Lorentzian and Newtonian spacetimes and their quantum (noncommutative) deformations

FRANCISCO J. HERRANZ

University of Burgos, Spain

XX-th Edition of Conference on Geometry,Integrability and Quantization
June 2-7, 2018, Varna - Bulgaria

Scheme

1. The nine two-dimensional Cayley–Klein geometries

The motion groups of the nine 2D Cayley–Klein (CK) geometries can be described in a unified setting by means of two real coefficients κ_1, κ_2 and are collectively denoted $SO_{\kappa_1,\kappa_2}(3)$.

The generators $\{P_1, P_2, J_{12}\}$ of the corresponding Lie algebras $\mathfrak{so}_{\kappa_1,\kappa_2}(3)$ have Lie commutators:

$$[J_{12}, P_1] = P_2 \qquad [J_{12}, P_2] = -\kappa_2 P_1 \qquad [P_1, P_2] = \kappa_1 J_{12}.$$

There is a single Lie algebra Casimir:

$$\mathcal{C} = P_2^2 + \kappa_2 P_1^2 + \kappa_1 J_{12}^2.$$

The CK algebras $\mathfrak{so}_{\kappa_1,\kappa_2}(3)$ can be endowed with a $\mathbb{Z}_2 \otimes \mathbb{Z}_2$ group of commuting automorphisms generated by:

$$\begin{aligned} \Pi_{(1)} &: (P_1, P_2, J_{12}) \to (-P_1, -P_2, J_{12}) \\ \Pi_{(2)} &: (P_1, P_2, J_{12}) \to (P_1, -P_2, -J_{12}). \end{aligned}$$

The two remaining involutions are the composition $\Pi_{(02)} = \Pi_{(1)} \cdot \Pi_{(2)}$ and the identity.

Each involution Π determines a subalgebra of $\mathfrak{so}_{\kappa_1,\kappa_2}(3)$ whose elements are invariant under Π leading to the following Cartan decompositions:

$$\mathfrak{so}_{\kappa_1,\kappa_2}(3) = \mathfrak{h}_{(1)} \oplus \mathfrak{p}_{(1)}, \qquad \mathfrak{h}_{(1)} = \langle J_{12} \rangle = \mathfrak{so}_{\kappa_2}(2), \qquad \mathfrak{p}_{(1)} = \langle P_1, P_2 \rangle.$$

$$\mathfrak{so}_{\kappa_1,\kappa_2}(3) = \mathfrak{h}_{(2)} \oplus \mathfrak{p}_{(2)}, \qquad \mathfrak{h}_{(2)} = \langle P_1 \rangle = \mathfrak{so}_{\kappa_1}(2), \qquad \mathfrak{p}_{(2)} = \langle P_2, J_{12} \rangle.$$

$$\mathfrak{so}_{\kappa_1,\kappa_2}(3) = \mathfrak{h}_{(02)} \oplus \mathfrak{p}_{(02)}, \qquad \mathfrak{h}_{(02)} = \langle P_2 \rangle = \mathfrak{so}_{\kappa_1\kappa_2}(2), \qquad \mathfrak{p}_{(02)} = \langle P_1, J_{12} \rangle.$$

The elements defining a 2D CK geometry are

• The **plane** as the set of points corresponds to the 2D symmetrical homogeneous space

$$S^2_{[\kappa_1],\kappa_2} \equiv SO_{\kappa_1,\kappa_2}(3)/H_{(1)} \equiv SO_{\kappa_1,\kappa_2}(3)/SO_{\kappa_2}(2) \qquad H_{(1)} = \langle J_{12} \rangle \approx SO_{\kappa_2}(2).$$

The generator J_{12} leaves a point O (the origin) invariant, thus J_{12} acts as the **rotation** around O. The involution $\Pi_{(1)}$ is the reflection around the origin. In this space P_1 and P_2 generate **translations** which move the origin point in two basic directions.

• The set of **lines** is identified as the 2D symmetrical homogeneous space

$$S^2_{\kappa_1,[\kappa_2]} \equiv SO_{\kappa_1,\kappa_2}(3)/H_{(2)} \equiv SO_{\kappa_1,\kappa_2}(3)/SO_{\kappa_1}(2) \qquad H_{(2)} = \langle P_1 \rangle \approx SO_{\kappa_1}(2).$$

In this space, the generator P_1 leaves invariant the 'origin' line l_1, which is moved in two basic directions by J_{12} and P_2. Therefore, within $S^2_{\kappa_1,[\kappa_2]}$, P_1 should be interpreted as the generator of **'rotations'** around l_1.

• There is a **second set of lines** corresponding to the 2D symmetrical homogeneous space

$$SO_{\kappa_1,\kappa_2}(3)/H_{(02)} \equiv SO_{\kappa_1,\kappa_2}(3)/SO_{\kappa_1\kappa_2}(2) \qquad H_{(02)} = \langle P_2 \rangle \approx SO_{\kappa_1\kappa_2}(2).$$

In this case, P_2 leaves invariant an 'origin' line l_2 in this space while J_{12} and P_1 do move l_2.

By a **two-dimensional CK geometry** we will understand the **set of three symmetrical homogeneous** spaces of points, lines of first-kind and lines of second-kind. The group $SO_{\kappa_1,\kappa_2}(3)$ acts transitively on each of these spaces.

The coefficients κ_1, κ_2 play a **twofold role**.

The space $S^2_{[\kappa_1],\kappa_2}$ has a **quadratic metric** coming from the Casimir, whose **signature** corresponds to the matrix $\mathrm{diag}(1, \kappa_2)$. This metric is riemannian (definite positive) for $\kappa_2 > 0$, lorentzian (indefinite) for $\kappa_2 < 0$ and degenerate for $\kappa_2 = 0$. This space has a canonical conexion which is compatible with the metric, and has **constant curvature** equal to κ_1.

In the notations $S^2_{[\kappa_1],\kappa_2}$, $S^2_{\kappa_1,[\kappa_2]}$ for the spaces, the κ_i in square brackets is the constant curvature, and the remaining constant determines the signature. Alternatively, the coefficients κ_1, κ_2 determine the **kind of measures of separation** amongst points and lines in the Klein sense:

• The pencil of points on a first-kind line is elliptical/parabolical/hyperbolical according to whether κ_1 is greater than/equal to/lesser than zero.

• Likewise for the pencil of points on a second-kind line depending on the product $\kappa_1\kappa_2$.

• Likewise for the pencil of lines through a point according to κ_2.

For κ_1 positive/zero/negative the **isotropy subgroup** $H_{(2)}$ is $SO(2)/\mathbb{R}/SO(1,1)$, and the same happens for $H_{(1)}$ (resp. $H_{(02)}$) according to the value of κ_2 (resp. $\kappa_1\kappa_2 \equiv \kappa_{02}$).

Whenever the coefficient κ_1 (resp. κ_2) is different from zero, a suitable choice of length unit (resp. angle unit) allows us to reduce it to either $+1$ or -1. Hence we obtain nine 2D real CK geometries.

There exists an 'automorphism' of the whole family, called **ordinary duality** $\mathcal{D}$, which is given by:

$$\mathcal{D} : (P_1, P_2, J_{12}) \to (-J_{12}, -P_2, -P_1) \qquad \mathcal{D} : (\kappa_1, \kappa_2) \to (\kappa_2, \kappa_1).$$

The map $\mathcal{D}$ leaves the commutation rules invariant while it interchanges the space of points with the space of first-kind lines, $S^2_{[\kappa_1],\kappa_2} \leftrightarrow S^2_{\kappa_1,[\kappa_2]}$, and the corresponding curvatures $\kappa_1 \leftrightarrow \kappa_2$, preserving the space of second-kind lines.

The vanishment of a coefficient κ_i corresponds to an **Inönü–Wigner contraction**. The limit $\kappa_1 \to 0$ is a local-contraction (around a point), while the limit $\kappa_2 \to 0$ is an axial-contraction (around a line).

	Measure of distance		
Measure of angle	Elliptic $\kappa_1 = 1$	Parabolic $\kappa_1 = 0$	Hyperbolic $\kappa_1 = -1$
Elliptic $\kappa_2 = 1$	Elliptic $SO(3)$ $[J_{12}, P_1] = P_2$ $[J_{12}, P_2] = -P_1$ $[P_1, P_2] = J_{12}$ $\mathcal{C} = P_2^2 + P_1^2 + J_{12}^2$ $H_{(1)} = SO(2)$ $H_{(2)} = SO(2)$ $H_{(02)} = SO(2)$	Euclidean $ISO(2)$ $[J_{12}, P_1] = P_2$ $[J_{12}, P_2] = -P_1$ $[P_1, P_2] = 0$ $\mathcal{C} = P_2^2 + P_1^2$ $H_{(1)} = SO(2)$ $H_{(2)} = \mathbb{R}$ $H_{(02)} = \mathbb{R}$	Hyperbolic $SO(2,1)$ $[J_{12}, P_1] = P_2$ $[J_{12}, P_2] = -P_1$ $[P_1, P_2] = -J_{12}$ $\mathcal{C} = P_2^2 + P_1^2 - J_{12}^2$ $H_{(1)} = SO(2)$ $H_{(2)} = SO(1,1)$ $H_{(02)} = SO(1,1)$
Parabolic $\kappa_2 = 0$	Co-Euclidean Oscillating NH $ISO(2)$ $[J_{12}, P_1] = P_2$ $[J_{12}, P_2] = 0$ $[P_1, P_2] = J_{12}$ $\mathcal{C} = P_2^2 + J_{12}^2$ $H_{(1)} = \mathbb{R}$ $H_{(2)} = SO(2)$ $H_{(02)} = \mathbb{R}$	Galilean $IISO(1)$ $[J_{12}, P_1] = P_2$ $[J_{12}, P_2] = 0$ $[P_1, P_2] = 0$ $\mathcal{C} = P_2^2$ $H_{(1)} = \mathbb{R}$ $H_{(2)} = \mathbb{R}$ $H_{(02)} = \mathbb{R}$	Co-Minkowskian Expanding NH $ISO(1,1)$ $[J_{12}, P_1] = P_2$ $[J_{12}, P_2] = 0$ $[P_1, P_2] = -J_{12}$ $\mathcal{C} = P_2^2 - J_{12}^2$ $H_{(1)} = \mathbb{R}$ $H_{(2)} = SO(1,1)$ $H_{(02)} = \mathbb{R}$
Hyperbolic $\kappa_2 = -1$	Co-Hyperbolic Anti-de Sitter $SO(2,1)$ $[J_{12}, P_1] = P_2$ $[J_{12}, P_2] = P_1$ $[P_1, P_2] = J_{12}$ $\mathcal{C} = P_2^2 - P_1^2 + J_{12}^2$ $H_{(1)} = SO(1,1)$ $H_{(2)} = SO(2)$ $H_{(02)} = SO(1,1)$	Minkowskian $ISO(1,1)$ $[J_{12}, P_1] = P_2$ $[J_{12}, P_2] = P_1$ $[P_1, P_2] = 0$ $\mathcal{C} = P_2^2 - P_1^2$ $H_{(1)} = SO(1,1)$ $H_{(2)} = \mathbb{R}$ $H_{(02)} = \mathbb{R}$	Doubly Hyperbolic De Sitter $SO(2,1)$ $[J_{12}, P_1] = P_2$ $[J_{12}, P_2] = P_1$ $[P_1, P_2] = -J_{12}$ $\mathcal{C} = P_2^2 - P_1^2 - J_{12}^2$ $H_{(1)} = SO(1,1)$ $H_{(2)} = SO(1,1)$ $H_{(02)} = SO(2)$

1.1. Spacetimes as Cayley–Klein spaces

Let $\mathcal{H}$, $\mathcal{P}$ and $\mathcal{K}$ be the generators of **time translations, space translations and boosts**, respectively, in the most simple $(1+1)$D homogeneous spacetime. Under the identification

$$P_1 \equiv \mathcal{H} \qquad P_2 \equiv \mathcal{P} \qquad J_{12} \equiv \mathcal{K}$$

the six CK groups with $\kappa_2 \leq 0$ (second and third rows of table ; NH means Newton–Hooke) are the motion groups of $(1+1)$D spacetimes:

• $S^2_{[\kappa_1],\kappa_2}$ is a $(1+1)$D spacetime, and points in $S^2_{[\kappa_1],\kappa_2}$ are **spacetime events**; the spacetime curvature equals κ_1 and is related to the usual universe (time) radius τ by $\kappa_1 = \pm 1/\tau^2$.

• The space of first-kind lines $S^2_{\kappa_1,[\kappa_2]}$ corresponds to the space of **time-like lines**. The coefficient κ_2 is the curvature of the space of time-like lines, linked to the relativistic constant c as $\kappa_2 = -1/c^2$. Relativistic spacetimes occur for $\kappa_2 < 0$ (the signature of the metric is $\mathrm{diag}(1, -1/c^2)$) and their non-relativistic limits correspond to $\kappa_2 = 0$.

• The space of second-kind lines $SO_{\kappa_1,\kappa_2}(3)/H_{(02)}$ is the 2D space of **space-like lines**.

1.2. Matrix realization and vector model

The following 3D real matrix representation of the CK algebra $\mathfrak{so}_{\kappa_1,\kappa_2}(3)$:

$$P_1 = \begin{pmatrix} 0 & -\kappa_1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad P_2 = \begin{pmatrix} 0 & 0 & -\kappa_1\kappa_2 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad J_{12} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\kappa_2 \\ 0 & 1 & 0 \end{pmatrix}$$

gives rise to a natural realization of the CK group $SO_{\kappa_1,\kappa_2}(3)$ as a group of linear transformations in an **ambient linear space** $\mathbb{R}^3 = (x^0, x^1, x^2)$ in which $SO_{\kappa_1,\kappa_2}(3)$ acts as the group of linear isometries of a bilinear form with matrix:

$$\Lambda = \text{diag}\,(1, \kappa_1, \kappa_1\kappa_2).$$

Their exponential leads to a representation of the **one-parametric subgroups** $H_{(2)}$, $H_{(02)}$ and $H_{(1)}$ generated by P_1, P_2 and J_{12} as:

$$\exp(\alpha P_1) = \begin{pmatrix} C_{\kappa_1}(\alpha) & -\kappa_1 S_{\kappa_1}(\alpha) & 0 \\ S_{\kappa_1}(\alpha) & C_{\kappa_1}(\alpha) & 0 \\ 0 & 0 & 1 \end{pmatrix}, \exp(\gamma J_{12}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & C_{\kappa_2}(\gamma) & -\kappa_2 S_{\kappa_2}(\gamma) \\ 0 & S_{\kappa_2}(\gamma) & C_{\kappa_2}(\gamma) \end{pmatrix}$$

$$\exp(\beta P_2) = \begin{pmatrix} C_{\kappa_1\kappa_2}(\beta) & 0 & -\kappa_1\kappa_2 S_{\kappa_1\kappa_2}(\beta) \\ 0 & 1 & 0 \\ S_{\kappa_1\kappa_2}(\beta) & 0 & C_{\kappa_1\kappa_2}(\beta) \end{pmatrix}$$

where the generalized cosine $C_\kappa(x)$ and sine $S_\kappa(x)$ functions are defined by

$$C_\kappa(x) := \sum_{l=0}^{\infty}(-\kappa)^l \frac{x^{2l}}{(2l)!} = \begin{cases} \cos\sqrt{\kappa}\,x & \kappa > 0 \\ 1 & \kappa = 0 \\ \cosh\sqrt{-\kappa}\,x & \kappa < 0 \end{cases}$$

$$S_\kappa(x) := \sum_{l=0}^{\infty}(-\kappa)^l \frac{x^{2l+1}}{(2l+1)!} = \begin{cases} \frac{1}{\sqrt{\kappa}}\sin\sqrt{\kappa}\,x & \kappa > 0 \\ x & \kappa = 0 \\ \frac{1}{\sqrt{-\kappa}}\sinh\sqrt{-\kappa}\,x & \kappa < 0 \end{cases} .$$

Two other useful functions are the 'versed sine' $V_\kappa(x)$ and the tangent $T_\kappa(x)$:

$$V_\kappa(x) := \frac{1}{\kappa}(1 - C_\kappa(x)) \qquad T_\kappa(x) := \frac{S_\kappa(x)}{C_\kappa(x)}.$$

These curvature-dependent trigonometric functions coincide with the usual circular and hyperbolic ones for $\kappa = 1$ and $\kappa = -1$, respectively; the case $\kappa = 0$ provides the parabolic or Galilean functions:

$$C_0(x) = 1, \qquad S_0(x) = x, \qquad V_0(x) = x^2/2.$$

The CK group $SO_{\kappa_1,\kappa_2}(3)$ can be seen as a group of linear transformations in an ambient space $\mathbb{R}^3 = (x^0, x^1, x^2)$, acting as the group of isometries of a bilinear form

$$\Lambda = \mathrm{diag}(1, \kappa_1, \kappa_1\kappa_2).$$

An element $X \in SO_{\kappa_1,\kappa_2}(3)$ satisfies

$$X^T \, \Lambda \, X = \Lambda.$$

The action of $SO_{\kappa_1,\kappa_2}(3)$ on $\mathbb{R}^3$ is linear but not transitive, since it conserves the quadratic form

$$(x^0)^2 + \kappa_1(x^1)^2 + \kappa_1\kappa_2(x^2)^2.$$

The action becomes transitive if we restrict to the orbit in $\mathbb{R}^3$ of the point O, which is contained in the 'sphere' Σ:

$$\Sigma \equiv (x^0)^2 + \kappa_1(x^1)^2 + \kappa_1\kappa_2(x^2)^2 = 1.$$

This orbit is identified with the CK space $S^2_{[\kappa_1],\kappa_2}$, and (x^0, x^1, x^2) are called **Weierstrass coordinates**; these allow us to obtain a differential realization of the generators as **first-order vector fields** in $\mathbb{R}^3$ with $\partial_i = \partial/\partial x^i$:

$$P_1 = \kappa_1 x^1 \partial_0 - x^0 \partial_1 \qquad P_2 = \kappa_1\kappa_2 x^2 \partial_0 - x^0 \partial_2 \qquad J_{12} = \kappa_2 x^2 \partial_1 - x^1 \partial_2.$$

2. Metric structure and coordinate systems of the 2D CK spaces of points

Hereafter we consider the homogeneous **space of points**

$$S^2_{[\kappa_1],\kappa_2} \equiv SO_{\kappa_1,\kappa_2}(3)/H_{(1)} \equiv SO_{\kappa_1,\kappa_2}(3)/SO_{\kappa_2}(2) \qquad H_{(1)} = \langle J_{12} \rangle \approx SO_{\kappa_2}(2).$$

Table 1: The nine two-dimensional CK spaces $S^2_{[\kappa_1],\kappa_2} = SO_{\kappa_1,\kappa_2}(3)/SO_{\kappa_2}(2)$.

Elliptic: $\mathbf{S}^2$ $S^2_{[+],+} = SO(3)/SO(2)$	Euclidean: $\mathbf{E}^2$ $S^2_{[0],+} = ISO(2)/SO(2)$	Hyperbolic: $\mathbf{H}^2$ $S^2_{[-],+} = SO(2,1)/SO(2)$
Oscillating NH: $\mathbf{NH}^{1+1}_+$ (Co-Euclidean) $S^2_{[+],0} = ISO(2)/ISO(1)$	Galilean: $\mathbf{G}^{1+1}$ $S^2_{[0],0} = IISO(1)/ISO(1)$	Expanding NH: $\mathbf{NH}^{1+1}_-$ (Co-Minkowskian) $S^2_{[-],0} = ISO(1,1)/ISO(1)$
Anti-de Sitter: $\mathbf{AdS}^{1+1}$ (Co-Hyperbolic) $S^2_{[+],-} = SO(2,1)/SO(1,1)$	Minkowskian: $\mathbf{M}^{1+1}$ $S^2_{[0],-} = ISO(1,1)/SO(1,1)$	De Sitter: $\mathbf{dS}^{1+1}$ (Doubly Hyperbolic) $S^2_{[-],-} = SO(2,1)/SO(1,1)$

If both coefficients κ_i are different from zero, $SO_{\kappa_1,\kappa_2}(3)$ is a simple Lie group, and the space $S^2_{[\kappa_1],\kappa_2}$ is endowed with a **non-degenerate metric** g_0 coming from the non-singular Killing–Cartan form in the Lie algebra $so_{\kappa_1,\kappa_2}(3)$.

At the origin, g_0 is given by:

$$g_0(P_1, P_1) = -2\kappa_1 \qquad g_0(P_2, P_2) = -2\kappa_1\kappa_2 \qquad g_0(P_1, P_2) = 0.$$

To cover the cases with $\kappa_1 = 0$ where g_0 vanishes identically, we take out a factor $-2\kappa_1$ out of g_0, and introduce the space **main metric** g_1 as

$$-2g_1 := g_0/\kappa_1.$$

If $\kappa_2 = 0$, g_1 is a degenerate metric and the action of $SO_{\kappa_1,0}(3)$ on $S^2_{[\kappa_1],0}$ has an invariant foliation. The restriction of g_1 to each foliation leaf vanishes, but

$$g_2 = \frac{1}{\kappa_2} g_1$$

has a non-vanishing and well defined restriction to each leaf; we call g_2 the **subsidiary metric**.

Proposition. *The metric structure for a generic space* $S^2_{[\kappa_1],\kappa_2}$ *is characterized by:*

• *A connection* ∇ *which is invariant under* $SO_{\kappa_1,\kappa_2}(3)$*.*

• *A hierarchy of two metrics* g_1 *and* $g_2 = \frac{1}{\kappa_2} g_1$ *compatible with* ∇*. The action of* $SO_{\kappa_1,\kappa_2}(3)$ *on* $S^2_{[\kappa_1],\kappa_2}$ *is by isometries of both metrics.*

• *The main metric* g_1 *is actually a metric in the true sense and has constant curvature* κ_1 *and signature diag*$(+, \kappa_2)$*.*

• *If* $\kappa_2 \neq 0$*,* g_2 *is a true metric proportional to* g_1*. If* $\kappa_2 = 0$*, the subsidiary metric* g_2 *gives a true metric only in each leaf of the invariant foliation in* $S^2_{[\kappa_1],0}$*, whose set of leaves can be parametrized by* $(x^0)^2 + \kappa_1 (x^1)^2 = 1 \equiv S^1_{[\kappa_1]}$*;* g_2 *has signature* $(+)$*.*

In terms of Weierstrass coordinates in the linear ambient space $\mathbb{R}^3$, the two metrics in $S^2_{[\kappa_1],\kappa_2}$ come from the **flat ambient metric**

$$\mathrm{d}s^2 = (\mathrm{d}x^0)^2 + \kappa_1 (\mathrm{d}x^1)^2 + \kappa_1 \kappa_2 (\mathrm{d}x^2)^2$$

in the form

$$(\mathrm{d}s^2)_1 = \frac{1}{\kappa_1} \mathrm{d}s^2 \qquad (\mathrm{d}s^2)_2 = \frac{1}{\kappa_2} (\mathrm{d}s^2)_1.$$

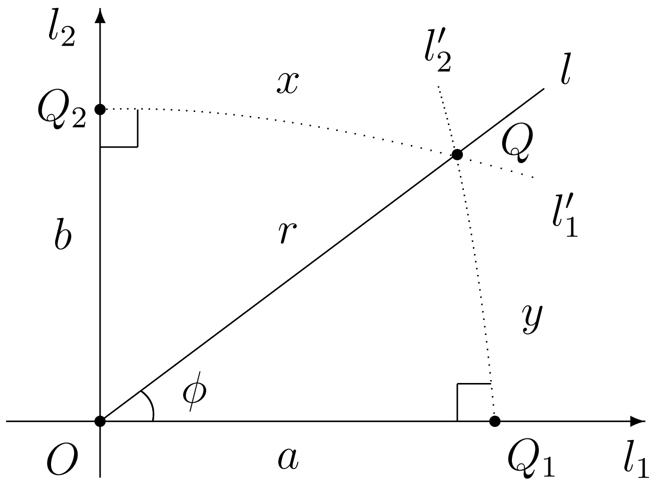
We introduce **three coordinate systems of geodesic type**.

Let us consider the origin $O \equiv (1,0,0)$, two (oriented) geodesics l_1, l_2 which are orthogonal through the origin, and a generic point Q with Weierstrass coordinates $\mathbf{x} = (x^0, x^1, x^2)$. We have:

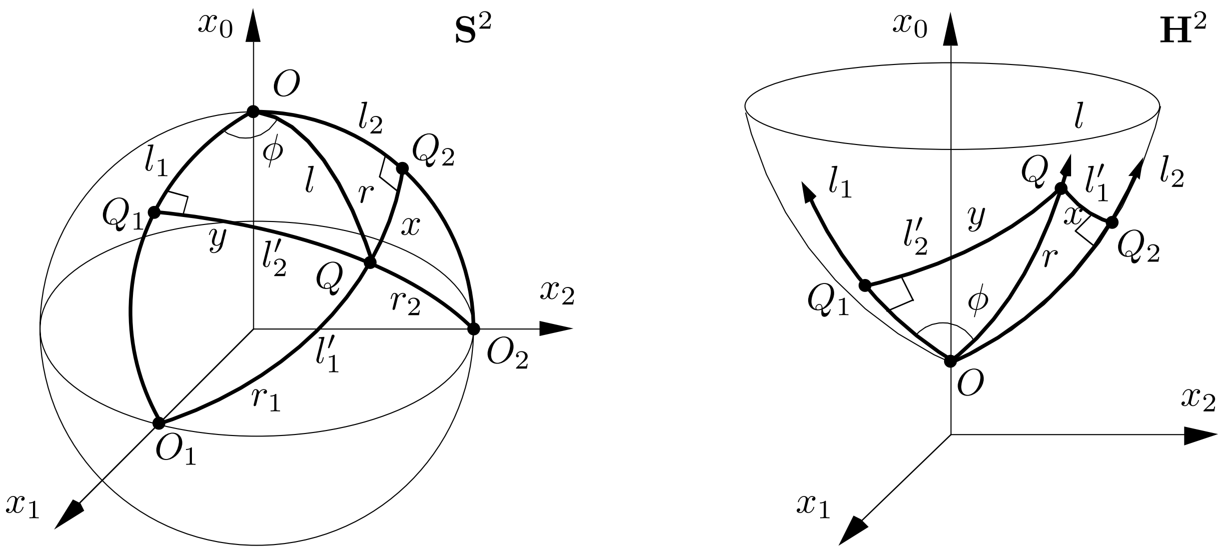
• If $\mathbf{x} = \exp(aP_1)\exp(yP_2)O$, we call (a, y) the **type I geodesic parallel coordinates of Q**.

• If $\mathbf{x} = \exp(bP_2)\exp(xP_1)O$, we call (x, b) the **type II geodesic parallel coordinates of Q**.

• The **geodesic polar coordinates** of the point Q are (r, ϕ) if $\mathbf{x} = \exp(\phi J_{12})\exp(rP_1)O$.



We compute the Weierstrass coordinates $\mathbf{x}$ of a generic point Q in the three geodesic



(a) (b)

coordinate systems.

By substitution in the expressions of the **metrics** in Weierstrass coordinates we find the main and subsidiary metrics in either geodesic coordinates.

From them we may compute the **conexion** symbols Γ^i_{jk}.

The **area** element $\mathrm{d}\mathcal{S}$ in coordinates say u^1, u^2 is $\sqrt{\det g_1/\kappa_2}\,\mathrm{d}u^1 \wedge \mathrm{d}u^2$.

Table 2: Weierstrass coordinates, metric, canonical connection and area element for $S^2_{[\kappa_1],\kappa_2}$ given in the three geodesic coordinate systems.

Parallel I (a, y)	Parallel II (x, b)	Polar (r, ϕ)
$x^0 = C_{\kappa_1}(a)C_{\kappa_1\kappa_2}(y)$ $x^1 = S_{\kappa_1}(a)C_{\kappa_1\kappa_2}(y)$ $x^2 = S_{\kappa_1\kappa_2}(y)$	$x^0 = C_{\kappa_1}(x)C_{\kappa_1\kappa_2}(b)$ $x^1 = S_{\kappa_1}(x)$ $x^2 = C_{\kappa_1}(x)S_{\kappa_1\kappa_2}(b)$	$x^0 = C_{\kappa_1}(r)$ $x^1 = S_{\kappa_1}(r)C_{\kappa_2}(\phi)$ $x^2 = S_{\kappa_1}(r)S_{\kappa_2}(\phi)$
$(\mathrm{d}s^2)_1 = C^2_{\kappa_1\kappa_2}(y)\mathrm{d}a^2 + \kappa_2\,\mathrm{d}y^2$ $(\mathrm{d}s^2)_2 = \mathrm{d}y^2$ for $a = a_0$	$(\mathrm{d}s^2)_1 = \mathrm{d}x^2 + \kappa_2 C^2_{\kappa_1}(x)\mathrm{d}b^2$ $(\mathrm{d}s^2)_2 = C^2_{\kappa_1}(x)\mathrm{d}b^2$ for $x = x_0$	$(\mathrm{d}s^2)_1 = \mathrm{d}r^2 + \kappa_2 S^2_{\kappa_1}(r)\mathrm{d}\phi^2$ $(\mathrm{d}s^2)_2 = S^2_{\kappa_1}(r)\mathrm{d}\phi^2$ for $r = r_0$
$\Gamma^y_{aa} = \kappa_1 S_{\kappa_1\kappa_2}(y)C_{\kappa_1\kappa_2}(y)$ $\Gamma^a_{ay} = -\kappa_1\kappa_2 T_{\kappa_1\kappa_2}(y)$	$\Gamma^x_{bb} = \kappa_1\kappa_2 S_{\kappa_1}(x)C_{\kappa_1}(x)$ $\Gamma^b_{bx} = -\kappa_1 T_{\kappa_1}(x)$	$\Gamma^r_{\phi\phi} = -\kappa_2 S_{\kappa_1}(r)C_{\kappa_1}(r)$ $\Gamma^\phi_{\phi r} = 1/T_{\kappa_1}(r)$
$\mathrm{d}\mathcal{S} = C_{\kappa_1\kappa_2}(y)\,\mathrm{d}a\wedge\mathrm{d}y$	$\mathrm{d}\mathcal{S} = C_{\kappa_1}(x)\,\mathrm{d}x\wedge\mathrm{d}b$	$\mathrm{d}\mathcal{S} = S_{\kappa_1}(r)\,\mathrm{d}r\wedge\mathrm{d}\phi$

$\mathbf{S}^2 = S^2_{[+],+}$	$\mathbf{E}^2 = S^2_{[0],+}$	$\mathbf{H}^2 = S^2_{[-],+}$
$x^0 = \cos a\, \cos y$ $x^1 = \sin a\, \cos y$ $x^2 = \sin y$	$x^0 = 1$ $x^1 = a$ $x^2 = y$	$x^0 = \cosh a\, \cosh y$ $x^1 = \sinh a\, \cosh y$ $x^2 = \sinh y$
$(\mathrm{d}s^2)_1 = \cos^2 y\, \mathrm{d}a^2 + \mathrm{d}y^2$	$(\mathrm{d}s^2)_1 = \mathrm{d}a^2 + \mathrm{d}y^2$	$(\mathrm{d}s^2)_1 = \cosh^2 y\, \mathrm{d}a^2 + \mathrm{d}y^2$
$\Gamma^y_{aa} = \sin y\, \cos y$ $\Gamma^a_{ay} = -\tan y$	$\Gamma^y_{aa} = 0$ $\Gamma^a_{ay} = 0$	$\Gamma^y_{aa} = -\sinh y\, \cosh y$ $\Gamma^a_{ay} = \tanh y$
$\mathrm{d}\mathcal{S} = \cos y\, \mathrm{d}a \wedge \mathrm{d}y$	$\mathrm{d}\mathcal{S} = \mathrm{d}a \wedge \mathrm{d}y$	$\mathrm{d}\mathcal{S} = \cosh y\, \mathrm{d}a \wedge \mathrm{d}y$
$\mathbf{NH}^{1+1}_{+} = S^2_{[+1/\tau^2],0}$	$\mathbf{G}^{1+1} = S^2_{[0],0}$	$\mathbf{NH}^{1+1}_{-} = S^2_{[-1/\tau^2],0}$
$x^0 = \cos(t/\tau)$ $x^1 = \tau \sin(t/\tau)$ $x^2 = y$	$x^0 = 1$ $x^1 = t$ $x^2 = y$	$x^0 = \cosh(t/\tau)$ $x^1 = \tau \sinh(t/\tau)$ $x^2 = y$
$(\mathrm{d}s^2)_1 = \mathrm{d}t^2$ $(\mathrm{d}s^2)_2 = \mathrm{d}y^2 \qquad t = t_0$	$(\mathrm{d}s^2)_1 = \mathrm{d}t^2$ $(\mathrm{d}s^2)_2 = \mathrm{d}y^2 \quad t = t_0$	$(\mathrm{d}s^2)_1 = \mathrm{d}t^2$ $(\mathrm{d}s^2)_2 = \mathrm{d}y^2 \qquad t = t_0$
$\Gamma^y_{tt} = \frac{1}{\tau^2}\, y \qquad \Gamma^t_{ty} = 0$	$\Gamma^y_{tt} = 0 \qquad \Gamma^t_{ty} = 0$	$\Gamma^y_{tt} = -\frac{1}{\tau^2}\, y \quad \Gamma^t_{ty} = 0$
$\mathrm{d}\mathcal{S} = \mathrm{d}t \wedge \mathrm{d}y$	$\mathrm{d}\mathcal{S} = \mathrm{d}t \wedge \mathrm{d}y$	$\mathrm{d}\mathcal{S} = \mathrm{d}t \wedge \mathrm{d}y$
$\mathbf{AdS}^{1+1} = S^2_{[+1/\tau^2],-1/c^2}$	$\mathbf{M}^{1+1} = S^2_{[0],-1/c^2}$	$\mathbf{dS}^{1+1} = S^2_{[-1/\tau^2],-1/c^2}$
$x^0 = \cos(t/\tau)\cosh(y/c\tau)$ $x^1 = \tau \sin(t/\tau)\cosh(y/c\tau)$ $x^2 = c\tau \sinh(y/c\tau)$	$x^0 = 1$ $x^1 = t$ $x^2 = y$	$x^0 = \cosh(t/\tau)\cos(y/c\tau)$ $x^1 = \tau \sinh(t/\tau)\cos(y/c\tau)$ $x^2 = c\tau \sin(y/c\tau)$
$(\mathrm{d}s^2)_1 = \cosh^2(y/c\tau)\mathrm{d}t^2 - \frac{1}{c^2}\,\mathrm{d}y^2$	$(\mathrm{d}s^2)_1 = \mathrm{d}t^2 - \frac{1}{c^2}\,\mathrm{d}y^2$	$(\mathrm{d}s^2)_1 = \cos^2(y/c\tau)\mathrm{d}t^2 - \frac{1}{c^2}\,\mathrm{d}y^2$
$\Gamma^y_{tt} = \frac{c}{\tau}\sinh(y/c\tau)\cosh(y/c\tau)$ $\Gamma^t_{ty} = \frac{1}{c\tau}\tanh(y/c\tau)$	$\Gamma^y_{tt} = 0$ $\Gamma^t_{ty} = 0$	$\Gamma^y_{tt} = -\frac{c}{\tau}\sin(y/c\tau)\cos(y/c\tau)$ $\Gamma^t_{ty} = -\frac{1}{c\tau}\tan(y/c\tau)$
$\mathrm{d}\mathcal{S} = \cosh(y/c\tau)\,\mathrm{d}t \wedge \mathrm{d}y$	$\mathrm{d}\mathcal{S} = \mathrm{d}t \wedge \mathrm{d}y$	$\mathrm{d}\mathcal{S} = \cos(y/c\tau)\,\mathrm{d}t \wedge \mathrm{d}y$

2.1. Vector fields

The differential realization of the generators as first-order vector fields in $\mathbb{R}^3$

$$P_1 = \kappa_1 x^1 \partial_0 - x^0 \partial_1 \qquad P_2 = \kappa_1 \kappa_2 x^2 \partial_0 - x^0 \partial_2 \qquad J_{12} = \kappa_2 x^2 \partial_1 - x^1 \partial_2 .$$

Geodesic parallel I coordinates (a, y)

$$P_1 = -\partial_a \qquad P_2 = -\kappa_1\kappa_2 S_{\kappa_1}(a) T_{\kappa_1\kappa_2}(y)\,\partial_a - C_{\kappa_1}(a)\,\partial_y$$
$$J_{12} = \kappa_2 C_{\kappa_1}(a) T_{\kappa_1\kappa_2}(y)\,\partial_a - S_{\kappa_1}(a)\,\partial_y$$

Geodesic parallel II coordinates (x, b)

$$P_1 = -C_{\kappa_1\kappa_2}(b)\,\partial_x - \kappa_1 T_{\kappa_1}(x) S_{\kappa_1\kappa_2}(b)\,\partial_b \qquad P_2 = -\partial_b$$
$$J_{12} = \kappa_2 S_{\kappa_1\kappa_2}(b)\,\partial_x - T_{\kappa_1}(x) C_{\kappa_1\kappa_2}(b)\,\partial_b$$

Geodesic polar coordinates (r, ϕ)

$$P_1 = -C_{\kappa_2}(\phi)\,\partial_r + \frac{S_{\kappa_2}(\phi)}{T_{\kappa_1}(r)}\,\partial_\phi \qquad P_2 = -\kappa_2 S_{\kappa_2}(\phi)\,\partial_r - \frac{C_{\kappa_2}(\phi)}{T_{\kappa_1}(r)}\,\partial_\phi$$
$$J_{12} = -\partial_\phi$$

These vectors fields satisfy the Killing equations for the metrics g_1, g_2 of the space $S^2_{[\kappa_1],\kappa_2}$, that is, $L_X g_i = \mu_X\, g_i = 0$, where $L_X g_i$ is the Lie derivative of g_i.

Therefore we find an application to Lie systems.

2.2. Laplace/wave-type equations

Let us consider a 2D space with coordinates (u^1, u^2), a differential operator $E = E(u^1, u^2, \partial_1, \partial_2)$ acting on functions $\Phi(u^1, u^2)$ defined on the space ($\partial_i \equiv \partial/\partial u^i$), and consider the differential equation:

$$E\Phi(u^1, u^2) = 0.$$

An operator $\mathcal{O}$ is a **symmetry** if $\mathcal{O}$ transforms solutions into solutions:

$$E\,\mathcal{O} = \mathcal{Q}\,E \qquad \text{or} \qquad [E, \mathcal{O}] = \mathcal{Q}'\,E$$

where $\mathcal{Q}$ is another operator and $\mathcal{Q}' = \mathcal{Q} - \mathcal{O}$.

We consider the **differential equation** obtained by taking as E the **Casimir** $\mathcal{C}$ of the CK algebra $so_{\kappa_1,\kappa_2}(3)$ in the space $S^2_{[\kappa_1],\kappa_2}$: $\mathcal{C}\Phi = 0$.

Recall that

$$\mathcal{C} = P_2^2 + \kappa_2 P_1^2 + \kappa_1 J_{12}^2, \qquad [\mathcal{C}, X] = 0, \qquad X \in \{P_1, P_2, J_{12}\}.$$

In the three geodesic coordinate systems, such an equation turns out to be

$$\left(\frac{\kappa_2}{C^2_{\kappa_1\kappa_2}(y)}\, \partial_a^2 + \partial_y^2 - \kappa_1\kappa_2 T_{\kappa_1\kappa_2}(y)\, \partial_y \right) \Phi(a, y) = 0$$

$$\left(\kappa_2 \partial_x^2 - \kappa_1\kappa_2 T_{\kappa_1}(x)\, \partial_x + \frac{1}{C^2_{\kappa_1}(x)}\, \partial_b^2 \right) \Phi(x, b) = 0$$

$$\left(\kappa_2\, \partial_r^2 + \frac{\kappa_2}{T_{\kappa_1}(r)} \partial_r + \frac{1}{S^2_{\kappa_1}(r)} \partial_\phi^2 \right) \Phi(r, \phi) = 0.$$

Table 3: The Laplace–Beltrami operator $\mathcal{C}$ giving rise to differential Laplace and wave-type equations $\mathcal{C}\Phi = 0$ in geodesic parallel I $\Phi(a, y) \equiv \Phi(t, y)$ and polar $\Phi(r, \phi) \equiv \Phi(r, \chi)$ coordinates for the nine CK spaces (when $\kappa_2 < 0$ the angle is denoted as χ and is a rapidity in the kinematical interpretation).

$so(3):\ \mathbf{S}^2 = S^2_{[+],+}$	$iso(2):\ \mathbf{E}^2 = S^2_{[0],+}$	$so(2,1):\ \mathbf{H}^2 = S^2_{[-],+}$
$\frac{1}{\cos^2 y}\,\partial_a^2 + \partial_y^2 - \tan y\,\partial_y$	$\partial_a^2 + \partial_y^2$	$\frac{1}{\cosh^2 y}\,\partial_a^2 + \partial_y^2 + \tanh y\,\partial_y$
$\frac{1}{\sin^2 r}\,\partial_\phi^2 + \partial_r^2 + \frac{1}{\tan r}\,\partial_r$	$\frac{1}{r^2}\,\partial_\phi^2 + \partial_r^2 + \frac{1}{r}\,\partial_r$	$\frac{1}{\sinh^2 r}\,\partial_\phi^2 + \partial_r^2 + \frac{1}{\tanh r}\,\partial_r$
$iso(2):\ \mathbf{NH}_+^{1+1} = S^2_{[+1/\tau^2],0}$	$iiso(1):\ \mathbf{G}^{1+1} = S^2_{[0],0}$	$iso(1,1):\ \mathbf{NH}_-^{1+1} = S^2_{[-1/\tau^2],0}$
∂_y^2	∂_y^2	∂_y^2
$\frac{1}{\tau^2\sin^2(r/\tau)}\,\partial_\chi^2$	$\frac{1}{r^2}\,\partial_\chi^2$	$\frac{1}{\tau^2\sinh^2(r/\tau)}\,\partial_\chi^2$
$so(2,1):\ \mathbf{AdS}^{1+1} = S^2_{[+1/\tau^2],-1/c^2}$	$iso(1,1):\ \mathbf{M}^{1+1} = S^2_{[0],-1/c^2}$	$so(2,1):\ \mathbf{dS}^{1+1} = S^2_{[-1/\tau^2],-1/c^2}$
$\frac{-1}{c^2\cosh^2(y/c\tau)}\,\partial_t^2 + \partial_y^2 + \frac{\tanh(y/c\tau)}{c\tau}\,\partial_y$	$-\frac{1}{c^2}\,\partial_t^2 + \partial_y^2$	$\frac{-1}{c^2\cos^2(y/c\tau)}\,\partial_t^2 + \partial_y^2 - \frac{\tan(y/c\tau)}{c\tau}\,\partial_y$
$\frac{1}{\tau^2\sin^2(r/\tau)}\,\partial_\chi^2 - \frac{1}{c^2}\,\partial_r^2 - \frac{1}{c^2\tau\tan(r/\tau)}\,\partial_r$	$\frac{1}{r^2}\,\partial_\chi^2 - \frac{1}{c^2}\,\partial_r^2 - \frac{1}{c^2 r}\,\partial_r$	$\frac{1}{\tau^2\sinh^2(r/\tau)}\,\partial_\chi^2 - \frac{1}{c^2}\,\partial_r^2 - \frac{1}{c^2\tau\tanh(r/\tau)}\,\partial_r$

• The usual 2D Laplace equation in $\mathbf{E}^2$ and the corresponding non-zero curvature **Laplace–Beltrami** versions in the sphere and hyperbolic plane.

• An equation which does not involve time in the three non-relativistic spacetimes (indeed reducing to a 1D 'Laplace' equation). This agrees with the known absence of a true Galilean invariant wave equation and is the main reason precluding a further development of non-relativistic electromagnetic theories, where only two separate electric and magnetic essentially static limits are allowed.

• The proper $(1+1)$D **wave equation** is associated to $\mathbf{M}^{1+1}$; its curvature versions correspond to anti-de Sitter and de Sitter electromagnetisms in both $\mathbf{AdS}^{1+1}$ and $\mathbf{dS}^{1+1}$.

3. Conformal symmetries

It is possible to enlarge the CK algebra by considering:

- A **dilation** generator D.

- Two specific **conformal transformations** G_1, G_2.

Procedure: by imposing **cycle-preserving transformations**.

Geodesic parallel I coordinates (a, y)

$$P_1 = -\partial_a \qquad P_2 = -\kappa_1\kappa_2 S_{\kappa_1}(a) T_{\kappa_1\kappa_2}(y)\,\partial_a - C_{\kappa_1}(a)\,\partial_y$$

$$J_{12} = \kappa_2 C_{\kappa_1}(a) T_{\kappa_1\kappa_2}(y)\,\partial_a - S_{\kappa_1}(a)\,\partial_y \qquad D = -\frac{S_{\kappa_1}(a)}{C_{\kappa_1\kappa_2}(y)}\,\partial_a - C_{\kappa_1}(a) S_{\kappa_1\kappa_2}(y)\,\partial_y$$

$$G_1 = \frac{1}{C_{\kappa_1\kappa_2}(y)}\left(V_{\kappa_1}(a) - \kappa_2 V_{\kappa_1\kappa_2}(y)\right)\partial_a + S_{\kappa_1}(a) S_{\kappa_1\kappa_2}(y)\,\partial_y$$

$$G_2 = \kappa_2 S_{\kappa_1}(a) T_{\kappa_1\kappa_2}(y)\,\partial_a - \left(V_{\kappa_1}(a) - \kappa_2 V_{\kappa_1\kappa_2}(y)\right)\partial_y$$

The commutation rules and Casimirs of $\mathrm{conf}_{\kappa_1,\kappa_2}$ read

$$
\begin{array}{lll}
[J_{12}, P_1] = P_2 & [J_{12}, P_2] = -\kappa_2 P_1 & [P_1, P_2] = \kappa_1 J_{12} \\
[J_{12}, G_1] = G_2 & [J_{12}, G_2] = -\kappa_2 G_1 & [G_1, G_2] = 0 \\
[D, P_i] = P_i + \kappa_1 G_i & [D, G_i] = -G_i & [D, J_{12}] = 0 \\
[P_1, G_1] = D & [P_2, G_2] = \kappa_2 D & \\
[P_1, G_2] = -J_{12} & [P_2, G_1] = J_{12} &
\end{array}
$$

$$
\begin{aligned}
\mathcal{C}_1 &= -J_{12}^2 + \kappa_2 D^2 + \kappa_2(P_1 G_1 + G_1 P_1) + (P_2 G_2 + G_2 P_2) + \kappa_1(\kappa_2 G_1^2 + G_2^2) \\
\mathcal{C}_2 &= J_{12} D + (G_1 P_2 - P_1 G_2)
\end{aligned}
$$

All spaces in the family $S^2_{[\kappa_1],\kappa_2}$ with the **same** κ_2 have **isomorphic** conformal algebras. These are:

- $so(3,1)$ ($(2+1)$D de Sitter algebra) as the conformal algebra of the three 2D **Riemannian spaces** with $\kappa_2 > 0$.
- $iso(2,1)$ ($(2+1)$D Poincaré) for the $(1+1)$D **non-relativistic spacetimes** with $\kappa_2 = 0$.
- $so(2,2)$ ($(2+1)$D anti-de Sitter) for the $(1+1)$D **relativistic spacetimes** with $\kappa_2 < 0$.

In relation with the usual approach to conformal groups by solving the conformal Killing equations we state the following:

Proposition. *All the conformal vectors fields X satisfy the conformal Killing equations for the metrics g_1, g_2 of the space $S^2_{[\kappa_1],\kappa_2}$, that is, $L_X g_i = \mu_X\, g_i$, where $L_X g_i$ is the Lie derivative of g_i. In Weierstrass coordinates the conformal factors μ_X are given by*

$$\mu_{P_1} = \mu_{P_2} = \mu_{J_{12}} = 0 \qquad \mu_D = -2x^0 \qquad \mu_{G_1} = 2x^1 \qquad \mu_{G_2} = 2\kappa_2 x^2.$$

Hence the conformal vector fields of the CK spaces would allow one the construction of **new Lie systems**.

4. N-dimensional CK spaces

4.1. Orthogonal CK algebras

Let us consider the real Lie algebra $so(N+1)$ whose $\frac{1}{2}N(N+1)$ generators J_{ab} $(a, b = 0, 1, \ldots, N,\, a < b)$ satisfy the non-vanishing Lie brackets given by

$$[J_{ab}, J_{ac}] = J_{bc}, \quad [J_{ab}, J_{bc}] = -J_{ac}, \quad [J_{ac}, J_{bc}] = J_{ab}, \quad a < b < c.$$

A grading group $\mathbf{Z}_2^{\otimes N}$ of $so(N+1)$ is spanned by the following N commuting **involutive automorphisms** $\Theta^{(m)}$ $(m = 1, \ldots, N)$:

$$\Theta^{(m)}(J_{ab}) = \begin{cases} J_{ab}, & \text{if either } m \leq a \text{ or } b < m; \\ -J_{ab}, & \text{if } a < m \leq b. \end{cases}$$

A large family of contracted real Lie algebras can be obtained from $so(N+1)$; this depends on $2^N - 1$ **real contraction parameters** which includes from the simple pseudo-orthogonal algebras $so(p, q)$ (the B_l and D_l Cartan series) (when all the contraction parameters are different from zero) up to the Abelian algebra at the opposite case (when all the parameters are equal to zero).

Properties associated with the simplicity of the algebra are lost at some point beyond the simple algebras in the contraction sequence.

There exists a **particular subset of contrated Lie algebras** which are “close to” to the simple ones, whose members are called CK or quasi-simple orthogonal algebras: all the CK algebras share, in any dimension, the same **rank** defined as the number of (functionally independent) Casimir invariants.

This **orthogonal CK family**, here denoted $so_\kappa(N+1)$, depends on N **real contraction coefficients** $\kappa = (\kappa_1, \ldots, \kappa_N)$:

$$[J_{ab}, J_{ac}] = \kappa_{ab} J_{bc}, \quad [J_{ab}, J_{bc}] = -J_{ac}, \quad [J_{ac}, J_{bc}] = \kappa_{bc} J_{ab}, \quad a < b < c,$$

without sum over repeated indices and where the two-index coefficients κ_{ab} are expressed in terms of the N basic ones through

$$\kappa_{ab} = \kappa_{a+1}\kappa_{a+2}\cdots\kappa_b, \quad a, b = 0, 1, \ldots, N, \quad a < b.$$

Each non-zero real coefficient κ_m can be reduced to either $+1$ or -1 by a rescaling of the Lie generators.

There are 3^N CK algebras.

The case $\kappa_m = 0$ can be interpreted as an **Inönü–Wigner contraction**, with parameter $\varepsilon_m \to 0$, and defined by the map

$$\Gamma^{(m)}(J_{ab}) = \begin{cases} J_{ab}, & \text{if either } m \leq a \text{ or } b < m; \\ \varepsilon_m J_{ab}, & \text{if } a < m \leq b. \end{cases}$$

Each involution $\Theta^{(m)}$ provides a **Cartan decomposition** as a direct sum of anti-invariant and invariant subspaces, denoted $p^{(m)}$ and $h^{(m)}$, respectively:

$$so_\kappa(N+1) = p^{(m)} \oplus h^{(m)},$$

with the linear sum referring to the linear structure; Lie commutators fulfil:

$$[h^{(m)}, h^{(m)}] \subset h^{(m)}, \quad [h^{(m)}, p^{(m)}] \subset p^{(m)}, \quad [p^{(m)}, p^{(m)}] \subset h^{(m)},$$

and thus $h^{(m)}$ is always a **Lie subalgebra** with a direct sum structure:

$$h^{(m)} = so_{\kappa_1,\dots,\kappa_{m-1}}(m) \oplus so_{\kappa_{m+1},\dots,\kappa_N}(N+1-m),$$

while the vector subspace $p^{(m)}$ is generally not a subalgebra.

The Cartan decomposition can be visualized in array form as follows:

$$
\begin{array}{cccc|cccc}
J_{01} & J_{02} & \cdots & J_{0\,m-1} & J_{0m} & J_{0\,m+1} & \cdots & J_{0N} \\
 & J_{12} & \cdots & J_{1\,m-1} & J_{1m} & J_{1\,m+1} & \cdots & J_{1N} \\
 & & \ddots & \vdots & \vdots & \vdots & & \vdots \\
 & & & J_{m-2\,m-1} & J_{m-2\,m} & J_{m-2\,m+1} & \cdots & J_{m-2\,N} \\
 & & & & J_{m-1\,m} & J_{m-1\,m+1} & \cdots & J_{m-1\,N} \\
\cline{5-8}
\multicolumn{1}{c}{} & & & \multicolumn{1}{c}{} & & J_{m\,m+1} & \cdots & J_{m\,N} \\
\multicolumn{1}{c}{} & & & \multicolumn{1}{c}{} & & & \ddots & \vdots \\
\multicolumn{1}{c}{} & & & \multicolumn{1}{c}{} & & & & J_{N-1\,N}
\end{array}
$$

The subspace $p^{(m)}$ is spanned by the $m(N+1-m)$ generators inside the rectangle; the left and down triangles correspond, in this order, to the subalgebras $so_{\kappa_1,\ldots,\kappa_{m-1}}(m)$ and $so_{\kappa_{m+1},\ldots,\kappa_N}(N+1-m)$ of $h^{(m)}$.

• When all $\kappa_a \neq 0\ \forall a$, $so_\kappa(N+1)$ is a **(pseudo-)orthogonal algebra** $so(p,q)$ $(p+q=N+1)$ and (p,q) are the number of positive and negative terms in the invariant quadratic form with matrix $(1, \kappa_{01}, \kappa_{02}, \ldots, \kappa_{0N})$.

• When $\kappa_1 = 0$ we recover the **inhomogeneous algebras** with semidirect sum structure

$$so_{0,\kappa_2,\ldots,\kappa_N}(N+1) \equiv t_N \odot so_{\kappa_2,\ldots,\kappa_N}(N) \equiv iso(p,q), \quad p+q=N,$$

where the Abelian subalgebra t_N is spanned by $\langle J_{0b};\ b=1,\ldots,N\rangle$ and $so_{\kappa_2,\ldots,\kappa_N}(N)$ preserves the quadratic form with matrix $\mathrm{diag}(+, \kappa_{12}, \ldots, \kappa_{1N})$.

• When $\kappa_1 = \kappa_2 = 0$ we get a **"twice-inhomogeneous" pseudo-orthogonal algebra**

$$so_{0,0,\kappa_3,\ldots,\kappa_N}(N+1) \equiv t_N \odot (t_{N-1} \odot so_{\kappa_3,\ldots,\kappa_N}(N-1)) \equiv iiso(p,q), \quad p+q=N-1,$$

where the metric of the subalgebra $so_{\kappa_3,\ldots,\kappa_N}(N-1)$ is $(1, \kappa_{23}, \kappa_{24}, \ldots, \kappa_{2N})$.

• When $\kappa_a = 0$, $a \notin \{1, N\}$, these contracted algebras can be described as

$$t_{a(N+1-a)} \odot (so_{\kappa_1,\ldots,\kappa_{a-1}}(p,q) \oplus so_{\kappa_{a+1},\ldots,\kappa_N}(p',q')), \quad p+q=a, \quad p'+q'=N+1-a.$$

• The fully contracted case in the CK family corresponds to setting all $\kappa_a = 0$. This is the so called **flag algebra** $so_{0,\ldots,0}(N+1) \equiv i\ldots iso(1)$ such that $iso(1) \equiv \mathbb{R}$.

4.2. Symmetrical homogeneous CK spaces

If we now consider the CK group $SO_\kappa(N+1)$ with Lie algebra $so_\kappa(N+1)$ we find that each Lie subalgebra $h^{(m)}$ generates a subgroup $H^{(m)}$ leading to the homogeneous coset space denoted by:

$$\mathcal{S}^{(m)} \equiv SO_\kappa(N+1)\,/\big(SO_{\kappa_1,\dots,\kappa_{m-1}}(m)\otimes SO_{\kappa_{m+1},\dots,\kappa_N}(N+1-m)\big)\ .$$

The **dimension** of $\mathcal{S}^{(m)}$ is that of $p^{(m)}$:

$$\dim(\mathcal{S}^{(m)}) = m(N+1-m).$$

Then $\mathcal{S}^{(m)}$ is a symmetrical homogeneous space and there are N such symmetrical homogeneous spaces $\mathcal{S}^{(m)}$ $(m=1,\dots,N)$ for each CK group $SO_\kappa(N+1)$.

We define the **rank** of the CK space $\mathcal{S}^{(m)}$ as the number of independent invariants under the action of the CK group for each generic pair of elements in $\mathcal{S}^{(m)}$:

$$\mathrm{rank}(\mathcal{S}^{(m)}) = \min(m, N+1-m).$$

The sectional **curvature** of $\mathcal{S}^{(m)}$ turns out to be constant and equal to κ_m.

Table 4: Isotopy subgroup, sectional curvature, dimension and rank of the set of N symmetrical homogeneous spaces $\mathcal{S}^{(m)} \equiv SO_\kappa(N+1)/H^{(m)}$.

Isotopy subgroup	Curv.	Dimension	Rank
$H^{(1)} = SO_{\kappa_2,\dots,\kappa_N}(N)$	κ_1	N	1
$H^{(2)} = SO_{\kappa_1}(2) \otimes SO_{\kappa_3,\dots,\kappa_N}(N-1)$	κ_2	$2(N-1)$	2
$H^{(3)} = SO_{\kappa_1,\kappa_2}(3) \otimes SO_{\kappa_4,\dots,\kappa_N}(N-2)$	κ_3	$3(N-2)$	3
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$H^{(m)} = SO_{\kappa_1,\dots,\kappa_{m-1}}(m) \otimes SO_{\kappa_{m+1},\dots,\kappa_N}(N+1-m)$	κ_m	$m(N+1-m)$	$\min(m, N+1-m)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$H^{(N-2)} = SO_{\kappa_1,\dots,\kappa_{N-3}}(N-2) \otimes SO_{\kappa_{N-1},\kappa_N}(3)$	κ_{N-2}	$(N-2)3$	3
$H^{(N-1)} = SO_{\kappa_1,\dots,\kappa_{N-2}}(N-1) \otimes SO_{\kappa_N}(2)$	κ_{N-1}	$(N-1)2$	2
$H^{(N)} = SO_{\kappa_1,\dots,\kappa_{N-1}}(N)$	κ_N	N	1

5. Conclusions

• We have provided a review on CK spaces and their vector fields.

• These results could be applied to the field of Lie and Lie-Hamilton systems.

• The case for the vector fields coming from isometries is currently in progress.

• The case for the vector fields coming from conformal symmetries is devoted for the future.

References

- Yaglom I M, Rozenfel'd B A and Yasinskaya E U 1966 *Sov. Math. Surveys* **19** 49

- Yaglom I M 1979 *A Simple Non-Euclidean Geometry and its Physical Basis* (New York: Springer)

- Ballesteros A, FJH, del Olmo M A and Santander M 1993 *J. Phys. A: Math. Gen.* **26** 5801

- FJH, Ortega R and Santander M 2000 *J. Phys. A: Math. Gen.* **33** 4525

- FJH and Santander M 2002 *J. Phys. A: Math. Gen.* **35** 6601

- Ballesteros A, FJH, Ragnisco O, and Santander M 2008 *Int .J. Theor. Phys* **47** 649