xv^{-1} = x-(xv+vx)v^{-1} = x - 2\frac{g(x,v)}{g(v,v)}v, \quad v \in C\ell^{1\times}_{p,q}, \quad x \in C\ell^1_{p,q};$$

- Theorem (Cartan-Diedonné): Every orthogonal transformation on a nongenerate space (V,g) is a product of reflections (the number $\leq \dim V$) in hyperplanes.
- Group $\Gamma^2_{p,q} := \{v_1 v_2 \cdots v_k \mid v_1,\ldots,v_k \in C\ell^{1\times}_{p,q}\}$.
- **Statement:** $\widetilde{\mathrm{Ad}}(\Gamma^2_{p,q}) = \mathrm{O}(p,q)$ (surjectivity). **Proof:** $f \in \mathrm{O}(p,q) \quad \Rightarrow$

$$\begin{aligned} f(x) &= \widetilde{\mathrm{Ad}}_{v_1} \circ \cdots \circ \widetilde{\mathrm{Ad}}_{v_k}(x) = \hat{v}_1 \cdots \hat{v}_k x v_k^{-1} \cdots v_1^{-1} = \widehat{v_1 \cdots v_k} x (v_1 \cdots v_k)^{-1} \\ &= \widetilde{\mathrm{Ad}}_{v_1 \cdots v_k}(x), \quad \text{for } v_1,\ldots,v_k \in V^\times \text{ and } x \in V. \quad \blacksquare \end{aligned}$$

- Group $\Gamma^1_{p,q} := \{T \in C\ell^\times_{p,q} \mid \forall x \in C\ell^1_{p,q}\ \hat{T} x T^{-1} \in C\ell^1_{p,q}\}$,
- Norm mapping (norm function) $N : C\ell_{p,q} \to C\ell_{p,q}$, $N(U) := \widehat{\widetilde{U}} U$.
- **Statement:** $N : \Gamma^1_{p,q} \to C\ell^{0\times}_{p,q} \cong \mathbb{R}^\times$. **Proof:**

$$T \in \Gamma^1_{p,q}, x \in C\ell^1_{p,q} \quad \Rightarrow \quad \hat{T} x T^{-1} = \widetilde{\hat{T} x T^{-1}} = \widetilde{T^{-1}} x \widetilde{\hat{T}} = (\widetilde{T})^{-1} x \widetilde{\hat{T}},$$

$$\Rightarrow \quad \widehat{\widetilde{\widehat{T}}}\, T x = x \widehat{\widetilde{T}}\, T \quad \Rightarrow \quad \widehat{\widetilde{T}}\, T \in \ker \widetilde{\mathrm{Ad}} = C\ell^{0\times}_{p,q}. \qquad \blacksquare$$

- **Statement:** $N : \Gamma^1_{p,q} \to \mathbb{R}^\times$ is a group homomorphism: $N(UV) = N(U)N(V)$, $N(U^{-1}) = (N(U))^{-1}$, $U, V \in \Gamma^1_{p,q}$. **Proof:**

$$N(UV) = \widehat{\widetilde{UV}}\, UV = \widehat{\widetilde{V}}\, \widehat{\widetilde{U}}\, UV = \widehat{\widetilde{V}}\, N(U) V = N(U)N(V),$$
$$e = N(e) = N(UU^{-1}) = N(U)N(U^{-1}). \quad \blacksquare$$

- **Statement:** $\widetilde{\mathrm{Ad}} : \Gamma^1_{p,q} \to \mathrm{O}(p,q)$. **Proof:**

$$N(\hat{T}) = \widehat{\widetilde{\widehat{T}}}\, \hat{T} = \widehat{\widehat{\widetilde{T}}\, T} = \widehat{N(T)} = N(T),$$
$$N(\widetilde{\mathrm{Ad}}_T(x)) = N(\hat{T} x T^{-1}) = N(\hat{T}) N(x) N(T^{-1}) = N(T) N(x) (N(T))^{-1} = N(x),$$
$$N(x) = \widehat{\widetilde{x}}\, x = -x^2 = -Q(x), \qquad Q(\widetilde{\mathrm{Ad}}_T(x)) = Q(x). \quad \blacksquare$$

- **Statement:** $\Gamma^1_{p,q}=\Gamma^2_{p,q}$.
  Proof: We know that $\Gamma^2_{p,q}\subseteq\Gamma^1_{p,q}$. Let us prove that $\Gamma^1_{p,q}\subseteq\Gamma^2_{p,q}$.
$$T\in\Gamma^1_{p,q}\;\Rightarrow\;\widetilde{\mathrm{Ad}}_T\in\mathrm{O}(p,q)\;\Rightarrow\;\exists S\in\Gamma^2_{p,q}:\widetilde{\mathrm{Ad}}_S=\widetilde{\mathrm{Ad}}_T$$
$$\Rightarrow\;\widetilde{\mathrm{Ad}}_{TS^{-1}}=\mathrm{id}\;\Rightarrow\;TS^{-1}=\lambda e,\lambda\in\mathbb{R}\;\Rightarrow\;T=\lambda S\in\Gamma^2_{p,q}.\quad\blacksquare$$
- Lipschitz group
$$\Gamma^{\pm}_{p,q}:=\Gamma^1_{p,q}=\Gamma^2_{p,q}=\{T\in C\!\ell^{(0)\times}_{p,q}\cup C\!\ell^{(1)\times}_{p,q}\,|\,\forall x\in C\!\ell^1_{p,q}\;TxT^{-1}\in C\!\ell^1_{p,q}\}$$
$$=\{v_1v_2\cdots v_k\,|\,v_1,\ldots,v_k\in C\!\ell^{1\times}_{p,q}\}.$$
- Clifford group $\Gamma_{p,q}:=\{T\in C\!\ell^{\times}_{p,q}\,|\,\forall x\in C\!\ell^1_{p,q}\;TxT^{-1}\in C\!\ell^1_{p,q}\}\supseteq\Gamma^{\pm}_{p,q}$;
- $\widetilde{\mathrm{Ad}}(\Gamma^{\pm}_{p,q})=\mathrm{O}(p,q)$, i.e.
$$\forall P=||p^a_b||\in\mathrm{O}(p,q)\quad\exists T\in\Gamma^{\pm}_{p,q}:\widehat{T}e_aT^{-1}=p^b_ae_b.$$
- subgroup $\Gamma^{+}_{p,q}:=\{T\in C\!\ell^{(0)\times}_{p,q}\,|\,\forall x\in C\!\ell^1_{p,q}\;TxT^{-1}\in C\!\ell^1_{p,q}\}\subset\Gamma^{\pm}_{p,q}$.
- $\widetilde{\mathrm{Ad}}(\Gamma^{+}_{p,q})=\mathrm{Ad}(\Gamma^{+}_{p,q})=\mathrm{SO}(p,q)$, i.e.
$$\forall P=||p^a_b||\in\mathrm{SO}(p,q)\quad\exists T\in\Gamma^{+}_{p,q}:\widehat{T}e_aT^{-1}=Te_aT^{-1}=p^b_ae_b.$$

$$\mathrm{Pin}(p,q) := \{T \in \Gamma^{\pm}_{p,q} \mid \widetilde{T}T = \pm e\} = \{T \in \Gamma^{\pm}_{p,q} \mid \widehat{\widetilde{T}}T = \pm e\},$$

$$\mathrm{Pin}_{\uparrow}(p,q) := \{T \in \Gamma^{\pm}_{p,q} \mid \widehat{\widetilde{T}}T = +e\}, \qquad \text{(Spin groups)}$$

$$\mathrm{Pin}_{\downarrow}(p,q) := \{T \in \Gamma^{\pm}_{p,q} \mid \widetilde{T}T = +e\},$$

$$\mathrm{Spin}(p,q) := \{T \in \Gamma^{+}_{p,q} \mid \widetilde{T}T = \pm e\} = \{T \in \Gamma^{+}_{p,q} \mid \widehat{\widetilde{T}}T = \pm e\},$$

$$\mathrm{Spin}_{\uparrow\downarrow}(p,q) := \{T \in \Gamma^{+}_{p,q} \mid \widetilde{T}T = +e\} = \{T \in \Gamma^{+}_{p,q} \mid \widehat{\widetilde{T}}T = +e\}.$$

$\mathrm{Pin}(p,q) = \mathrm{Spin}_{\uparrow\downarrow}(p,q) \sqcup \mathrm{Pin}_{\uparrow}(p,q)' \sqcup \mathrm{Pin}_{\downarrow}(p,q)' \sqcup \mathrm{Spin}(p,q)'$, (4 components)
$\mathrm{Pin}_{\uparrow}(p,q) = \mathrm{Spin}_{\uparrow\downarrow}(p,q) \sqcup \mathrm{Pin}_{\uparrow}(p,q)'$, $\quad \mathrm{Pin}_{\downarrow}(p,q) = \mathrm{Spin}_{\uparrow\downarrow}(p,q) \sqcup \mathrm{Pin}_{\downarrow}(p,q)'$,
$\mathrm{Spin}(p,q) = \mathrm{Spin}_{\uparrow\downarrow}(p,q) \sqcup \mathrm{Spin}(p,q)'$.

Euclidian case (2 components):

$$\mathrm{Pin}(n) := \mathrm{Pin}(n,0) = \mathrm{Pin}_{\downarrow}(0,n), \quad \mathrm{Spin}(n,0) = \mathrm{Pin}_{\uparrow}(n,0) = \mathrm{Spin}_{\uparrow\downarrow}(n,0),$$
$$\mathrm{Pin}(0,n) := \mathrm{Pin}(0,n) = \mathrm{Pin}_{\uparrow}(0,n), \quad \mathrm{Spin}(0,n) = \mathrm{Pin}_{\downarrow}(0,n) = \mathrm{Spin}_{\uparrow\downarrow}(0,n).$$

Subgroups are normal $(\mathrm{Spin}_{\uparrow\downarrow}(p,q) \triangleleft \mathrm{Spin}(p,q),\ \mathrm{Spin}(p,q) \triangleleft \mathrm{Pin}(p,q),\ \ldots)$

Proof: $H = \mathrm{Spin}_{\uparrow\downarrow}(p,q)$, $G = \mathrm{Spin}(p,q) \Rightarrow \widetilde{g}g = \pm e\, \forall g \in G$, $\widetilde{h}h = e\, \forall h \in H$
$\Rightarrow \widetilde{ghg^{-1}}ghg^{-1} = \widetilde{g^{-1}}\widetilde{h}\widetilde{g}ghg^{-1} = +e \Rightarrow ghg^{-1} \in H.$ ■

Theorem

The following homomorphisms are surjective with kernel $\{\pm 1\}$*:*

$$\widetilde{\mathrm{Ad}}:\mathrm{Pin}(p,q)\to\mathrm{O}(p,q),\qquad \widetilde{\mathrm{Ad}}:\mathrm{Spin}(p,q)\to\mathrm{SO}(p,q),$$
$$\widetilde{\mathrm{Ad}}:\mathrm{Spin}_{\uparrow\downarrow}(p,q)\to\mathrm{SO}_{\uparrow\downarrow}(p,q),\qquad \widetilde{\mathrm{Ad}}:\mathrm{Pin}_{\uparrow}(p,q)\to\mathrm{O}_{\uparrow}(p,q),$$
$$\widetilde{\mathrm{Ad}}:\mathrm{Pin}_{\downarrow}(p,q)\to\mathrm{O}_{\downarrow}(p,q).$$

$$\widehat{T}e_aT^{-1}=p_a^be_b,\qquad P=||p_a^b||\in\mathrm{O}(p,q),\quad \pm T\in\mathrm{Pin}(p,q).$$

$$\mathcal{C}\ell^{(0)}_{p,q}\cong\mathcal{C}\ell^{(0)}_{q,p}\Rightarrow \mathrm{Spin}(p,q)\cong\mathrm{Spin}(q,p),\qquad \mathrm{Pin}(p,q)\ncong\mathrm{Pin}(q,p),$$
$$\mathrm{Pin}(1,0)=\{\pm e,\pm e_1\}\cong\mathbb{Z}_2\times\mathbb{Z}_2,\quad \mathrm{Pin}(0,1)\cong\mathbb{Z}_4,\quad \mathrm{Spin}(1)=\{\pm e\}=\mathbb{Z}_2.$$

Details:

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- Lawson H. B., Michelsohn M. L., Spin Geometry, Princeton Math. Ser., 38, Princeton Univ. Press, Princeton, NJ (1989).

Another proof using Generalized Pauli theorem (not Cartan-Diedonné theorem):

- D. Sh., The use of the generalized Pauli's theorem for odd elements of Clifford algebra to analyze relations between spin and orthogonal groups of arbitrary dimension, Vestn.Samar.Gos.Tekhn.Univ. Ser.Fiz.-Mat.Nauki, 1(30) (2013) [in Russian]
- N. Marchuk, D. Sh., Introduction to the theory of Clifford algebras [in Russian], Phasis, Moscow (2012) 590 pp.
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Spin groups in small dimensions

Theorem

Condition $TxT^{-1} \in C\ell^1_{p,q}, \forall x \in C\ell^1_{p,q}$ holds automatically in the cases $n \leq 5$ for all 5 spin groups, i.e.

$$\mathrm{Pin}(p,q) = \{T \in C\ell^{(0)}_{p,q} \cup C\ell^{(1)}_{p,q} \mid \widetilde{T}T = \pm e\}, \quad n = p+q \leq 5.$$

Proof $T \in C\ell^{(0)}_{p,q} \cup C\ell^{(1)}_{p,q} \Rightarrow TxT^{-1} \in C\ell^1_{p,q} \oplus C\ell^3_{p,q} \oplus C\ell^5_{p,q}$,

$\widetilde{T}T = \pm e \Rightarrow \widetilde{TxT^{-1}} = \pm \widetilde{Tx\widetilde{T}} = \pm Tx\widetilde{T} \Rightarrow TxT^{-1} \in C\ell^1_{p,q} \oplus C\ell^5_{p,q}$,

$n = 5$: suppose $TxT^{-1} = v + \lambda e_{1\ldots5}, v \in C\ell^1_{p,q}, \lambda \in \mathbb{R}^\times \Rightarrow$

$\lambda = (TxT^{-1}e^{-1}_{1\ldots5} - ve^{-1}_{1\ldots5})|_{e\to1} = \mathrm{Tr}(TxT^{-1}e^{-1}_{1\ldots5}) = \mathrm{Tr}(xe^{-1}_{1\ldots5}) = 0.$ ■

Example: $n = 6, \quad T = \frac{1}{\sqrt{2}}(e_{12} + e_{3456}) \in C\ell^{(0)}_{6,0}$,

$\widetilde{T}T = e, \qquad Te_1T^{-1} = -e_{23456} \notin C\ell^1_{6,0}$.

Theorem

$Spin_{\uparrow\downarrow}(p,q)$ is isomorphic to the following groups in the cases $n = p + q \leq 6$:

(p,q)	0	1	2	3	4	5	6
0	$\mathrm{O}(1)$	$\mathrm{O}(1)$	$\mathrm{U}(1)$	$\mathrm{SU}(2)$	${}^2\mathrm{SU}(2)$	$\mathrm{Sp}(2)$	$\mathrm{SU}(4)$
1	$\mathrm{O}(1)$	$\mathrm{GL}(1,\mathbb{R})$	$\mathrm{SU}(1,1)$	$\mathrm{Sp}(1,\mathbb{C})$	$\mathrm{Sp}(1,1)$	$\mathrm{SL}(2,\mathbb{H})$	
2	$\mathrm{U}(1)$	$\mathrm{SU}(1,1)$	${}^2\mathrm{SU}(1,1)$	$\mathrm{Sp}(2,\mathbb{R})$	$\mathrm{SU}(2,2)$		
3	$\mathrm{SU}(2)$	$\mathrm{Sp}(1,\mathbb{C})$	$\mathrm{Sp}(2,\mathbb{R})$	$\mathrm{SL}(4,\mathbb{R})$			
4	${}^2\mathrm{SU}(2)$	$\mathrm{Sp}(1,1)$	$\mathrm{SU}(2,2)$				
5	$\mathrm{Sp}(2)$	$\mathrm{SL}(2,\mathbb{H})$					
6	$\mathrm{SU}(4)$						

$$\mathrm{U}(1)\simeq\mathrm{SO}(2),\quad \mathrm{SU}(2)\simeq\mathrm{Sp}(1),\quad \mathrm{SU}(1,1)\simeq\mathrm{SL}(2,\mathbb{R})\simeq\mathrm{Sp}(1,\mathbb{R}),\quad \mathrm{SL}(2,\mathbb{C})\simeq\mathrm{Sp}(1,\mathbb{C}).$$

Lie algebras, two-sheeted coverings

- The following Lie groups have the following Lie algebras:

Lie group	Lie algebra
$\mathcal{C}\!\ell^{\times}_{p,q}$	$\mathcal{C}\!\ell_{p,q}$
Clifford group $\Gamma_{p,q}$	$\begin{cases} \mathcal{C}\!\ell^0_{p,q} \oplus \mathcal{C}\!\ell^2_{p,q}, & \text{if } n \text{ is even;} \\ \mathcal{C}\!\ell^0_{p,q} \oplus \mathcal{C}\!\ell^2_{p,q} \oplus \mathcal{C}\!\ell^n_{p,q}, & \text{if } n \text{ is odd.} \end{cases}$
Lipschitz group $\Gamma^{\pm}_{p,q}$, $\Gamma^{+}_{p,q}$	$\mathcal{C}\!\ell^0_{p,q} \oplus \mathcal{C}\!\ell^2_{p,q}$
5 spinor groups $\mathrm{Pin}(p,q), \mathrm{Spin}(p,q), \ldots$	$\mathcal{C}\!\ell^2_{p,q}$

- Spin groups are two-sheeted coverings of the orthogonal groups.
- The groups $\mathrm{Spin}_{\uparrow\downarrow}(p,q)$ are pathwise connected for $p \geq 2$ or $q \geq 2$. They are nontrivial covering groups of the corresponding orthogonal groups. Example: $\mathrm{Spin}_{\uparrow\downarrow}(1,1) = \{ue + ve_{12} \mid u^2 - v^2 = 1\}$ - two branches of hyperbole (is not pathwise connected).
- The groups $\mathrm{Spin}_{\uparrow\downarrow}(n)$, $n \geq 3$ and $\mathrm{Spin}_{\uparrow\downarrow}(1, n-1) \cong \mathrm{Spin}_{\uparrow\downarrow}(n-1, 1)$, $n \geq 4$ are simply connected. They are the universal covering groups of the corresponding orthogonal groups.

Other Lie groups and Lie algebras

	Lie group	Lie algebra	dimension
1	$(\mathbb{C}\otimes C\!\ell_{p,q})^\times=\{U\in\mathbb{C}\otimes C\!\ell_{p,q}\,\vert\,\exists U^{-1}\}$	$\overline{\mathbf{0123}}\oplus i\overline{\mathbf{0123}}$	2^{n+1}
2	$C\!\ell^\times_{p,q}=\{U\in C\!\ell_{p,q}\,\vert\,\exists U^{-1}\}$	$\overline{\mathbf{0123}}$	2^n
3	$C\!\ell^{(0)\times}_{p,q}=\{U\in C\!\ell^{(0)}_{p,q}\,\vert\,\exists U^{-1}\}$	$\overline{\mathbf{02}}$	2^{n-1}
4	$(\mathbb{C}\otimes C\!\ell^{(0)}_{p,q})^\times=\{U\in\mathbb{C}\otimes C\!\ell_{p,q}\,\vert\,\exists U^{-1}\}$	$\overline{\mathbf{02}}\oplus i\overline{\mathbf{02}}$	2^n
5	$(C\!\ell^{(0)}_{p,q}\oplus iC\!\ell^{(1)}_{p,q})^\times=\{U\in C\!\ell^{(0)}_{p,q}\oplus iC\!\ell^{(1)}_{p,q}\,\vert\,\exists U^{-1}\}$	$\overline{\mathbf{02}}\oplus i\overline{\mathbf{13}}$	2^n
6	$\mathrm{G}^{23i01}_{p,q}=\{U\in\mathbb{C}\otimes C\!\ell_{p,q}\,\vert\,\overline{\tilde{U}}U=e\}$	$\overline{\mathbf{23}}\oplus i\overline{\mathbf{01}}$	2^n
7	$\mathrm{G}^{12i03}_{p,q}=\{U\in\mathbb{C}\otimes C\!\ell_{p,q}\,\vert\,\overline{\hat{\tilde{U}}}U=e\}$	$\overline{\mathbf{12}}\oplus i\overline{\mathbf{03}}$	2^n
8	$\mathrm{G}^{2i0}_{p,q}=\{U\in C\!\ell^{(0)}_{p,q}\,\vert\,\overline{\tilde{U}}U=e\}$	$\overline{\mathbf{2}}\oplus i\overline{\mathbf{0}}$	2^{n-1}
9	$\mathrm{G}^{23i23}_{p,q}=\{U\in\mathbb{C}\otimes C\!\ell_{p,q}\,\vert\,\tilde{U}U=e\}$	$\overline{\mathbf{23}}\oplus i\overline{\mathbf{23}}$	$2^n-2^{\frac{n+1}{2}}\sin\frac{\pi(n+1)}{4}$
10	$\mathrm{G}^{12i12}_{p,q}=\{U\in\mathbb{C}\otimes C\!\ell_{p,q}\,\vert\,\hat{\tilde{U}}U=e\}$	$\overline{\mathbf{12}}\oplus i\overline{\mathbf{12}}$	$2^n-2^{\frac{n+1}{2}}\cos\frac{\pi(n+1)}{4}$
11	$\mathrm{G}^{2i2}_{p,q}=\{U\in\mathbb{C}\otimes C\!\ell^{(0)}_{p,q}\,\vert\,\tilde{U}U=e\}$	$\overline{\mathbf{2}}\oplus i\overline{\mathbf{2}}$	$2^{n-1}-2^{\frac{n}{2}}\cos\frac{\pi n}{4}$
12	$\mathrm{G}^{2i1}_{p,q}=\{U\in C\!\ell^{(0)}_{p,q}\oplus iC\!\ell^{(1)}_{p,q}:\overline{\tilde{U}}U=e\}$	$\overline{\mathbf{2}}\oplus i\overline{\mathbf{1}}$	$2^{n-1}-2^{\frac{n-1}{2}}\cos\frac{\pi(n+1)}{4}$
13	$\mathrm{G}^{2i3}_{p,q}=\{U\in C\!\ell^{(0)}_{p,q}\oplus iC\!\ell^{(1)}_{p,q}:\overline{\hat{\tilde{U}}}U=e\}$	$\overline{\mathbf{2}}\oplus i\overline{\mathbf{3}}$	$2^{n-1}-2^{\frac{n-1}{2}}\sin\frac{\pi(n+1)}{4}$
14	$\mathrm{G}^{23}_{p,q}=\{U\in C\!\ell_{p,q}\,\vert\,\tilde{U}U=e\}$	$\overline{\mathbf{23}}$	$2^{n-1}-2^{\frac{n-1}{2}}\sin\frac{\pi(n+1)}{4}$
15	$\mathrm{G}^{12}_{p,q}=\{U\in C\!\ell_{p,q}\,\vert\,\hat{\tilde{U}}U=e\}$	$\overline{\mathbf{12}}$	$2^{n-1}-2^{\frac{n-1}{2}}\cos\frac{\pi(n+1)}{4}$
16	$\mathrm{G}^{2}_{p,q}=\{U\in C\!\ell^{(0)}_{p,q},\,\vert\,\tilde{U}U=e\}$	$\overline{\mathbf{2}}$	$2^{n-2}-2^{\frac{n-2}{2}}\cos\frac{\pi n}{4}$

$$
\mathrm{G}^{23i01}_{p,q} \cong \begin{cases} \mathrm{U}(2^{\frac{n}{2}}), & \text{if } p \text{ is even and } q=0;\\ \mathrm{U}(2^{\frac{n-1}{2}})\oplus\mathrm{U}(2^{\frac{n-1}{2}}), & \text{if } p \text{ is odd and } q=0;\\ \mathrm{U}(2^{\frac{n-2}{2}},2^{\frac{n-2}{2}}), & \text{if } n \text{ is even and } q\neq 0;\\ \mathrm{U}(2^{\frac{n-3}{2}},2^{\frac{n-3}{2}})\oplus\mathrm{U}(2^{\frac{n-3}{2}},2^{\frac{n-3}{2}}), & \text{if } p \text{ is odd and } q\neq 0 \text{ is even};\\ \mathrm{GL}(2^{\frac{n-1}{2}},\mathbb{C}), & \text{if } p \text{ is even and } q \text{ is odd}. \end{cases}
$$

- Snygg J., Clifford Algebra - A Computational Tool For Physicists, Oxford University Press, New York (1997). (*c*-unitary groups)
- Porteous I.R., Clifford Algebras and the Classical Groups, Cambridge Univ. Press (1995).
- D. Sh., Symplectic, Orthogonal and Linear Lie Groups in Clifford Algebra, Advances in Applied Clifford Algebras, 25:3 (2015), arXiv: 1409.2452
- D. Sh., On Some Lie Groups Containing Spin Group in Clifford Algebra, Journal of Geometry and Symmetry in Physics, 42 (2016), arXiv: 1607.07363
- D. Sh., Classification of Lie algebras of specific type in complexified Clifford algebras, arXiv:1704.03713

Lecture 5

Dirac Equation

Dirac equation in Clifford algebras. Dirac-Hestenes equation. Spinors in n dimensions.

Dirac equation

$\mathbb{R}^{1,3}, \quad x^\mu,\ \mu = 0,1,2,3, \quad \eta = \mathrm{diag}(1,-1,-1,-1), \quad \partial_\mu := \frac{\partial}{\partial x^\mu},$

$a_\mu : \mathbb{R}^{1,3} \to \mathbb{R}, \quad$ (electromagnetic 4-vector potential),

$m \geq 0 \in \mathbb{R}, \quad$ (mass of an electron),

$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}\mathbf{1}, \quad \gamma^\mu \in \mathrm{Mat}(4,\mathbb{C}), \quad$ (Dirac gamma-matrices),

$\psi : \mathbb{R}^{1,3} \to \mathbb{C}^4, \quad$ (wave function, Dirac spinor)

$$\boxed{i\gamma^\mu(\partial_\mu\psi - ia_\mu\psi) - m\psi = 0}.$$

Dirac P.A.M., Proc. Roy. Soc. Lond. A117 (1928).

Dirac P.A.M., Proc. Roy. Soc. Lond. A118 (1928).

$(p,q) = (3,0)$: Pauli spinors, $\psi : \mathbb{R}^3 \to \mathbb{C}^2$, Pauli matrices σ^μ.

Gauge invariance

$$\boxed{i\gamma^\mu(\partial_\mu\psi - ia_\mu\psi) - m\psi = 0}$$

$$a_\mu \to a'_\mu = a_\mu + \lambda(x), \quad \psi \to \psi' = \psi e^{i\lambda(x)}, \qquad \lambda(x) \in \mathbb{R},$$

$$i\gamma^\mu(\partial_\mu\psi' - ia'_\mu\psi') - m\psi' = i\gamma^\mu(\partial_\mu(e^{i\lambda}\psi) - i(a_\mu + \partial_\mu\lambda)(e^{i\lambda}\psi)) - m(e^{i\lambda}\psi) =$$
$$= i\gamma^\mu(i(\partial_\mu\lambda)e^{i\lambda}\psi + e^{i\lambda}(\partial_\mu\psi) - ia_\mu e^{i\lambda}\psi - i(\partial_\mu\lambda)e^{i\lambda}\psi) - me^{i\lambda}\psi =$$
$$= e^{i\lambda}(i\gamma^\mu(\partial_\mu\psi - ia_\mu\psi) - m\psi) = 0.$$

$$\mathrm{U}(1) = \{e^{i\lambda} \,|\, \lambda \in \mathbb{R}\}, \qquad \mathfrak{u}(1) = \{i\lambda \,|\, \lambda \in \mathbb{R}\}.$$

Relativistic invariance

$$\boxed{i\gamma^\mu(\partial_\mu\psi - ia_\mu\psi) - m\psi = 0}$$

$$x^\mu \to x^{\mu\prime} = p^\mu_\nu x^\nu, \quad P = ||p^\mu_\nu|| \in \mathrm{O}(1,3),$$
$$\partial_\mu \to \partial'_\mu = q^\nu_\mu\partial_\nu, \quad a_\mu \to a'_\mu = q^\nu_\mu a_\nu, \quad Q = ||q^\mu_\nu|| = P^{-1},$$

1) $\gamma^\mu \to \gamma^{\mu\prime} = p^\mu_\nu\gamma^\nu, \quad \psi \to \psi' = \psi,$

2) $\gamma^\mu \to \gamma^{\mu\prime} = \gamma^\mu, \quad \psi \to \psi' = S\psi, \quad S^{-1}\gamma^\mu S = p^\mu_\nu\gamma^\nu,$

$$i\gamma^{\mu\prime}(\partial'_\mu\psi' - ia'_\mu\psi') - m\psi' = i\gamma^\mu(q^\nu_\mu\partial_\nu(S\psi) - iq^\nu_\mu a_\nu S\psi) - mS\psi) =$$
$$= S(iS^{-1}q^\nu_\mu\gamma^\mu S(\partial_\nu\psi - ia_\mu\psi) - m\psi) = S(i\gamma^\nu(\partial_\nu\psi - ia_\mu\psi) - m\psi) = 0.$$

A. Sommerfeld, Atombau und Spektrallinien, Vol. 2, F. Vieweg und Sohn, Braunschweig, 1951

N. Marchuk, Field theory equations, Amazon, CreateSpace, 2012, 290 pp (tensor approach)

Dirac equation in Clifford algebra

$$\mathbb{C}\otimes C\ell_{1,3}, \qquad \{e^0, e^1, e^2, e^3\},$$

$$t=\frac{1}{2}(e+e^0)\frac{1}{2}(e+ie^{12}) \leftrightarrow \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad t^2=t=t^\dagger,$$

$$\psi \leftrightarrow \begin{pmatrix} \psi_1 & 0 & 0 & 0\\ \psi_2 & 0 & 0 & 0\\ \psi_3 & 0 & 0 & 0\\ \psi_4 & 0 & 0 & 0 \end{pmatrix} \in \mathrm{I}(t)=(\mathbb{C}\otimes C\ell_{1,3})t \quad \text{(spinor space)},$$

$$ie^\mu(\partial_\mu\psi - ia_\mu\psi)-m\psi=0.$$

If n is odd: double spinor space (provides a faithful but reducible representation, idempotent $t+\hat{t}$, where t is primitive).

Similarly for the real Clifford algebra $C\ell_{p,q}$: spinor spaces or double spinor spaces (if $p-q \mod 4=1$, see Cartan's periodicity).

Lounesto P., Clifford Algebras and Spinors, Cambridge Univ. Press (1997).

Weyl spinors

Chirality operator (pseudoscalar): $\omega = \begin{cases} e^{1\ldots n}, & p-q=0,1 \mod 4; \\ ie^{1\ldots n}, & p-q=2,3 \mod 4. \end{cases}$

$\omega = \omega^{-1} = \omega^{\dagger}$,

Orthogonal idempotents $P_L := \frac{1}{2}(e-\omega), \quad P_R := \frac{1}{2}(e+\omega)$,

$P_L^2 = P_L, \qquad P_R^2 = P_R, \qquad P_L P_R = P_R P_L = 0.$

If n is odd, then $\mathbb{C}\otimes C\ell_{p,q}$ is the direct sum of two ideals:

$\mathbb{C}\otimes C\ell_{p,q} = P_L(\mathbb{C}\otimes C\ell_{p,q}) \oplus P_R(\mathbb{C}\otimes C\ell_{p,q}), \quad \mathbb{C}\otimes C\ell_{p,q} \cong {}^2\mathrm{Mat}(2^{\frac{n-1}{2}}, \mathbb{C}).$

Let us consider the case of even n.

The set of Dirac spinors: $E_D = \{\psi \in \mathrm{I}(t)\}, \qquad E_D = E_{LW} \oplus E_{RW}$,

left Weyl spinors: $E_{LW} := \{\psi \in E_D : P_L\psi = \psi\}, \qquad P_L\psi = \psi \Leftrightarrow \omega\psi = -\psi$,

right Weyl spinors: $E_{RW} := \{\psi \in E_D : P_R\psi = \psi\}, \qquad P_R\psi = \psi \Leftrightarrow \omega\psi = \psi$.

Dirac conjugation

$(e^a)^\dagger = \pm A_\pm^{-1} e^a A_\pm \Leftrightarrow U^\dagger = A_+^{-1}\overline{\tilde{U}}A_+, \quad U^\dagger = A_-^{-1}\overline{\widehat{\tilde{U}}}A_-, \quad U \in \mathbb{C}\otimes C\ell_{p,q},$

n is even: $\exists A_\pm$, p is odd, q is even: $\exists A_+$, p is even, q is odd: $\exists A_-$,

Dirac conjugation : $\psi^{D_\pm} := \psi^\dagger (A_\pm)^{-1}$

Example: $(p,q) = (1,3)$, $\psi^{D_+} = \psi^\dagger \gamma^0$, $\psi^{D-} = \psi^\dagger \gamma^{123}$,

Bilinear covariants: $j_\pm^A = \psi^{D_\pm} e^A \psi$,

The law of conservation of the Dirac current: $\boxed{\partial_\mu(\psi^{D_+} e^\mu \psi) = 0}$.

Proof: $ie^\mu(\partial_\mu \psi - ia_\mu \psi) - m\psi = 0 \Rightarrow^\dagger -i(\partial_\mu(\psi)^\dagger + ia_\mu \psi^\dagger)(e^\mu)^\dagger - m\psi^\dagger = 0 | A_+^{-1}$

$\Rightarrow i(\partial_\mu(\psi^{D_+})^\dagger + ia_\mu(\psi^{D_+})^\dagger)e^\mu + m(\psi^{D_+})^\dagger = 0|\psi, \; \psi^{D_+}| ie^\mu(\partial_\mu\psi - ia_\mu\psi) - m\psi = 0$

$\Rightarrow i(\psi^{D_+} e^\mu \partial_\mu \psi + \partial_\mu(\psi^{D_+})^\dagger e^\mu \psi) = 0.$ ■

Benn I. M., Tucker R. W., An introduction to Spinors and Geometry with Applications in Physics, Bristol (1987)

D. Sh., Pauli theorem in the description of n-dimensional spinors in the Clifford algebra formalism, Theoret. and Math. Phys., 175:1 (2013) (see References)

Majorana and charge conjugations

$$(e^a)^{\mathrm{T}} = \pm C_\pm^{-1} e^a C_\pm \Leftrightarrow U^{\mathrm{T}} = C_+^{-1} \tilde{U} C_+, \quad U^{\mathrm{T}} = C_-^{-1} \hat{\tilde{U}} C_-, \quad U \in \mathbb{C} \otimes \mathcal{C}\!\ell_{p,q},$$

n is even: $\exists C_\pm$, $\quad n = 1 \mod 4$: $\exists C_+$, $\quad n = 3 \mod 4$: $\exists C_-$,

$$(C_\pm)^{\mathrm{T}} = \lambda_\pm C_\pm, \qquad \overleftarrow{C_\pm} C_\pm = \lambda_\pm e,$$

$$\lambda_+ = \begin{cases} +1, & n \equiv 0, 1, 2 \mod 8; \\ -1, & n \equiv 4, 5, 6 \mod 8, \end{cases} \qquad \lambda_- = \begin{cases} +1, & n \equiv 0, 6, 7 \mod 8; \\ -1, & n \equiv 2, 3, 4 \mod 8. \end{cases}$$

$$\overleftarrow{e^a} = \pm B_\pm^{-1} e^a B_\pm \Leftrightarrow \overleftarrow{U} = B_+^{-1} \overline{U} B_+, \quad \overleftarrow{U} = B_-^{-1} \hat{\overline{U}} B_-, \quad U \in \mathbb{C} \otimes \mathcal{C}\!\ell_{p,q},$$

n is even: $\exists B_\pm$, $\quad p - q = 1 \mod 4$: $\exists B_+$, $\quad p - q = 3 \mod 4$: $\exists B_-$,

$B_\pm^T = \epsilon_\pm B_\pm$, $\qquad \overleftarrow{B_\pm} B_\pm = \epsilon_\pm e$, where "$\leftarrow$" is matrix complex conjugation,

$$\epsilon_+ = \begin{cases} +1, & p - q \equiv 0, 1, 2 \mod 8; \\ -1, & p - q \equiv 4, 5, 6 \mod 8, \end{cases} \quad \epsilon_- = \begin{cases} +1, & p - q \equiv 0, 6, 7 \mod 8; \\ -1, & p - q \equiv 2, 3, 4 \mod 8, \end{cases}$$

Majorana conjugation : $\psi^{M_\pm} := \psi^{\mathrm{T}} (C_\pm)^{-1}$

Example: $(p, q) = (1, 3)$, $\qquad \psi^{M_+} = \psi^\dagger (\gamma^{13})^{-1}, \psi^{M_-} = \psi^\dagger (\gamma^{02})^{-1}$

Charge conjugation : $\psi^{ch_\pm} := B_\pm \overleftarrow{\psi}$

Example: $(p, q) = (1, 3)$, $\qquad \psi^{ch_+} = \gamma^{013} \overleftarrow{\psi}, \psi^{ch_-} = \gamma^2 \overleftarrow{\psi}$.

Majorana and Majorana-Weyl spinors

Relation between different conjugations:

$$B_+ = \widetilde{A_+^{-1}} C_+, \quad B_+ = \widehat{\widetilde{A_-^{-1}}} C_-, \quad B_- = \widetilde{A_-^{-1}} C_+, \quad B_- = \widehat{\widetilde{A_+^{-1}}} C_-,$$

$$\psi^{ch_+} = C_+(\psi^{D_+})^{\mathrm{T}} = C_-(\psi^{D_-})^{\mathrm{T}}, \qquad \psi^{ch_-} = C_-(\psi^{D_+})^{\mathrm{T}} = C_+(\psi^{D_-})^{\mathrm{T}},$$

Majorana spinors : $E_M := \{\psi \in E_D \,|\, \psi^{ch_-} = \pm\psi\}$, $p-q = 0, 6, 7 \mod 8$,

pseudo-Majorana spinors : $E_{psM} := \{\psi \in E_D \,|\, \psi^{ch_+} = \pm\psi\}$, $p-q = 0, 1, 2 \mod 8$,

Proof: $E_{psM}: \psi = \pm B_+ \overleftarrow{\psi}, \quad \pm B_+^{-1}\psi = \overleftarrow{\psi} = \pm \overleftarrow{B}_+ \psi = \pm\epsilon_+ B_+^{-1}\psi,$

$(1-\epsilon_+)\psi = 0, \quad \epsilon_+ = 1.$ ■ E_M : analogously $\epsilon_- = 1.$ ■

left Majorana-Weyl spinors : $E_{LMW} := \{\psi \in E_{LW} \,|\, \psi^{ch_-} = \pm\psi\} =$

$= \{\psi \in E_{LW} \,|\, \psi^{ch_+} = \pm\psi\}, \quad p-q = 0 \mod 8,$

right Majorana-Weyl spinors : $E_{RMW} := \{\psi \in E_{RW} \,|\, \psi^{ch_-} = \pm\psi\} =$

$= \{\psi \in E_{RW} \,|\, \psi^{ch_+} = \pm\psi\}, \quad p-q = 0 \mod 8,$

Proof: $E_W \Rightarrow n$ is even. Let $p-q = 2 \mod 4$, $\omega = ie^{1\ldots n}$, $E_{LW}: ie^{1\ldots n}\psi = -\psi$

$\Rightarrow B_+^{-1}(-ie^{1\ldots n})B_+\overleftarrow{\psi} = -\overleftarrow{\psi}$, $B_+\overleftarrow{\psi} = \pm\psi$, $\Rightarrow ie^{1\ldots n}\psi = \psi \Rightarrow E_{RW} \Rightarrow$ (?!) ■

$B_+\overleftarrow{\psi} = \pm\psi, \quad e^{1\ldots n}\psi = \psi \Rightarrow B_-\overleftarrow{\psi} = \pm\psi \Rightarrow$ (?!) ■

Dirac-Hestenes equation

$$\mathbb{R}^{1,3}, \qquad \mathbb{C}\otimes C\ell_{1,3}, \qquad \{e^0, e^1, e^2, e^3\}, \qquad t=\frac{1}{4}(e+E)(e-iI)$$

$$E:=e^0, \qquad I:=-e^{12}, \qquad t^2=t=t^\dagger, \qquad it=It, \qquad t=Et.$$

Theorem. $\forall U\in \mathrm{I}(t)$ the equation $Xt=U$ has unique solution $X\in C\ell^{(0)}_{1,3}$ and unique solution $X\in C\ell^{(1)}_{1,3}$.

Proof Orthonormal basis of left ideal $\mathrm{I}(t)$:

$$\tau_k=F_k t, \quad k=1,2,3,4, \quad F_1=2e, F_2=2e^{13}, F_3=2e^{03}, F_4=2e^{01}\in C\ell^{0)}_{1,3},$$

$$U=(\alpha^k+i\beta^k)\tau_k, \qquad \alpha^k,\beta^k\in\mathbb{R},$$

1) Using $it=It$, we obtain that $X=F_k(\alpha^k+I\beta^k)\in C\ell^{(0)}_{1,3}$ is solution of $Xt=U$.
2) Let us prove: if the element $Y\in C\ell^{(0)}_{1,3}$ is solution of equation $Yt=0$, then $Y=0$. We have for element $Yt\in \mathrm{I}(t)$:

$$Yt=\frac{1}{2}((y-iy_{12})\tau_1+(-y_{13}-iy_{23})\tau_2+(y_{03}-iy_{0123})\tau_3+(y_{01}+iy_{02})\tau_4)=0.$$

3) Using $t=Et$, we obtain that $X=F_kE(\alpha^k+I\beta^k)\in C\ell^{(1)}_{1,3}$ is also solution of equation $Xt=U$. 4) The proof of uniqueness in this case is similar. ■

$$ie^\mu(\partial_\mu\psi - ia_\mu\psi) - m\psi = 0 \Rightarrow e^\mu(\partial_\mu\psi - ia_\mu\psi) + im\psi = 0, \quad \psi \in \mathrm{I}(t), \qquad (5)$$

Dirac-Hestenes equation: $$\boxed{e^\mu(\partial_\mu\Psi - a_\mu\Psi I)E + m\Psi I = 0, \quad \Psi \in C\ell^{(0)}_{1,3}} \qquad (6)$$

$$(\Leftarrow) \qquad (6)|t; \qquad Et = t, \qquad It = it, \qquad \Psi t = \psi \Rightarrow (5)$$

$$(\Rightarrow) \qquad (5), \quad \psi \in \mathrm{I}(t) \Rightarrow \exists\Psi \in C\ell^{(0)}_{1,3} : \Psi t = \psi \qquad Et = t, \qquad It = it \Rightarrow$$

$$\underbrace{(e^\mu(\partial_\mu\Psi - a_\mu\Psi I)E + m\Psi I)}_{\in C\ell^{(0)}_{1,3}} t = 0 \Rightarrow (6). \qquad \blacksquare$$

Hestenes D., Space-Time Algebra, Gordon and Breach, New York, (1966).

$$\dim \mathrm{I}(t) = \dim \mathbb{C}^4 = 8, \qquad \dim C\ell^{(0)}_{1,3} = 8, \qquad \text{(real dimension)}$$

Thank you for your attention!