The Fractional Zener Model of the Spacetime

Viorel Laurentiu Cartas
MIAA, Down Danube University of Galati, Romania
viorel.cartas@ugal.ro

ABSTRACT

In the last decade topnotch experiments (LIGO and GP-B) have putted into evidence the viscoelastic nature of the space time. In the present work we have applied the viscoelastic constitutive equations for a spcetime model, based on the fractional Zener representation, which is the most general way of thinking about materials. Dispersion and dissipation are discussed in the frame of the spacetime, considered as a viscoelastic material

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Introduction

In the last ten years two very significant experiments have occurred: The LIGO Laser Interferometer Gravitational-Wave Observatory (there are twin LIGO detectors at Livingston Louisiana and Hanford Washington) and the Gravity Probe B (GP-B), both of them dealing with the General Relativity (GR) theory. LIGO is an experiment concerning with the existence of the gravitational waves (GW), a phenomenon which was anticipated by Einstein decades ago. GPB has aimed to put into evidence how the Earth's rotation drags the local spacetime's frame with it (frame dragging was foresighted by Einstein as well).

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These two experiments were thought many years ago, the only one reason they were not achieved was the technological unsatisfactory level. It was needed a high accuracy because the expected experimental data was very small. For LIGO, it was expected to be finding a change in the arm length (the arm length of the GW detector is 4 km) of the order of 10^{-17} cm. For GP-B they have struggled to measure the drift of the orientation of a gyroscope's spin axis of 6,614.4 milli-arc-seconds per year in the orbital plane of the satellite caring the experiment, due to the curvature of Earth's local spacetime, and the drift of the spin axes of 40.9 milli-arc-seconds per year in a perpendicular plane (that is, the plane of Earth's rotation), due to the frame-dragging effect. In order to visualize such a little angle you can imagine how little we see, from one side to the other, a strand of hair, from thirty kilometers away. Another important commune features for LIGO and GP-B is how expensive they are and the fact that they are at cosmic scale. The key players were Kip S. Thorne (Caltech) for LIGO and Leonard Schiff (Stanford) for GP-B and both experiments have led to a Nobel Prize..

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LIGO and GP-B undoubtedly prove that the space time is not anymore only a mathematical tool but it is rather, a real physical object featuring peculiar properties. The space time is “elastic” in the sense of being a proper environment considering the existence of the gravitational wave. The spacetime is “viscous” in the sense that it could be dragged by another material body moving inside.

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Generally speaking, a viscoelastic object is essentially described by two phenomena: the creep and the relaxation. The first one is the variation of the strain when the stress is constant. I have represented in the Fig.1. a diagram, describing this phenomenon. Firstly, increases because of a constant . Then, maintaining constant, exponentially decreases till a constant value.

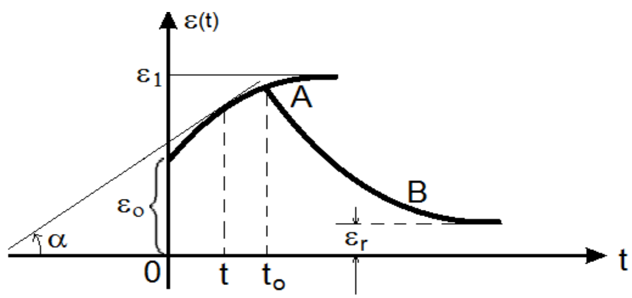

Fig.1.

The creep phenomenon

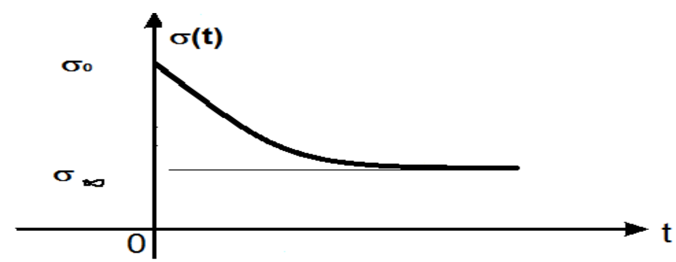

Fig. 2.

The relaxation phenomenon

The second one is the relaxation phenomenon: considering the time evolution, when we have a constant strain the stress exponentially decreases. This behavior is represented into the Fig.2.

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Let’s consider the Hook low in the constitutive Riemann tensor form :

$$\sigma_{ij} = \lambda\, \varepsilon\, \delta_{ij} + 2\mu\, \varepsilon_{ij}$$

σ the stress tensor

ε the strain tensor

λ μ the elastic Lame constants

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High similitude to the Einstein equation:

$$T_{ij} = R_{ij} - \frac{1}{2} g_{ij} R$$

T_{ij} the stress-energy tensor

R_{ij} the curvature

$$\eta_{ij} \equiv \delta_{ij}$$

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In the frame of the viscoelastic body theory we generalize

$\sigma_{ij}(\vec{r},t) = L_{ijkl}\varepsilon_{kl}(\vec{r},t)$ $\;i,j,k,l = 1,2,3$ $\quad L_{ijkl}$ tensorfunctional

$$L_{ijkl} = L_{jikl} = L_{ijlk}$$

36 components

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Using the Stieltjes integral

$$\sigma_{ij}(\vec{r},t) = \int_0^\infty \varepsilon_{kl}(\vec{r},t-\tau)\,d\psi_{ijkl}(\vec{r},\tau)$$

retarded strain tensor

relaxation tensor

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A new form in the frame of the distribution theory

$$\sigma_{ij}(\vec{r},t) = \varepsilon_{kl}(\vec{r},t) \star \frac{\partial \psi_{ijkl}(\vec{r},t)}{\partial t}$$

convolution
product

Using the Laplace transform we can write the partial derivative of the psi 's tensor components:

$$\frac{\partial \psi_{ijkl}(\vec{r},t)}{\partial t} = \lambda(\vec{r},t)\delta_{ij}\delta_{kl} + \mu(\vec{r},t)\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right)$$

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Slide 4

A new form in the frame of the distribution theory

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$$\sigma_{ij}(\vec{r},t) = \lambda(\vec{r},t)\delta_{ij} \star \varepsilon(\vec{r},t) + \mu(\vec{r},t) \star \varepsilon_{ij}(\vec{r},t)$$

$$T_{ij} = \lambda\, g_{ij} \star R + \mu \star R_{ij}$$

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basic formulas

Integral:

$$ {}_0I_t^n f(t) := f_n(t) = \frac{1}{(n-1)!}\int_0^t (t-\tau)^{n-1} f(\tau)d\tau, n \in \mathbb{N}, t > 0 \rightarrow $$

$$ \rightarrow \quad {}_0I_t^\alpha f(t) := f_\alpha(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} f(\tau)d\tau, \alpha \in \mathbb{R}^+, t > 0 $$

The first formula is the Cauchy formula (an n-fold primitive of a certain function is a convolution integral) and the second one is the Riemann-Liouville fractional integral of order α

$$ {}_0I_t^0 := \hat{1} $$

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basic formulas

diferential:

$$
{}_0D_t^{\alpha} f(t) := \begin{cases} \dfrac{1}{\Gamma(m-\alpha)} \dfrac{d^m}{dt^m} \displaystyle\int_0^t \dfrac{f(\tau)d\tau}{(t-\tau)^{\alpha+1-m}}, & m-1 < \alpha < m, \\ \dfrac{d^m}{dt^m} f(t), & \alpha = m \end{cases}
$$

This is the Riemann-Liouville fractional derivative of order α

$$
{}_0D_t^0 := \hat{1}
$$

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<u>The linear viscoelasticity theory</u>

In this theory there are two functions which describe the creep and the relaxation : M(t) and N(t) the creep compliance and the relaxation modulus.

They are strain, respectively stress responses to the unit step of stress, respectively to the unit step of strain.

When t goes to the zero value then we have a glassy behavior of the viscoelastic body.

$$M_g = \lim_{t \to 0} M \qquad N_g = \lim_{t \to 0} N$$

When t goes to the infinite then we have an equilibrium behavior of the viscoelastic body

$$M_e = \lim_{t \to \infty} M \qquad N_e = \lim_{t \to \infty} N$$

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creep representation

$$R_{ij} = \int_{-\infty}^{t} M(t-\tau)\, dT_{ij}$$

relaxation representation

$$T_{ij} = \int_{-\infty}^{t} N(t-\tau)\, dR_{ij}$$

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creep representation $$R_{ij} = \int_{-\infty}^{t} M(t-\tau)\,dT_{ij}$$

relaxation representation $$T_{ij} = \int_{-\infty}^{t} N(t-\tau)\,dR_{ij}$$

$\Downarrow$

The constitutive equations into the linear GR theory

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<u>**The Zener model of the General Relativity**</u>

$N_e, M_g \in (0,\infty);\ M_e, N_g$ are finite $\Rightarrow$

$\Rightarrow$ a parallel connection between a spring and a dashpot and this system then, is connected in series with another spring
or, equivalently, a spring connected in series with a dashpot and then in parallel with another spring

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The Zener model of the GR equation, in the most general form

$$\left[1+\zeta\frac{d^{\upsilon}}{d{x_k}^{\upsilon}}\right]T_{ij}=\left[e\delta^{ij}+\eta\frac{d^{\upsilon}}{d{x_k}^{\upsilon}}\right]R_{ij}\qquad \upsilon\in(0,1)$$

ζ η are related to the viscosity and e is related to the elasticity

A study on this equation’s solutions is a work in progress

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Concluding remarks

The number of papers, dealing with the elasticity of the spacetime, constantly increases due to the recent astronomical scale experiments (LIGO-USA, VIRGO-Italy, future LISA-ESA, the Einstein telescope, etc.). Because of another astronomical scale experiment (GP-B), I have tried to complete the idea of the spacetime's elasticity considering the viscosity, as well.

A theory of the spacetime's viscoelasticity has to take into consideration some ”material” features of the spacetime and has to introduce some material coefficients in order to conceive a constitutive relation determining in this way a spacetime Rheology.

All this peculiar ideas concerning the spacetime, address to the exceptionally small variations; nevertheless, finding a way to influence, somehow, the spacetime as a physical object, is of great importance