Some Cheeger-Gromov-Taylor Type Compactness Theorems for Ricci Solitons

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Aim & Plan

1. **Introduction**

   A Brief Review of **Ricci Solitons**

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   A Brief Review of Some Classical Compactness Theorems

2. **Results**

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   – **Ambrose** and **Cheeger-Gromov-Taylor Type Theorems** via **Bakry-Èmery** and **Modified Ricci Curvatures**

Notation

(M, g) : an n-dimensional **connected** Riemannian manifold **without boundary**,

∇ : the Levi-Civita connection with respect to g,

$\mathfrak{X}(M)$: the set of **smooth** vector fields on M,

$\mathcal{C}^\infty(M)$: the set of **smooth** functions on M.

A **gradient vector field** and a **Hessian** of $f \in \mathcal{C}^\infty(M)$ are defined by

$$g(\nabla f, X) = df(X) \quad \text{and} \quad \mathrm{Hess}\, f(X, Y) = g(\nabla_X \nabla f, Y), \quad X, Y \in \mathfrak{X}(M),$$

respectively. A **curvature tensor**, a **Ricci curvature**, and a **scalar curvature** are defined by

$$R(X, Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,$$

$$\mathrm{Ric}_g(X, Y) := \sum_{i=1}^{n} g(R(e_i, X)Y, e_i), \quad \text{and} \quad R := \sum_{i=1}^{n} \mathrm{Ric}_g(e_i, e_i),$$

respectively. Here $X, Y, Z \in \mathfrak{X}(M)$ and $\{e_i\}_{i=1}^n$ is an orthonormal frame of M.

Ricci Solitons

Definition (Hamilton 1982) A **Ricci soliton** is a Riemannian manifold (M, g) admitting a **vector field** $V \in \mathfrak{X}(M)$ and a **real constant** $\lambda \in \mathbb{R}$ such that

$$\mathrm{Ric}_g + \frac{1}{2}\mathcal{L}_V g = \lambda g,$$

where $\mathcal{L}_V$ is the Lie derivative in the direction of V. The soliton (M, g) is

shrinking if $\lambda > 0$, **steady** if $\lambda = 0$, **expanding** if $\lambda < 0$.

Remark A typical example of Ricci solitons is an **Einstein manifold**, where V is given by a Killing vector field. In this case, we say that the soliton is **trivial**.

If $V = \nabla f$ for some smooth function $f : M \to \mathbb{R}$, then the soliton (M, g) is called a **gradient Ricci soliton**. In such a case, the soliton satisfies

$$\mathrm{Ric}_g + \mathrm{Hess}\, f = \lambda g.$$

We refer to f as a **potential function** of the gradient Ricci soliton.

A **gradient Ricci soliton** $\boxed{\mathrm{Ric}_g + \mathrm{Hess}\, f = \lambda g}$

A typical example of gradient Ricci solitons is the **Gaussian soliton** $(\mathbb{R}^n, g_0)$.

- g_0 is the canonical flat metric on $\mathbb{R}^n$.
- a potential function $f : \mathbb{R}^n \to \mathbb{R}$ is given by $f(\boldsymbol{x}) = \frac{\lambda}{2}|\boldsymbol{x}|^2$ ($\lambda > 0$ or $\lambda < 0$).

Remark The Gaussian soliton is **non-compact**.

A **Ricci soliton** $\boxed{\mathrm{Ric}_g + \frac{1}{2}\mathcal{L}_V g = \lambda g}$

Given a Ricci soliton (M, g), we may define a time-dependent vector field by $W_t := -\frac{1}{2\lambda t}V$. We denote by φ_t the flow generated by W_t. The metric

$$g(t) = -2\lambda t \varphi_t^* g$$

is a solution to the **Ricci flow**

$$\frac{\partial g}{\partial t}(t) = -2\,\mathrm{Ric}_{g(t)} \,.$$

Background and Motivation

A **Ricci soliton**

$$\boxed{\mathrm{Ric}_g + \frac{1}{2}\mathcal{L}_V g = \lambda g}$$

- Introduced by Hamilton (1982)
- A **natural generalization** of an Einstein manifold
- The Ricci flow has achieved great success in **finding canonical metrics**
- Ricci solitons play important roles in the Ricci flow
  - Correspond to **self-similar solutions** to the flow
  - Arise as **singularity models** of the flow
- Intimately related to Li-Yau-Hamilton type estimates
  - Steady Ricci solitons **achieve the equality** in the estimates
- Ricci solitons play important roles in **Superstring Theory**

Properties of Ricci Solitons

A **Ricci soliton** $\boxed{\mathrm{Ric}_g + \frac{1}{2}\mathcal{L}_V g = \lambda g}$ is

shrinking if $\lambda > 0$, **steady** if $\lambda = 0$, **expanding** if $\lambda < 0$.

Theorem (Perelman 2002) Any compact Ricci soliton must be **gradient**.

Theorem (Hamilton 1993) Any compact **steady** or **expanding** **gradient** Ricci soliton must be **trivial**.

Theorem (Hamilton 1986 for $n = 2$, Ivey 1992 for $n = 3$) In **dimension $n \leqslant 3$**, any compact **shrinking** Ricci soliton must be **trivial**.

Some Examples of Ricci Solitons

Compact Case

- Koiso 1990, Cao 1994 : **shrinking** gradient **Kähler** Ricci solitons
  – $\mathbb{P}^1(\mathbb{C})$-bundles over Kähler-Einstein manifolds, e.g. $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$
- Wang and Zhu 2003 : **shrinking** gradient **Kähler** Ricci solitons
  – Toric Fano Kähler manifolds, e.g. $\mathbb{CP}^2 \# \overline{2\mathbb{CP}^2}$

Complete Non-Compact Case

- Hamilton 1986 : **steady** gradient Ricci solitons
  – Cigar solitons $\left(\mathbb{R}^2, \frac{dx^2+dy^2}{1+x^2+y^2}\right)$ with $f = -\log(1 + x^2 + y^2)$
- Futaki and Wang 2009 : **expanding** gradient **Kähler** Ricci solitons
  – Cone manifolds over compact Sasaki manifolds

Curvature and Topology

One of the most natural and important topics in Riemannian geometry is the relation between **curvature** and **topology** of underlying manifolds.

Theorem (Lohkamp 1992) Any n-dimensional manifold M, $n \geqslant 3$, admits a complete Riemannian metric g whose Riemannian curvature satisfies

$$-a(n) \leqslant \mathrm{Rm} \leqslant -b(n),$$

where $a(n) > b(n) > 0$ are positive constants depending only on n.

Corollary (Lohkamp 1992) Any n-dimensional manifold M, $n \geqslant 3$, admits a complete Riemannian metric g whose Ricci curvature is everywhere negative.

Remark Corollary above says that there are **no obstructions** to the existence of complete Riemannian metrics with negative Ricci curvature.

Myers's Theorem

Natural questions to ask about a complete Riemannian manifold (M, g) are

- When is M **compact**?
- How large is the **diameter** of M?

Theorem (Myers 1941) Let (M, g) be an n-dimensional complete Riemannian manifold. If there exists **a positive constant $\lambda > 0$** such that

$$\mathrm{Ric}_g(X, X) \geqslant \lambda g(X, X), \quad X \in \mathfrak{X}(M),$$

then M is compact with finite fundamental group. Moreover,

$$\mathrm{diam}(M, g) \leqslant \pi \sqrt{\frac{n-1}{\lambda}}.$$

Ambrose's Theorem

Theorem (Ambrose 1956) Let (M, g) be a complete Riemannian manifold. Suppose that there exists **a point $p \in M$** for which every geodesic $\gamma : [0, +\infty) \to M$ emanating from p satisfies

$$\int_0^{+\infty} \mathrm{Ric}_g(\dot{\gamma}(t), \dot{\gamma}(t))dt = +\infty.$$

Then (M, g) is compact.

Remark Ambrose's theorem above **does not require** the Ricci curvature to be everywhere non-negative. Moreover, since

$$\mathrm{Ric}_g \geqslant \lambda g \quad (\lambda > 0) \quad \Longrightarrow \quad \int_0^{+\infty} \mathrm{Ric}_g(\dot{\gamma}(t), \dot{\gamma}(t))dt = +\infty,$$

the compactness result in Myers's theorem follows from Ambrose's theorem.

Cheeger-Gromov-Taylor's Theorem

Theorem (Cheeger, Gromov, and Taylor 1981) Let (M, g) be an n-dimensional complete Riemannian manifold. Suppose that there exist **a point $p \in M$** and **positive constants $r_0 > 0$ and $v > 0$** such that

$$\mathrm{Ric}_g(x) \geqslant (n-1)\frac{\left(\frac{1}{4}+v^2\right)}{d^2(x,p)}$$

for all $x \in M$ satisfying $d(x,p) \geqslant r_0$, where $d(x,p)$ is the distance between x and p. Then (M, g) is compact. Moreover, the diameter from p satisfies

$$\mathrm{diam}_p(M,g) \leqslant r_0 \exp\left(\frac{\pi}{v}\right).$$

Remark Theorem above is **not true** if $v = 0$. In fact, the Euclidean space $\mathbb{R}^n$ equipping with the metric $dr^2 + rg(\theta)$ outside some compact set is not compact and the Ricci curvature satisfies the condition as in Theorem with $v = 0$, where $g(\theta)$ is the standard metric on the sphere $\mathbb{S}^{n-1}$.

Modified and Bakry-Émery Ricci Curvatures

Let (M, g) be an n-dimensional complete Riemannian manifold, $V \in \mathfrak{X}(M)$ and $f \in \mathcal{C}^\infty(M)$. We put

$$\mathrm{Ric}_V := \mathrm{Ric}_g + \frac{1}{2}\mathcal{L}_V g, \quad \mathrm{Ric}_f := \mathrm{Ric}_g + \mathrm{Hess}\, f$$

and call them a **modified Ricci curvature** and a **Bakry-Émery Ricci curvature**, respectively. We also put

$$\Delta_V := \Delta_g - V \cdot \nabla, \quad \Delta_f := \Delta_g - \nabla f \cdot \nabla$$

and call them a **V-Laplacian** and a **Witten-Laplacian**, respectively.

- **Good substitutes** of the Ricci curvature and the Laplacian to establish

Eigenvalue estimates, **Li-Yau Harnack inequalities**, **Liouville theorems**, $\cdots$

Question Are classical results for Einstein manifolds true for the solitons ?

Remark The shrinking Gaussian soliton $(\mathbb{R}^n, g_0)$ is **non-compact**.

A Compactness Theorem for Ricci Solitons

Theorem (Fernández-López and García-Río 2004) Let (M, g) be a complete Riemannian manifold satisfying

$$\mathrm{Ric}_V := \mathrm{Ric}_g + \frac{1}{2}\mathcal{L}_V g \geqslant \lambda g$$

for **a positive constant $\boldsymbol{\lambda > 0}$**. Then

$$M \text{ is compact} \quad \Longleftrightarrow \quad |V| \text{ is bounded on } M.$$

Moreover, if M is compact, then the fundamental group satisfies

$$|\pi_1(M)| < +\infty.$$

Theorem (Wylie 2007) The finiteness of $\pi_1(M)$ remains valid under the completeness of (M, g) and a positive lower bound on Ric_V.

A Diameter Bound via Ric_{V}

Theorem (Limoncu 2009) Let (M, g) be an n-dimensional complete Riemannian manifold satisfying

$$\mathrm{Ric}_V := \mathrm{Ric}_g + \frac{1}{2}\mathcal{L}_V g \geqslant \lambda g$$

for **a positive constant $\lambda > 0$**. If $|V| \leqslant k$ for a non-negative constant $k \geqslant 0$, then M is compact. Moreover,

$$\mathrm{diam}(M, g) \leqslant \frac{\pi}{\lambda}\left(\frac{k}{\sqrt{2}} + \sqrt{\frac{k^2}{2} + (n-1)\lambda}\right).$$

Remark Recently, under the same assumption as in Theorem above, the upper diameter bound above was improved to

$$\mathrm{diam}(M, g) \leqslant \frac{1}{\lambda}\left(2k + \sqrt{4k^2 + (n-1)\lambda\pi^2}\right).$$

By taking $k = 0$, Theorem above is reduced to the Myers's theorem.

A Diameter Bound via Ric_{f}

Theorem (Wei and Wylie 2007) Let (M, g) be an n-dimensional complete Riemannian manifold satisfying

$$\mathrm{Ric}_f := \mathrm{Ric}_g + \mathrm{Hess}\, f \geqslant \lambda g$$

for **a positive constant $\lambda > 0$**. If $|f| \leqslant k$ for a non-negative constant $k \geqslant 0$, then M is compact. Moreover,

$$\mathrm{diam}(M, g) \leqslant \pi\sqrt{\frac{n-1}{\lambda}} + \frac{4k}{\sqrt{(n-1)\lambda}}.$$

Remark Recently, under the same assumption as in Theorem above, the upper diameter bound above was improved to

$$\mathrm{diam}(M, g) \leqslant \frac{\pi}{\sqrt{\lambda}}\sqrt{n - 1 + \frac{8k}{\pi}}.$$

By taking $k = 0$, Theorem above is reduced to the Myers's theorem.

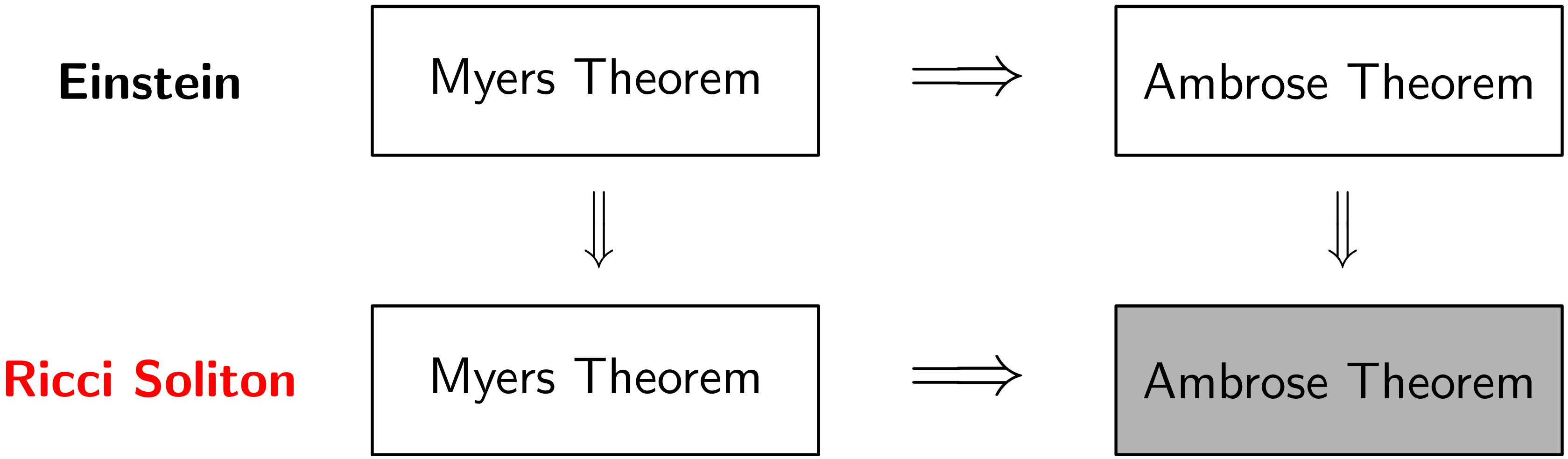
Einstein	Myers Theorem	$\Longrightarrow$	Ambrose Theorem
	$\Downarrow$		$\Downarrow$
Ricci Soliton	Myers Theorem	$\Longrightarrow$	Ambrose Theorem

Theorem (— 2015) Let (M, g) be a complete Riemannian manifold. Suppose that there exists **a point** $\boldsymbol{p \in M}$ for which every geodesic $\gamma : [0, +\infty) \to M$ emanating from p satisfies

$$\int_0^{+\infty} \mathrm{Ric}_V(\dot{\gamma}(t), \dot{\gamma}(t))dt = +\infty$$

and $|V| \leqslant \boldsymbol{k}$ for a non-negative constant $\boldsymbol{k \geqslant 0}$. Then (M, g) is compact.

Remark An Ambrose type theorem above was already proved by Zhang (2013) in the case where $\boldsymbol{V = \nabla f}$.



Theorem (— 2016) Let (M, g) be a complete Riemannian manifold. Suppose that there exists **a point** $\boldsymbol{p} \in \boldsymbol{M}$ for which every geodesic $\gamma : [0, +\infty) \to M$ emanating from p satisfies

$$\int_0^{+\infty} \mathrm{Ric}_f(\dot\gamma(t), \dot\gamma(t))dt = +\infty$$

and $|\boldsymbol{f}| \leqslant \boldsymbol{k}$ for a non-negative constant $\boldsymbol{k} \geqslant \mathbf{0}$. Then (M, g) is compact.

Some Cheeger-Gromov-Taylor Type Compactness Theorems via Modified Ricci Curvature

Theorem (Soylu 2016) Let (M, g) be an n-dimensional complete Riemannian manifold. Suppose that there exist **a point $p \in M$** and **positive constants $r_0 > 0$ and $v > 0$** such that

$$\mathrm{Ric}_f(x) \geqslant (n-1)\frac{\left(\frac{1}{4}+\boldsymbol{v}^2\right)}{d^2(x,p)}$$

for all $x \in M$ satisfying $d(x,p) \geqslant r_0$, where $d(x,p)$ is the distance between x and p. If $|\boldsymbol{f}| \leqslant (\boldsymbol{n}-\mathbf{1})\boldsymbol{k}$ for a non-negative constant $\boldsymbol{k} \geqslant \mathbf{0}$, then (M, g) is compact. Moreover, the diameter from p satisfies

$$\mathrm{diam}_p(M,g) \leqslant r_0 \exp\left(\frac{1}{v^2}\sqrt{8\boldsymbol{k}^2+\pi^2v^2+4\boldsymbol{k}\sqrt{4\boldsymbol{k}^2+\pi^2v^2\left(1+4v^2\right)}}\right).$$

Remark By taking $\boldsymbol{k} = \mathbf{0}$, Theorem above is reduced to the Cheeger-Gromov-Taylor's compactness theorem.

Recall Wei and Wylie (2007) proved that a complete Riemannian manifold (M, g) with a positive lower bound on the Bakry-Émery Ricci curvature $\mathbf{Ric}_{\boldsymbol{f}} \geqslant \boldsymbol{\lambda g}$ $(\lambda > 0)$ is compact if $|\boldsymbol{f}| \leqslant \boldsymbol{k}$ for some non-negative constant $\boldsymbol{k} \geqslant \mathbf{0}$.

Theorem (Limoncu 2011) Let (M, g) be an n-dimensional complete Riemannian manifold satisfying

$$\mathrm{Ric}_f \geqslant \lambda g$$

for **a positive constant** $\boldsymbol{\lambda} > \mathbf{0}$. Suppose that there exist **a point** $\boldsymbol{p} \in \boldsymbol{M}$ and a non-negative constant $\boldsymbol{k} \geqslant \mathbf{0}$ such that

$$|\boldsymbol{\nabla f}|(\boldsymbol{x}) \leqslant \frac{\boldsymbol{k}}{\boldsymbol{d(x, p)}}$$

for all $x \in M \setminus \{p\}$, where $d(x, p)$ is the distance between x and p. Then (M, g) is compact. Moreover, the diameter from p satisfies

$$\mathrm{diam}_p(M, g) \leqslant \frac{\pi}{\sqrt{\lambda}} \sqrt{n - 1 + 4\boldsymbol{k}}.$$

Theorem (— 2016) Let (M, g) be an n-dimensional complete Riemannian manifold. Suppose that there exist **a point $p \in M$** and **positive constants $r_0 > 0$ and $v > 0$** such that

$$\mathrm{Ric}_V(x) \geqslant (n-1)\frac{\left(\frac{1}{4} + v^2\right)}{d^2(x,p)}$$

for all $x \in M$ satisfying $d(x,p) \geqslant r_0$, where $d(x,p)$ is the distance between x and p. If there exists a non-negative constant $k \geqslant 0$ such that

$$|V|(x) \leqslant \frac{(n-1)k}{d(x,p)} \quad \text{and} \quad k < v^2$$

for all $x \in M \setminus \{p\}$. Then (M, g) is compact. Moreover, the diameter from p satisfies

$$\mathrm{diam}_p(M,g) \leqslant r_0 \exp\left(\frac{2k + \sqrt{4k^2 + (v^2 - k)\pi^2}}{v^2 - k}\right).$$

Remark By taking $k = 0$, Theorem above is reduced to the Cheeger-Gromov-Taylor's compactness theorem.

m-Modified and m-Bakry-Émery Ricci Curvatures

Let (M, g) be an n-dimensional complete Riemannian manifold, $V \in \mathfrak{X}(M)$ and $f \in \mathcal{C}^\infty(M)$. For $m \in [n, +\infty)$, we put

$$\mathrm{Ric}_V^m := \begin{cases} \mathrm{Ric}_g & m = n, \\ \mathrm{Ric}_g + \frac{1}{2}\mathcal{L}_V g - \frac{1}{m-n} V^* \otimes V^* & m > n, \end{cases}$$

$$\mathrm{Ric}_f^m := \begin{cases} \mathrm{Ric}_g & m = n, \\ \mathrm{Ric}_g + \mathrm{Hess}\, f - \frac{1}{m-n} df \otimes df & m > n \end{cases}$$

and call them an **m-modified Ricci curvature** and an **m-Bakry-Émery Ricci curvature**, respectively. Here V^* is the metric dual of V with respect to g.

- **Good substitutes** of the Ricci curvature
- Important tools in **Optimal Transport Theory** by Lott, Sturm and Villani
- Play important roles in **Perelman's entropy formulas** for the Ricci flow

Theorem (Limoncu 2009) Let (M, g) be an n-dimensional complete Riemannian manifold. Suppose that there exists **a positive constant $\lambda > 0$** such that

$$\mathrm{Ric}_V^m \geqslant \lambda g,$$

where $m \in [n, +\infty)$. Then (M, g) is compact. Moreover,

$$\mathrm{diam}(M, g) \leqslant \frac{\pi}{\sqrt{\lambda}} \sqrt{m - 1}.$$

Remark The Myers type theorem above was already proved by Qian (1995) in the case where $V = \nabla f$.

Ric_g	**Myers type**	**Ambrose type**	**C-G-T type**
Ric_V	FL-GR, Limoncu, —	Zhang, —	—
Ric_f	Wei-Wylie, Limoncu, —	—	Soylu
Ric_V^m	**Limoncu**	???	???
Ric_f^m	Qian	???	???

Theorem (— 2015) Let (M, g) be an n-dimensional complete Riemannian manifold. Suppose that there exists **a point $p \in M$** for which every geodesic $\gamma : [0, +\infty) \to M$ emanating from p satisfies

$$\int_0^{+\infty} \mathrm{Ric}_V^m(\dot{\gamma}(t), \dot{\gamma}(t))dt = +\infty,$$

where $m \in [n, +\infty)$. Then (M, g) is compact.

Remark The Ambrose type theorem above was already proved by Cavalcante, Oliveira and Santos (2015) in the case where $V = \nabla f$. The **key ingredient** in proving an Ambrose type theorem above is the Riccati inequality for the m-modified Ricci curvature

$$\boxed{\mathrm{Ric}_V^m(\dot{\gamma}, \dot{\gamma}) \leqslant -\dot{m}_V - \frac{(m_V)^2}{m-1}, \quad m_V := \Delta_V d},$$

which may be derived by applying the Bochner-Weitzenböck formula

$$\frac{1}{2}\Delta_V |\nabla u|^2 = |\operatorname{Hess} u|^2 + \mathrm{Ric}_V(\nabla u, \nabla u) + g(\nabla \Delta_V u, \nabla u), \quad u \in \mathcal{C}^\infty(M)$$

to the distance function $u(x) = d(x, p)$. This formula was proved by Li (2014).

A Cheeger-Gromov-Taylor Type Compactness Theorem via m-Modified Ricci Curvature

Theorem (— 2016) Let (M, g) be an n-dimensional complete Riemannian manifold. Suppose that there exist **a point $p \in M$** and **positive constants $r_0 > 0$ and $v > 0$** such that

$$\mathrm{Ric}_V^m(x) \geqslant (m-1)\frac{\left(\frac{1}{4} + \boldsymbol{v}^2\right)}{d^2(x,p)}$$

for all $x \in M$ satisfying $d(x,p) \geqslant r_0$, where $m \in [n, +\infty)$ and $d(x,p)$ is the distance between x and p. Then (M, g) is compact. Moreover, the diameter from p satisfies

$$\mathrm{diam}_p(M,g) \leqslant r_0 \exp\left(\frac{\pi}{v}\right).$$

Remark A similar Cheeger-Gromov-Taylor type compactness theorem via m-Bakry-Émery Ricci curvature was established by Wang (2013).

Some Myers Type Theorems via (m-) Bakry-Émery and (m-) Modified Ricci Curvatures

Ric_g	Myers	Ambrose	Cheeger-Gromov-Taylor
Ric_V	FL-GR, Limoncu, ——	Zhang, ——	——
Ric_f	Wei-Wylie, Limoncu, ——	——	Soylu
Ric_V^m	Limoncu	——	Wang, ——
Ric_f^m	Qian	C-O-S	——

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Thank You for Your Attention!

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