A non-geometric interpretation of relativistic spin structure

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Playing field

Let M be a 4-manifold, local coordinates $x = (x^1, x^2, x^3, x^4)$.

A half-density is a quantity $M \to \mathbb{C}$ which under changes of local coordinates transforms as the square root of a density.

Will work with 2-columns $v : M \to \mathbb{C}^2$ of half-densities.

Inner product $\langle v, w \rangle := \int_M w^* v \, dx$, where $dx = dx^1 dx^2 dx^3 dx^4$.

Note: if M is equipped with a prescribed positive density ρ then half-densities can be identified with scalar fields. Just a matter of multiplying or dividing by $\sqrt{\rho}$.

Want to study a formally self-adjoint first order linear differential operator L acting on 2-columns of complex-valued half-densities.

Need an invariant analytic description of my differential operator.

In local coordinates my operator reads

$$L = F^\alpha(x)\frac{\partial}{\partial x^\alpha} + G(x),$$

where $F^\alpha(x)$ and $G(x)$ are some 2×2 matrix-functions.

The principal and subprincipal symbols are defined as

$$L_{\mathrm{prin}}(x,p) := iF^\alpha(x)\, p_\alpha,$$

$$L_{\mathrm{sub}}(x) := G(x) + \frac{i}{2}(L_{\mathrm{prin}})_{x^\alpha p_\alpha}(x) = G(x) - \frac{1}{2}(F^\alpha)_{x^\alpha}(x),$$

where $p=(p_1,p_2,p_3,p_4)$ is the dual variable (momentum).

Fact: L_{prin} and L_{sub} are invariantly defined 2×2 Hermitian matrix-functions on T^*M and M respectively.

Fact: L_{prin} and L_{sub} uniquely determine the operator L.

Emergence of parallelizability

We say that our operator L is *non-degenerate* if

$$L_{\mathsf{prin}}(x,p) \neq 0, \qquad \forall (x,p) \in T^*M \setminus \{0\}.$$

Lemma 1 A manifold M admits a non-degenerate operator L if and only if it is parallelizable.

Lorentzian metric appears out of thin air

The determinant of the principal symbol is a quadratic form in momentum

$$\det L_{\mathrm{prin}}(x,p) = -g^{\alpha\beta}(x)\,p_\alpha p_\beta \tag{1}$$

and the coefficients $g^{\alpha\beta}(x) = g^{\beta\alpha}(x)$, $\alpha,\beta = 1,2,3,4$, can be interpreted as components of a (contravariant) metric tensor.

Lemma 2 My metric is Lorentzian, i.e. the metric tensor $g^{\alpha\beta}(x)$ has three positive eigenvalues and one negative eigenvalue.

Time-orientability

Lemma 3 A parallelizable Lorentzian manifold (M, g) admits a non-degenerate operator L satisfying condition (1) if and only if it is time-orientable.

Extracting more geometry and topology from operators

Assume that our time-orientable Lorentzian metric is fixed. Work with all possible 2×2 formally self-adjoint non-degenerate first order linear differential operators corresponding, in the sense of formula (1), to the given metric.

Want to classify operators corresponding to the given metric.

Take an arbitrary smooth matrix-function

$$R : M \to SL(2,\mathbb{C}) \tag{2}$$

and consider the transformation of the differential operator

$$L \mapsto R^* L R.$$

This induces the transformation of the principal symbol

$$L_{\mathsf{prin}} \mapsto R^* L_{\mathsf{prin}} R.$$

Note: $\det L_{\mathsf{prin}}$ is preserved, hence, metric is preserved.

Definition 1 We say that the operators L and $\tilde{L}$ are *equivalent* if

$$\tilde{L}_{\mathsf{prin}} = R^* L_{\mathsf{prin}} R$$

for some smooth matrix-function (2). An equivalence class of operators is called *spin structure*.

Main result

Theorem 1 For parallelizable time-orientable Lorentzian 4-manifolds the two definitions of spin structure, our analytic definition and the traditional one, are equivalent.

In collaboration with topologist Nikolai Saveliev (University of Miami).

Z. Avetisyan, Y.-L. Fang, N. Saveliev and D. Vassiliev, Analytic definition of spin structure. Preprint arXiv:1611.08297 (2016).

Classification beyond spin structure

Subprincipal symbol transforms as

$$L_{\mathrm{sub}} \mapsto R^* L_{\mathrm{sub}} R + \frac{i}{2}\left(R^*_{x^\alpha}(L_{\mathrm{prin}})_{p_\alpha} R - R^*(L_{\mathrm{prin}})_{p_\alpha} R_{x^\alpha}\right).$$

Problem: subprincipal symbol does not transform covariantly.

Solution: define *covariant* subprincipal symbol $L_{\mathsf{csub}}(x)$ as

$$L_{\mathsf{csub}} := L_{\mathsf{sub}} + \frac{i}{16}\, g_{\alpha\beta}\{L_{\mathsf{prin}}, \operatorname{adj} L_{\mathsf{prin}}, L_{\mathsf{prin}}\}_{p_\alpha p_\beta}\,,$$

where

$$\{U,V,W\} := U_{x^\alpha} V W_{p_\alpha} - U_{p_\alpha} V W_{x^\alpha}$$

is the generalised Poisson bracket on matrix-functions and adj is the operator of matrix adjugation

$$U = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} =: \operatorname{adj} U$$

from elementary linear algebra.

Electromagnetic covector potential appears out of thin air

The covariant subprincipal symbol can be uniquely represented as

$$L_{\mathsf{csub}}(x) = L_{\mathsf{prin}}(x, A(x)), \tag{3}$$

where A is a real-valued covector field which is invariant under gauge transformations.

Explanation: the matrices $(L_{\mathsf{prin}})_{p_\alpha}$, $\alpha = 1, 2, 3, 4$, are Pauli matrices and these form a basis in the real vector space of 2×2 Hermitian matrices. Formula (3) is simply an expansion of the matrix L_{csub} with respect to the basis of Pauli matrices.

3-dimensional Riemannian geometry

1. More restrictive choice of operators: $\operatorname{tr} L_{\text{prin}}(x,p) = 0$.

2. My non-degeneracy condition is now equivalent to the more familiar ellipticity condition $\det L_{\text{prin}}(x,p) \neq 0$.

3. A 3-manifold admits a 2×2 first order elliptic operator with trace-free principal symbol if and only if it is parallelizable.

4. A 3-manifold is parallelizable if and only if it is orientable.

5. My metric is automatically Riemannian: $\det L_{\text{prin}}(x,p) < 0$.

6. More restrictive choice of gauge transformations:

$$R : M \to SU(2).$$

Examples from 3-dimensional Riemannian geometry

1. $\mathbb{S}^3$ has a unique spin structure.

2. $\mathbb{T}^3$ has eight distinct spin structures.

Two different spin structures on $\mathbb{T}^3$

$$L_{\mathrm{prin}}(x,p)=\begin{pmatrix} p_3 & p_1-ip_2 \\ p_1+ip_2 & -p_3 \end{pmatrix},$$

$$\begin{aligned} L_{\mathrm{prin}}(x,p) &= \begin{pmatrix} p_3 & e^{ix^3}(p_1-ip_2) \\ e^{-ix^3}(p_1+ip_2) & -p_3 \end{pmatrix} \\ &= \begin{pmatrix} e^{\frac{i}{2}x^3} & 0 \\ 0 & e^{-\frac{i}{2}x^3} \end{pmatrix} \begin{pmatrix} p_3 & p_1-ip_2 \\ p_1+ip_2 & -p_3 \end{pmatrix} \begin{pmatrix} e^{-\frac{i}{2}x^3} & 0 \\ 0 & e^{\frac{i}{2}x^3} \end{pmatrix}. \end{aligned}$$

Here we use cyclic coordinates x^α, $\alpha=1,2,3$, of period 2π.

Special unitary matrix-function in latter formula is discontinuous.

Massless Dirac operator on Riemannian 3-manifold

Elliptic self-adjoint 2×2 first order linear differential operator with trace-free principal symbol and zero covariant subprincipal symbol.

Two different massless Dirac operators on $\mathbb{T}^3$

$$-i\begin{pmatrix} \partial_3 & \partial_1 - i\partial_2 \\ \partial_1 + i\partial_2 & -\partial_3 \end{pmatrix},$$

$$-i\begin{pmatrix} \partial_3 & e^{ix^3}(\partial_1 - i\partial_2) \\ e^{-ix^3}(\partial_1 + i\partial_2) & -\partial_3 \end{pmatrix} - \frac{1}{2}I.$$

Their spectra are different.