Natural Discretization in Noncommutative Field Theory

Ciprian Acatrinei
Department of Theoretical Physics
Horia Hulubei National Institute for Nuclear Physics
Bucharest, Romania

7 June 2018

Geometry, Integration, Quantization – Varna, 2-7 June 2018

Scalars

Consider a (2+1)-dimensional scalar field ϕ, depending on space coordinates forming a Heisenberg algebra (time remains commutative):

$$\phi(t, \hat{x}, \hat{y}), \qquad [\hat{x}, \hat{y}] = i\theta \hat{\mathbb{1}} \ . \tag{1}$$

θ is a constant with the dimensionality of an area. ϕ is a time-dependent scalar operator acting on the Hilbert space $\mathcal{H}$ on which the algebra is represented. (1) implies $[\hat{x}, \phi(\hat{x}, \hat{y})] = i\theta \frac{\partial \phi}{\partial \hat{y}}$ and $[\hat{y}, \phi(\hat{x}, \hat{y})] = -i\theta \frac{\partial \phi}{\partial \hat{x}}$. The field action reads (allow for $\phi^\dagger \neq \phi$)

$$S = \int \mathrm{d}t \, \mathrm{Tr}_{\mathcal{H}} \left(\frac{1}{2} \dot{\phi}^\dagger \dot{\phi} + \frac{1}{2} [\hat{x}, \phi^\dagger][\hat{x}, \phi] + \frac{1}{2} [\hat{x}, \phi^\dagger][\hat{y}, \phi] - V(\phi^\dagger \phi) \right) \tag{2}$$

If $V(\phi^\dagger \phi) = 0$, the equations of motion for ϕ are

$$\ddot{\phi} + \frac{1}{\theta^2} [\hat{x}, [\hat{x}, \phi]] + \frac{1}{\theta^2} [\hat{y}, [\hat{y}, \phi]] = 0. \tag{3}$$

In Cartesian coordinates, the solutions are plane waves

$$\phi \sim e^{i(k_x \hat{x} + k_y \hat{y}) - i\omega t}, \quad k_x^2 + k_y^2 = \omega^2, \tag{4}$$

only formally identical to the commutative ones: due to the operators in the exponent, such waves have bilocal character. If a mass term $m^2 \phi^\dagger \phi / 2$ is inserted in the action, $\omega^2 \longrightarrow \omega^2 - m^2$.

Continuous representation

Consider again $\phi(t, \hat{x}, \hat{y})$; $[\hat{x}, \hat{y}] = i\theta\hat{I}$; $\hat{x}$, $\hat{y}$: $\mathcal{H} \to \mathcal{H}$. Choose the basis $\{|x>\}$ of eigenstates of $\hat{x}$: $\hat{x}|x>= x|x>$, $\hat{y}|x>= -i\theta\frac{\partial}{\partial x}|x>$.
To quantize ϕ, promote normal modes expansion coefficients a and a^*. to annihilation/creation operators a, $a^\dagger$ on a standard Fock space $\mathcal{F}$.
To introduce NC space, apply Weyl (not Weyl-Moyal!) quantization to the exponentials $e^{i(k_x x + k_y y)}$ (the normal modes). The result is

$$\phi = \int\int \frac{dk_x dk_y}{2\pi\sqrt{2\omega_{\vec{k}}}} \left[\hat{a}_{k_x k_y} e^{i(\omega_{\vec{k}} t - k_x\hat{x} - k_y\hat{y})} + \hat{a}^\dagger_{k_x k_y} e^{-i(\omega_{\vec{k}} t - k_x\hat{x} - k_y\hat{y})}\right]. \quad (5)$$

ϕ acts on a direct product of two Hilbert spaces, $\phi : \mathcal{F}\otimes\mathcal{H} \to \mathcal{F}\otimes\mathcal{H}$.
Saturate the action of ϕ on $\mathcal{H}$ by working with expectation values $< x'|\phi|x >$: $\mathcal{F} \to \mathcal{F}$. Bilocality appears explicitely due to

$$< x'|e^{i(k_x\hat{x} + k_y\hat{y})}|x >= e^{ik_x(x + k_y\theta/2)}\delta(x' - x - k_y\theta) = e^{ik_x\frac{x+x'}{2}}\delta(x' - x - k_y\theta). \quad (6)$$

The span along the x axis is $(x' - x) = \theta k_y$; the energy is

$$\omega_{\vec{k}} = \sqrt{k_x^2 + \frac{\Delta x^2}{\theta^2} + m^2}. \quad (7)$$

Notice the intrinsic **IR/UV**-dual character of the dipoles: both big momentum (UV) and big extension (IR) give big energy.

Symmetries

Reintroduce the commutative z direction and use the notation

$$< x'|\phi|x > \equiv \phi(x', x) \equiv \phi(\bar{x}, \Delta x), \qquad \bar{x} \equiv \frac{x + x'}{2}, \qquad \Delta x \equiv (x' - x)$$

The free equation of motion for $\phi(t, \hat{x}, \hat{y}, z)$ follows from the action

$$S = Tr_H \int dt \int dz \left((\dot{\phi})^2 + \frac{1}{\theta^2}[\hat{x}, \phi]^2 + \frac{1}{\theta^2}[\hat{y}, \phi]^2 - (\partial_z \phi)^2 + m^2\phi^2 \right),$$

and reads $(\partial_t^2 - \partial_z^2 + m^2)\phi + \frac{1}{\theta^2}[\hat{y}, [\hat{y}, \phi]] + \frac{1}{\theta^2}[\hat{x}, [\hat{x}, \phi]] = 0$. Sandwiching it between $|x>$ states, one gets rid of NC and obtains the wave equation

$$\left(\partial_t^2 - \partial_{\bar{x}}^2 - \partial_z^2 + \frac{(x' - x)^2}{\theta^2} + m^2 \right) \phi(x, x') = 0$$

for a dipole living in (2+1) commutative dimensions at $t, \bar{x}, z$ *and* having extension Δx. Notice the full agreement with the dispersion relation (7). In the interacting case, the relevant Lagrangian is thus

$$2L = (\partial_t \phi)^2 - (\partial_{\bar{x}} \phi)^2 + [(\theta^{-1}\Delta x)^2 + m^2]\phi^2 - 2V(\phi)$$

and is invariant under Lorentz boosts along the $\bar{x}$-axis, and along the z-axis, independently (recall the tensorial character of $\theta = \theta_{xy} \sim xy$ and $\Delta x \sim x$). These bilocal representation symmetries are at variance with the Moyal approach claim $O(2)_{x-y} \otimes O(1,1)_{t-z}$.

Interactions

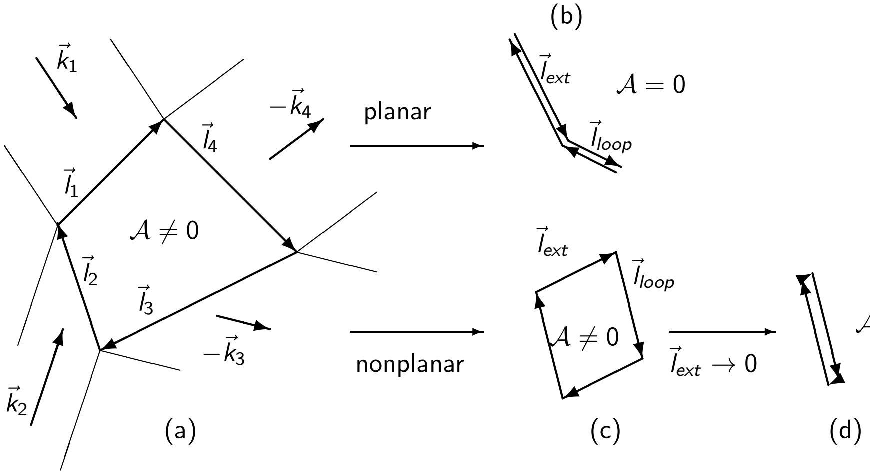

figure 1: Area versus finiteness

Radial Coordinates

If the physical situation requires polar coordinates (a source emiting radiation, a circular membrane oscillating), the harmonic oscillator (radial) basis $\{|n\rangle\}$

$$\hat{N}\,|n\rangle = n\,|n\rangle\,, \quad \hat{N} = \hat{a}^\dagger\hat{a}, \quad \hat{a} = \frac{1}{\sqrt{2\theta}}(\hat{x} + i\hat{y}), \quad n = 0, 1, 2, \ldots \qquad (8)$$

is the natural one. The equations of motion become

$$\ddot{\phi} + \frac{2}{\theta}[\hat{a}, [\hat{a}^\dagger, \phi]] = 0. \qquad (9)$$

$\hat{N} = \frac{1}{2}(\frac{\hat{x}^2+\hat{y}^2}{\theta} - 1)$ is basically (half) the radius square operator, in units of θ. Its eigenvalues n in (8) correspond to discrete points with radius growing like $\sqrt{n}$ ($n \sim \frac{r^2}{2\theta}$) for large n. The NC plane is realized via (8) as the semi-infinite discrete space of the points labeled by n.

Recurrence relation

Can sandwich any equation containing the operatorial field $\phi(\hat{x}, \hat{y}, t)$ between $\langle n'|$ and $|n\rangle$ states, eliminating NC in this way. The resulting field $\langle n'|\,\phi(t)\,|n\rangle \equiv \phi_{n',n}(t)$ is commutative and *bilocal*. If $\phi_{n',n}(t) = e^{i\omega t}\phi_{n',n}$ then $\hat{a}\,|n\rangle = \sqrt{n}\,|n-1\rangle$, $\hat{a}^\dagger\,|n\rangle = \sqrt{n+1}\,|n+1\rangle$ lead to the equation of motion for $\phi_{n',n}$

$$\sqrt{n'+1}\sqrt{n+1}\phi_{n'+1,n+1} + \sqrt{n'}\sqrt{n}\phi_{n'-1,n-1} - (n'+1+n-\lambda)\phi_{n',n} = 0. \tag{10}$$

Above, $\lambda = \frac{\theta}{2}\omega^2$. Eq. (10) is a recurrence relation of order two, describing the radial classical dynamics of a field which lives on a discrete space. The initial angular dependence of $\phi(\hat{x}, \hat{y}, t)$ is not lost. It is encoded in the two-index structure of $\phi_{n',n}$. The operator ϕ is reconstructed from the c-numbers $\phi_{n',n}$ via

$$\phi = \sum_{n',n\in\boldsymbol{N}} |n'\rangle\,\phi_{n',n}(t)\,\langle n|\,. \tag{11}$$

The bilocality of the fields $\phi_{n',n}$ thus implies the nonlocality of ϕ.

Radial symmetry

Recall

$$N|n\rangle = n|n\rangle\,, \quad N = a^\dagger a, \quad a = \frac{1}{\sqrt{2\theta}}(x+iy). \tag{12}$$

$N = \frac{1}{2}(\frac{x^2+y^2}{\theta} - 1)$ is basically the radius square operator, in units of θ.
If ϕ depends only on the combination of $\hat{x}$ and $\hat{y}$ given by $\hat{N}$, $\phi = \phi(\hat{N})$, have radial symmetry – recall that $\hat{N} = \frac{1}{2}(\frac{\hat{x}^2+\hat{y}^2}{\theta} - 1)$.
Then ϕ diagonal in the $|n\rangle$ basis: $\langle n'|\,\phi\,|n\rangle = \phi_{n',n}\delta_{n',n}$; *local* field!
Define $\phi_{n,n} \equiv \phi_n$. Then

$$(n+1)\phi_{n+1} + n\phi_{n-1} + (\lambda - 2n - 1)\phi_n = 0, \quad n = 0, 1, 2, \ldots \tag{13}$$

The expectation value $\phi_n \equiv \langle n|\,\phi\,|n\rangle$ characterizes $\phi = \sum_{n=0}^{\infty} |n\rangle\,\phi_n\,\langle n|$ uniformly at radius squared n. No angular dependence appears anymore.
If a single value ϕ_{n_0} is nonzero, then $|n_0\rangle\,\phi_{n_0}\,\langle n_0|$ describes a field located at n_0.
If the discrete derivative operator Δ is defined by

$$\Delta\phi_n = \phi_{n+1} - \phi_n. \tag{14}$$

we obtain the difference equation $(2\lambda/\theta = \omega^2 - m^2)$

$$n\Delta^2\phi_{n-1} + \Delta\phi_n + \lambda\phi_n = 0, \quad n = 0, 1, 2, \ldots \tag{15}$$

Solution of the equation of motion

$$(n+1)\phi_{n+1} + n\phi_{n-1} + (\lambda - 2n - 1)\phi_n = 0, \tag{16}$$

equivalent to the difference equation (15), describes travelling or stationary waves on the semi-infinite discrete space of points $n = 0, 1, 2, \ldots$ Two linearly independent solutions are (up to a multiplicative dimensionfull constant)

$$\phi_1(n) = \sum_{k=0}^{n} \frac{(-\lambda)^k}{k!}\ C_n^k\ , \tag{17}$$

$$\phi_2(n) = \sum_{k=0}^{n-1} \frac{(-\lambda)^k}{k!}\ \sum_{j=1}^{n-k} \frac{C_{n-j}^k}{k+j}\ . \tag{18}$$

They are *finite sums.* $\phi_2(n) = e^{-\lambda}\left(\tilde{\phi}_2(n) - \phi_1(n)\ \cdot \sum_{k=1}^{\infty} \frac{\lambda^k}{k!k}\right)$, where

$$\tilde{\phi}_2(n) = \sum_{k=0}^{n} \frac{(-\lambda)^k}{k!} C_n^k (H_{n-k} - 2H_k) + \frac{(-\lambda)^n}{n!} \sum_{s=1}^{\infty} \frac{\lambda^s (s-1)!}{[(n+s)!/n!]^2}. \tag{19}$$

H_k is a discrete version of the logarithmic function,

$$H_k = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k},\quad k = 1, 2, 3 \ldots\quad ; \qquad H_0 = 0. \tag{20}$$

Sources

$$(n+1)\phi(n+1) - (2n+1-\lambda)\phi(n) + n\phi(n-1) = j(n). \tag{21}$$

Consider first a nonzero source δ_{n,n_0}. Adapt the method of variation of constants to the discrete case

$$\phi_p(n) = c_1(n)\phi_1(n) + c_2(n)\phi_2(n). \tag{22}$$

Assuming $c_{1,2}(n)$ constant except for a jump at n_0,

$$c_i(n+1) - c_i(n) = f_i\delta_{n_0,n}, \qquad i = 1, 2, \tag{23}$$

obtain

$$f_1 = \frac{\phi_2(n_0)}{(n_0+1)W(n_0)} = \phi_2(n_0), \quad f_2 = -\frac{\phi_1(n_0)}{(n_0+1)W(n_0)} = -\phi_1(n_0). \tag{24}$$

$W(n)$ is the nonzero discrete Wronskian. In the physically most interesting case $n_0 = 0$ the difference equation (21) becomes first-order. The above method works the same, due to the simple Ansatz (23). From the physical point of view, the second solution ϕ_2, involved with radiation, is tied to a source at the origin.
The solution for an arbitrary distribution of charges $j(n), \forall n$, is now obtained by linear superposition of the above type of solutions.
It does **no**t display **singularities**.

Commutative limit

Consider the $n \to \infty$ limit (small θ limit). Using $\lambda = \theta\omega^2/2$ and $n = \frac{r^2}{2\theta} \to \infty$, $\phi_1(n)$ becomes, as a function of r,

$$\phi_1(n) \overset{n\to\infty}{\to} f_1(r) = \sum_{k=0}^{\infty} \frac{(-1)^k(\omega r)^{2k}}{(k!)^2 2^{2k}} = J_0(\omega r) \overset{r\to\infty}{\sim} \sqrt{\frac{2}{\pi\omega r}} cos(\omega r - \pi/4). \tag{25}$$

$f_1(r)$ is independent of θ. Similarly, ϕ_2 becomes

$$\phi_2(n) \to f_2(r) = \sum_{k=0}^{\infty} \frac{(-1)^k(\omega r)^{2k}}{(k!)^2 2^{2k}} [2ln(\omega r) - 2H_k + \gamma - ln(2\theta\omega^2)]. \tag{26}$$

γ is the Euler-Mascheroni constant, $\gamma = \lim_{k=\infty}(H_k - lnk) \simeq 0.5772$. $f_2(r)$ still depends on θ, via a logarithmic term; its $\theta \to 0$ limit is singular. Using the series expansion of the Bessel function of first (J_0) and second kind (Neumann function, Y_0), $f_2(r)/\pi = Y_0(\omega r) + (\gamma + ln(2\theta\omega^2))J_0(\omega r)$.

Dirac equation

Pick up 2-spinor in (2+1)-dimensions, $\Psi = \left(\begin{array}{c} \psi \\ \chi \end{array} \right)$. Commutative Dirac equation would be

$$i\partial_t \Psi = (-i\alpha_1 \partial_x - i\alpha_2 \partial_y + \beta m)\Psi. \tag{27}$$

Recall $[\hat{x}, \phi(\hat{x}, \hat{y})] = i\theta \frac{\partial \phi}{\partial \hat{y}}$, $[\hat{y}, \phi(\hat{x}, \hat{y})] = -i\theta \frac{\partial \phi}{\partial \hat{x}}$.
Take $\alpha_1 = \sigma_x, \alpha_2 = \sigma_y, \beta = \sigma_z$.

$$\begin{aligned} \theta(E - m)\,\psi &= [y,\ \chi] + i[x,\ \chi] &(28)\\ \theta(E + m)\,\chi &= [y,\ \psi] + i[x,\ \psi]. &(29)\end{aligned}$$

The NC Dirac equation is written conveniently as

$$\begin{aligned} \sqrt{\theta/2}\,(E - m)\,\psi &= +i[a^\dagger,\ \chi] &(30)\\ \sqrt{\theta/2}\,(E + m)\,\chi &= -i[a\ ,\ \psi]. &(31)\end{aligned}$$

Relation to scalar solutions
If $\lambda_1 \equiv \frac{\theta}{2}(E-m)^2$, $\lambda_2 \equiv \frac{\theta}{2}(E+m)^2$ then

$$
\begin{aligned}
\sqrt{\lambda_1}\,\psi &= +i[a^\dagger,\chi] & (32)\\
\sqrt{\lambda_2}\,\chi &= -i[a\,,\psi]. & (33)
\end{aligned}
$$

That leads immediately to

$$
\begin{aligned}
\frac{\theta}{2}\,(E^2-m^2)\,\psi &= [a\,,[a^\dagger,\ \psi] & (34)\\
\frac{\theta}{2}\,(E^2-m^2)\,\chi &= [a^\dagger,[a\,,\ \chi]. & (35)
\end{aligned}
$$

Both fermionic components obey the scalar wave equation, in complete analogy with the commutative case. Yet...

Nondiagonal solutions

Sandwich between $|n\rangle$ states, $\langle n'|\,\psi\,|n\rangle \equiv \psi_{n'n}$,

$$\sqrt{\theta/2}(E-m)\,\psi_{n'n} = +i\sqrt{n'}\,\chi_{n'-1,n} - i\sqrt{n+1}\,\chi_{n',n+1} \qquad (36)$$

$$\sqrt{\theta/2}(E+m)\,\chi_{n'n} = -i\sqrt{n'+1}\,\chi_{n'+1,n} + i\sqrt{n}\,\chi_{n',n-1} \qquad (37)$$

Notice that even for $n'=n$, not enough to have "diagonal" solutions!
Previous scalar solutions not enough. Must go to case without radial symmetry!
If find a generic scalar solution ϕ, can choose $\psi \sim \phi$ **and** $\chi \sim -i[a,\phi]$.
Then, have second spinor of the form $\chi \sim \phi$ **and** $\psi \sim +i[a^\dagger,\phi]$.
Then, normalize.

Back to scalars

General case: consider a more suggestive form

$$\phi_{n_1,n_2} \equiv \phi^{(n_1-n_2)}_{\min\{n_1,n_2\}}, \qquad m \equiv |n_1 - n_2|, \qquad n_1, n_2 \in \boldsymbol{N}. \tag{38}$$

Then $\phi_n^{(m)}$ and $\phi_n^{(-m)}$ obey the same second-order difference equation

$$\sqrt{n+m+1}\sqrt{n+1}\phi^{(m)}_{n+1} + \sqrt{n+m}\sqrt{n}\phi^{(m)}_{n-1} + (\lambda - 2n - m - 1)\phi^{(m)}_n = 0 \tag{39}$$

but their boundary/initial conditions are assigned independently.
The field ϕ turns into

$$\phi = \sum_{m=0}^{\infty} a_m \sum_{n=0}^{\infty} |n+m\rangle\, e^{i\omega t}\phi_n^{(m)}\, \langle n| + \sum_{m=0}^{\infty} b_m \sum_{n=0}^{\infty} |n\rangle\, e^{i\omega t}\phi_n^{(-m)}\, \langle n+m|\,. \tag{40}$$

Configurations with different m can be freely superposed in (40); the coefficients a_m and b_m are determined solely through initial/boundary conditions (e.g. which modes are excited). In contrast to $|n\rangle\, \phi_n^{(0)}\, \langle n|$, which associates a value $\phi_n^{(0)}$ to *one* point n, $|n+m\rangle\, \phi_n^{(m)}\, \langle n|$ associates a value $\phi_n^{(m)}$ to the *two* points $n+m$ and n. Thus, m is a measure of the delocalization of the field configuration it characterizes.

Operatorial Noether Theorem

The bilocal quantity $\phi_n^{(m)} \equiv \phi_{n',n}$ is described by a discrete radius squared n and an 'extension' $m \equiv |n'-n|$.

Claim: m is related to the quantity conjugated to the polar angle, i.e. to the planar angular momentum.

Proof: Adapt Noether's theorem to the operatorial set-up of $\phi(\hat{x},\hat{y})$ and obtain the expression for angular momentum in the $x-y$ plane, $J_z \equiv J$. First, identify the generator of rotations in the NC plane. Since

$$e^{i\alpha\hat{N}}\hat{x}e^{-i\alpha\hat{N}} = +\hat{x}\cos\alpha + \hat{y}\sin\alpha, \qquad e^{i\alpha\hat{N}}\hat{y}e^{-i\alpha\hat{N}} = -\hat{x}\sin\alpha + \hat{y}\cos\alpha,$$

$\hat{N} = \frac{1}{2}(\frac{\hat{x}^2+\hat{y}^2}{\theta} - 1) = \hat{a}^\dagger\hat{a}$ generates rotations in the NC $x-y$ plane. The variation of the field ϕ under an infinitesimal rotation is then

$$\delta\phi \;\equiv\; e^{i\alpha\hat{N}}\phi(\hat{x},\hat{y})e^{-i\alpha\hat{N}} - \phi(\hat{x},\hat{y}) \quad\simeq\quad i\alpha[\hat{N},\phi], \;\text{ if }\; \alpha\to 0. \tag{41}$$

The field action remains invariant under such unitary transformations. Second, adapt the Noether theorem to this operatorial set-up. The conserved charge associated to the invariance under rotations, namely angular momentum, turns out to be

$$J_z = Tr_H\; i\dot{\phi}^\dagger[\hat{a}^\dagger\hat{a},\phi]. \tag{42}$$

Angular Momentum versus Nonlocality

We show that states 'delocalized' by an amount m carry m units of angular momentum. Denoting the angular momentum of a field configuration ϕ by $J_z[\phi]$, obtain

$$J_z\left[\sum_{n\in N}|n+m\rangle\,\phi_n^{(m)}\,\langle n|\right] = +m\omega\,\sum_{n=0}^{\infty}[\phi_n^{(m)}]^2; \tag{43}$$

$$J_z\left[\sum_{n\in N}|n\rangle\,\phi_n^{(-m)}\,\langle n+m|\right] = -m\omega\,\sum_{n=0}^{\infty}[\phi_n^{(-m)}]^2. \tag{44}$$

Dividing by the normalization factor $N_m^+ = \sum_{n=0}^{\infty}[\phi_n^{(m)}]^2$, leads to

$$J_z\left[\phi^{(+m)} \equiv \frac{1}{\sqrt{N_m^+}}\sum_{n\in N}|n+m\rangle\,\phi_n^{(m)}\,\langle n|\right] = (+m\omega).$$

Similarly, $J_z\left[\phi^{(-m)}\right] = (-m\omega)$. Can write ϕ as an m-expansion

$$\phi = \sum_{m\in N}[a_m\phi^{(+m)} + b_m\phi^{(-m)}]. \tag{45}$$

In the 1D discrete view this is an expansion in field configurations with well-defined bilocality m. In the 2D NC FT view, Eq. (45) accounts for nonradial dependence of ϕ through an angular momentum expansion.

Bilocal waves via finite series

Parametrize the two independent solutions of the difference equation as:

$$\phi_n^{1(m)} = \sqrt{\frac{(n+m)!}{n!}} f_1(\lambda) u_n^{(m)}, \qquad \phi_n^{2(m)} = \sqrt{\frac{(n+m)!}{n!}} f_2(\lambda) v_n^{(m)}. \quad (46)$$

The continuum limit behaviour of $u_n^{(m)}$ and $v_n^{(m)}$ requires (see below)

$$f_1(\lambda) = \lambda^{\frac{m}{2}}, \qquad f_2(\lambda) = \lambda^{-\frac{m}{2}}. \quad (47)$$

$u_n^{(m)}$ and $v_n^{(m)}$, denoted collectively by $\tilde{\phi}_n^{(m)}$, satisfy the simpler recurrence

$$(n+m+1)\tilde{\phi}_{n+1}^{(m)} + n\tilde{\phi}_{n-1}^{(m)} + (\lambda - 2n - m - 1)\tilde{\phi}_n^{(m)} = 0. \quad (48)$$

If the discrete derivative operator Δ is defined by

$$\Delta\phi_n = \phi_{n+1} - \phi_n \quad (49)$$

and the shift operator $\hat{E}$ is defined by

$$\hat{E}\phi_n \equiv \phi_{n+1}, \quad (50)$$

then the homogeneous difference equation (48) can be rewritten as

$$\left[\hat{D}\right] \tilde{\phi}_n^{(m)} \equiv \left[n\Delta^2\hat{E}^{-1} + (m+1)\Delta + \lambda\right] \tilde{\phi}_n^{(m)} = 0. \quad (51)$$

The difference operator $\left[\hat{D}\right]$ annihilates the field $\tilde{\phi}_n^{(m)}$, $\forall n \in \mathbb{N}$.

First solution

Define for α real, the falling factorial power $n^{\underline{\alpha}}$,

$$n^{\underline{\alpha}} \equiv \frac{\Gamma(n+1)}{\Gamma(n+1-\alpha)}, \qquad \Longrightarrow \qquad \Delta n^{\underline{\alpha}} = \alpha n^{\underline{\alpha-1}}. \tag{52}$$

For natural $\alpha = k$, $n^{\underline{k}} = n(n-1)\cdots(n-k+1)$; this explains the name. Search for an expansion in falling factorial powers of n

$$\tilde{\phi}_n^{(m)} = a_0^m(\sigma,\lambda) n^{\underline{\sigma}} + a_1^m(\sigma,\lambda) n^{\underline{\sigma+1}} + a_2^m(\sigma,\lambda) n^{\underline{\sigma+2}} + \dots \tag{53}$$

From $\left[\hat{D}\right] \sum_{k=0}^{\infty} a_k^m(\sigma,\lambda) n^{\underline{\sigma+k}} = 0$ we obtain the indicial equation

$$\sigma(\sigma+m) = 0 \tag{54}$$

and the recurrence relation for the expansion coefficients $a_k^m(\sigma,\lambda)$

$$(k+\sigma)(k+\sigma+m)\, a_k^m(\sigma,\lambda) + \lambda\, a_{k-1}^m(\sigma,\lambda) = 0. \tag{55}$$

Eq. (55) guarantees that (53) is also an expansion in powers of λ. For $\sigma \to 0$ we obtain the first finite series solution

$$u_n^{(m)} = \sum_{k=0}^{n} \frac{(-\lambda)^k}{k!(m+k)!} \left[\frac{\Gamma(n+1)}{\Gamma(n+1-k)} = \frac{n!}{(n-k)!} = n^{\underline{k}}\right]. \tag{56}$$

From now on we drop a dimensionfull multiplicative constant a_0.

Second solution - infinite series

The second solution cannot be obtained easily since roots differ by an integer in the indicial equation: $\sigma = -m$ makes the coefficients diverge after some k. We therefore solve the inhomogeneous equation

$$\left[\hat{D}\right] \sum_{k=0}^{\infty} a_k(\sigma) n^{\underline{\sigma+k}} = \sigma(\sigma+m)^2 n^{\underline{\sigma}}, \tag{57}$$

take the derivative of its solution with respect to σ, then take the limit $\sigma + m \to 0$. Carefully tracking the poles of intervening digamma functions $\Psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$, we obtain the second solution

$$\begin{aligned} w_n^{(m)} &= \sum_{k=0}^{n} \frac{\lambda^m(-\lambda)^k}{(k+m)!} C_n^k (H_{n-k} - H_k - H_{k+m} + H_{m-1} - \gamma) \\ &\quad - n! \sum_{k=0}^{m-1} \frac{\lambda^k}{k!} \frac{(m-k-1)!}{(m-k+n)!} + \sum_{k=n+1}^{\infty} \frac{\lambda^{k+m}(-)^n n!(k-n-1)!}{k!(k+m)!}. \end{aligned} \tag{58}$$

H_k is a discrete version of the logarithmic function,

$$H_k = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k}, \qquad k = 1, 2, 3\ldots \quad ; \qquad H_0 = 0; \tag{59}$$

γ is the Euler-Mascheroni constant, $\gamma = \lim_{k=\infty}(H_k - ln k) \simeq 0.5772$.

Second solution - finite series

$w_n^{(m)}$ is an infinite convergent series in λ. To obtain a finite series, search for a linear combination of $u_n^{(m)}$ and $w_n^{(m)}$,

$$v_n^{(m)} = a(\lambda) u_n^{(m)} + b(\lambda) w_n^{(m)}, \tag{60}$$

that is independent of λ in $n = 0$ and $n = 1$. Convenient to impose $v_0^{(m)} = 0$ and $v_1^{(m)} = \frac{1}{m+1}$, in order to determine $a(\lambda)$ and $b(\lambda)$. Obtain

$$v_n^{(m)} = e^{-\lambda} w_n^{(m)} + e^{-\lambda} \left[\sum_{k=0}^{m-1} \frac{\lambda^k \; m!}{k!(m-k)} + \lambda^m \left(\frac{1}{m} + \gamma \right) - \sum_{k=1}^{\infty} \frac{\lambda^{k+m} \; m!}{(k+m)! k} \right] u_n^{(m)}.$$

A long calculation involving products and sums of finite and infinite power series and a variety of combinatorial identities leads to

$$v_n^{(m)} = \sum_{L=0}^{n-1} (-\lambda)^L \left\{ \sum_{s=1}^{n-L} \frac{(-)^{s-1} C_n^{s+L}}{(m+s)(m+s+1)\cdots(m+s+L)} \right\}. \tag{61}$$

This is our final result for $v_n^{(m)}$. The first $v_n^{(m)}$'s can be calculated easily. As expected, $v_0^{(m)} = 0$ and $v_1^{(m)} = \frac{1}{(m+1)}$ for any natural m; then,

$$v_2^{(m)} = \frac{(3+m) - \lambda}{(m+2)!/m!}, \quad v_3^{(m)} = \frac{(m^2 + 6m + 11) - (2m+8)\lambda + \lambda^2}{(m+3)!/m!}.$$

General solution

Putting everything together we obtain the two finite series solutions:

$$\phi_n^{1(m)} = \sqrt{\lambda^m \frac{(n+m)!}{n!}} \sum_{k=0}^{n} (-\lambda)^k \left[\frac{C_n^k}{(k+m)!} \right]; \tag{62}$$

$$\phi_n^{2(m)} = \sqrt{\lambda^{-m} \frac{(n+m)!}{n!}} \sum_{L=0}^{n-1} (-\lambda)^L \left\{ \sum_{s=1}^{n-L} \frac{(-)^{s-1} C_n^{s+L} (m+s-1)!}{(m+s+L)!} \right\}. \tag{63}$$

They are *finite sums* and are linearly independent, since their Casoratian (the discrete analogue of the Wronskian) is nonvanishing:

$$D_n^m \equiv \phi_n^{1(m)} \phi_{n+1}^{2(m)} - \phi_{n+1}^{1(m)} \phi_n^{2(m)} = \frac{1}{\sqrt{n+m+1}\sqrt{n+1}}. \tag{64}$$

The general solution is a linear combination with coefficients determined from appropriate boundary conditions. Since $\phi_n^{(-m)}$ obeys the same equation, it will also be a superposition of the solutions (62) and (63) but with coefficients determined from independently assigned boundary conditions.

Recall that m (respectively $-m$) characterizes simultaneously the degree of nonlocality and the angular momentum of each bilocal configuration $|n+m\rangle\, \phi_n^{(m)} \,\langle n|$ (respectively $|n\rangle\, \phi_n^{(-m)} \,\langle n+m|$).

Radially symmetric solutions

If $\phi = \phi(\hat{N})$, the relevant difference equation simplifies to the local form (13). Its first solution follows immediately

$$u_n \equiv u_n^{(m=0)} = \sum_{k=0}^{n} \frac{(-\lambda)^k}{k!} \; C_n^k \; , \tag{65}$$

Although the indicial equation $\sigma^2 = 0$ has now equal roots, the second (infinite series) solution

$$w_n = \sum_{k=0}^{n} \frac{(-\lambda)^k}{k!} C_n^k (H_{n-k} - 2H_k - \gamma) + \frac{(-\lambda)^n}{n!} \sum_{s=1}^{\infty} \frac{(+\lambda)^s (s-1)!}{[(n+s)!/n!]^2} \tag{66}$$

is still the specialization of $w_n^{(m)}$ to $m = 0$, namely $w_n \equiv w_n^{(0)}$. If $v_n = a(\lambda)u_n + b(\lambda)w_n$ obeys $v_0 = 0$ and $v_1 = 1$ then

$$v_n = \sum_{N=0}^{n-1} (-\lambda)^N \left\{ \sum_{s=1}^{n-N} \frac{(-)^{s-1} C_n^{s+N}}{s(s+1)\cdots(s+N)} \right\} \tag{67}$$

Again, $v_n \equiv v_n^{(0)}$. One can reach an even simpler expression

$$v_n = \sum_{k=0}^{n-1} \frac{(-\lambda)^k}{k!} \; \sum_{j=1}^{n-k} \frac{C_{n-j}^k}{k+j} \; . \tag{68}$$

Sources - pointlike

Sources can be introduced an inhomogeneous term j_n,

$$\sqrt{n+m+1}\sqrt{n+1}\phi_{n+1}^{(m)}+\sqrt{n+m}\sqrt{n}\phi_{n-1}^{(m)}+(\lambda-2n-m-1)\phi_n^{(m)}=j_n. \tag{69}$$

The solution for an arbitrary distribution of sources j_n, $\phi_n^{(m)}[j_n]$, is a linear superposition of solutions $\phi_n^{(m)}[j\delta_{n,n_0}]$ with sources $j_n = j\delta_{n,n_0}$ localized at an arbitrary but single point n_0.

If $n_0 = 0$ one has the most interesting case - a source at the origin, $j_n = j\delta_{n,0}$. If $j = \sqrt{m!}\lambda^{-m/2}$, the difference equation (69) is now solved precisely by $\phi_n^{2(m)}$ $(+c\phi_n^{1(m)}$, c arbitrary). Indeed, $\phi_n^{2(m)}$ does solve the homogeneous equation (39) everywhere *except* at $n = 0$, where the difference equation becomes first order and admits only one solution, $\phi_n^{1(m)}$. If a source $j = \sqrt{m!}\lambda^{-m/2}$ is added at $n = 0$ however, $\phi_n^{2(m)}$ is the particular solution of the resulting inhomogeneous equation. This is in line with the fact that v_n enters the description of radially propagating waves, which *require* a source at the origin. If $j_n = j_0\delta_{n,0}$, the solution is

$$\phi_n^{(m)}[j_0\delta_{n,0}] = j_0\frac{\lambda^{m/2}}{\sqrt{m!}}\ \phi_n^{2(m)}. \tag{70}$$

Sources - general

Consider then the case of a source at $n_0 \geq 1$ (a ring-like source), $j_n = j\delta_{n,n_0}$. Adaptation to the discrete case of the method of variation of constants (or of the method of Green functions) leads to

$$\phi_n^{(m)}[j\delta_{n,n_0}] = c_1\phi_n^{1(m)} + c_2\phi_n^{2(m)} - j\theta_{n,n_0}\left(\phi_n^{1(m)}\phi_{n_0}^{2(m)} - \phi_n^{2(m)}\phi_{n_0}^{1(m)}\right), \quad (71)$$

with the step function $\theta_{m,n} \equiv \theta(m-n)$ defined as

$$\theta_{m,n} = \begin{cases} 1 & \text{if } m-n \geq 0 \\ 0 & \text{if } m-n < 0. \end{cases} \quad (72)$$

Finally, denoting by $\phi_n^{(m)}[0] = c_1\phi_n^{1(m)} + c_2\phi_n^{2(m)}$ the general solution in the absence of sources, we obtain the general solution with an arbitrary source distribution j_n as

$$\phi_n^{(m)}[j_n] = \phi_n^{(m)}[0] - \sum_{n_0} j_{n_0}\theta_{n,n_0}\left(\phi_n^{1(m)}\phi_{n_0}^{2(m)} - \phi_n^{2(m)}\phi_{n_0}^{1(m)}\right). \quad (73)$$

The sum is taken over all points n_0 with nonzero sources, $j_{n_0} \neq 0$. If a source j_0 appears at the origin one notes that, due to $\phi_0^{1(m)} = \frac{\lambda^{m/2}}{m!}$ and $\phi_0^{2(m)} = 0$, the $n_0 = 0$ contribution in (73) reproduces (70). The general solution (73) does not display singularities, even at the location of the sources.

Commutative limit

As $\theta \to 0$, one has $n \simeq n' \simeq \frac{r^2}{2\theta} \to \infty$ and $\lambda = \frac{\theta\omega^2}{2} \to 0$, but $\lambda \cdot n \sim (\frac{\omega r}{2})^2$ is finite; m stays finite as well. Taking $n \to \infty$, expanding the square roots to order $O(\frac{1}{n^2})$ and replacing $\frac{\Delta}{\Delta n}$ by $\frac{d}{dn}$ one obtains

$$\left(n + \frac{m}{2}\right) \frac{d^2\phi_n^m}{dn^2} + \frac{d\phi_n^m}{dn} + \left(\lambda - \frac{m^2}{4n}\right) \phi_n^m = 0. \tag{74}$$

Recalling that $\lambda = \frac{\theta\omega^2}{2}$ and passing via $n \equiv \frac{r^2}{2\theta}$ from a function of n to a function of r, $\phi_n^{(m)} \to f^{(m)}(r)$, Eq. (74) becomes as $\theta \to 0$:

$$\frac{d^2 f^{(m)}}{dr^2} + \frac{1}{r}\frac{df^{(m)}}{dr} + (\omega^2 - \frac{m^2}{r^2}) f^{(m)}(r) = 0. \tag{75}$$

This is precisely the Bessel equation of order m for a function of independent variable ωr. Our solutions should therefore reduce at large distances to linear combinations of the cylindrical functions of order m.

Standing waves

Indeed, as $n \simeq \frac{r^2}{2\theta} \to \infty$, $\lambda = \frac{\theta\omega^2}{2} \to 0$ and $\lambda \cdot n \simeq \left(\frac{\omega r}{2}\right)^2$, a properly normalized $u_n^{(m)}$ becomes, as a function of r, the m-th order Bessel function $J_m(\omega r)$,

$$\sqrt{\lambda^m \frac{(n+m)!}{n!}}\; u_n^{(m)} \longrightarrow J_m(\omega r). \tag{76}$$

In this way we also establish the function f_1 in (46) to be $\lambda^{m/2}$, up to multiplicative factors which go to 1 when λ goes to 0. For instance, ortogonality of the $\phi_n^{1,m}$ in (62), seen as functions of $\lambda \in [0,\infty)$, requires further multiplication of the LHS of (76) by $\mathrm{e}^{-\frac{\lambda}{2}}$. Not being of immediate relevance for our purposes, such factors are omitted.

The correspondence between NC and usual waves can now be established through their behaviour at large distances. Given that the m-th order Bessel function $J_m(\omega r)$ describes usual radially standing waves (oscillations), Eq. (76) implies that its counterpart $\phi_n^{1(m)}$ describes radially standing NC waves (oscillations). Moreover, $J_m(\omega r)$ can carry angular momentum ωm or $-\omega m$ in usual planar field theory, in perfect agrement with the fact that $\phi_n^{1(m)}$ enters (40) in combination with either $|n+m\rangle\langle n|$ or $|n\rangle\langle n+m|$.

Travelling waves

For the second solution things are less immediate. In the end, one obtains

$$\sqrt{\lambda^{-m}\frac{(n+m)!}{n!}}v_n^{(m)} \longrightarrow \pi Y_m(\omega r) + J_m(\omega r)[H_{m-1} - 2\gamma - \log(\theta\omega^2/2)]. \tag{77}$$

This also establishes the function f_2 in (46) to be $\lambda^{-m/2}$.
On the other hand, the first Hankel function of order m

$$H_m^1(\omega r) = J_m(\omega r) + iY_m(\omega r) \tag{78}$$

describes waves which propagate outward radially and rotate angularly with frequency plus or minus ωm [unless waves with m and $-m$ dependence are superposed, e.g. to render the angular part of the wave standing]. In consequence, the linear combination of $\phi_n^{1(m)}$ and $\phi_n^{2(m)}$ which tends to $H_m^1(\omega r)$ as $\theta \to 0$ will describe a NC wave radially propagating outwards towards $n = \infty$ and carrying angular momentum $+m\omega$ or $-m\omega$ [unless two waves with opposite m are superposed]. This combination is easily found to be

$$\phi_n^{3(m)} = \phi_n^{1(m)} + \frac{i}{\pi}\left(\phi_n^{2(m)} - \phi_n^{1(m)}\left[H_{m-1} - 2\gamma - \log\frac{\theta\omega^2}{2}\right]\right) \tag{79}$$

and displays angular momentum $m\omega$ when it combines with $|n+m\rangle\langle n|$ and angular momentum $-m\omega$ when it combines with $|n\rangle\langle n+m|$.

Amusing (and Useful) Identities

Notation: n, k non-negative integers, i.e. $n, k \in \boldsymbol{N}$; $C_n^k = \frac{n!}{(n-k)!k!}$.

Discrete logarithm is $H_n \equiv 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$. Then (known identity)

$$\sum_{k=1}^{n} \frac{(-)^{k-1} C_n^k}{k} = H_n \tag{80}$$

together with two useful generalizations of it,

$$\sum_{k=1}^{n} \frac{(-)^{k-1}}{k} \frac{C_n^k \quad m!}{(k+1)(k+2)\dots(k+m)} = H_{n+m} - H_m, \tag{81}$$

$$\sum_{k=1}^{n} \frac{(-)^{k-1}}{k} \frac{C_{n+m}^{k+m} \quad p!}{(k+1)\dots(k+p)} = \sum_{j=1}^{n} \frac{C_{m+n-j}^{m}}{p+j}. \tag{82}$$

If translate the denominator k in (80) by a positive integer p, find

$$\sum_{k=0}^{n} \frac{(-)^{k-1} C_n^k}{k+p} = -\frac{(p-1)!n!}{(p+n)!} \qquad p = 1, 2, 3 \dots (p \geq 1). \tag{83}$$

and its generalization, again for p integer and $p \geq 1$,

$$\sum_{k=0}^{n} \frac{(-)^{k-1} C_n^k \ m!}{(k+p)(k+p+1)\dots(k+p+m)} = -\frac{(p-1)!(n+m)!}{(p+n+m)!}. \tag{84}$$

Some special functions

Digamma function: $\Psi(x) = \frac{d \log \Gamma(x)}{dx} = \frac{\Gamma'(x)}{\Gamma(x)}$, ($\Gamma(z+1) = z\Gamma(z)$; z complex but not a negative integer, n positive integer)

$$\Psi(z+1) \equiv \frac{\Gamma'(z+1)}{\Gamma(z+1)} = -\gamma + \sum_{l=1}^{\infty} \frac{z}{l(l+z)}, \quad \Psi(n+1) = -\gamma + H_n. \quad (86)$$

Bessel functions:

$$J_m(z) = \sum_{k=0}^{\infty} \frac{(-)^k (z/2)^{2k+m}}{k!(k+m)!}, \quad (87)$$

Neumann functions Y_m:

$$\pi Y_m(z) = 2J_m(z)(\gamma + \log\frac{z}{2}) - \sum_{k=0}^{m-1} \frac{(m-k-1)!}{k!} \left(\frac{z}{2}\right)^{2k-m} - \sum_{k=0}^{\infty} (-)^k \left(\frac{z}{2}\right)^{2k+m} \frac{H_{m+k} + H_k}{k!(k+m)!}. \quad (88)$$

Laguerre polynomials:

$$L_n^m(\lambda) = \sum_k \frac{\Gamma(n+m+1)}{\Gamma(k+m+1)} \frac{(-\lambda)^k}{k!(n-k)!}. \quad (89)$$

Acknowledgements

Financial support through PN 18 09 01 01/2018 is acknowledged.

References

- Fermions
  CSA (2018) to appear.
- Lattice programme
  CSA, TIM Proceedings AIP (2015)
- First solutions on discrete space
  CSA, JHEP 1302 (2013) 057; CSA, J. Mod. Phys. A41 (2008) 215401.
- Bilocality
  CSA, Phys. Rev. D67 (2003) 045020.
- IR/UV
  Minwalla, Seiberg and Van Raamsdonk, JHEP 0003 (2000) 035.
- NC Solitons
  Gopakumar, Minwalla and Strominger, JHEP 0005 (2000) 020. Polychronakos, Phys. Lett. B495 (2000) 407; Bak, Phys. Lett. B495 (2000) 251. For a review, see Harvey, hep-th/0102076.

THANK YOU