The Heisenberg group and $SL_2(\mathbb{R})$

a survival pack

Vladimir V. Kisil

School of Mathematics
University of Leeds (England)

email: kisilv@maths.leeds.ac.uk
Web: http://www.maths.leeds.ac.uk/~kisilv

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The theory of a complex variable has intimate connections with the harmonic analysis. It was pretty obvious from the beginning that several complex variables cannot serve a similar role to the theory of harmonic functions in many dimensions.

E. Stein and his school developed real variable technique to fill the gap. It was primary based on the Hardy-Littlewood maximal functions.

Clifford analysis, which flourished a bit later, provides the right path to treat harmonic functions in the spirit of complex variables.

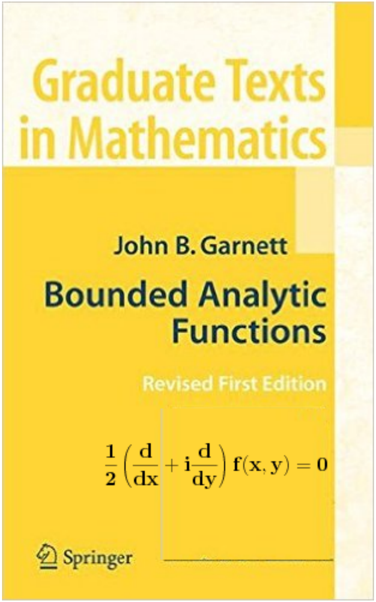
It seems, that two techniques are based on different ideologies and sometime people confront one to another.

Both methods seems to have clear advantages. The real variable technique:

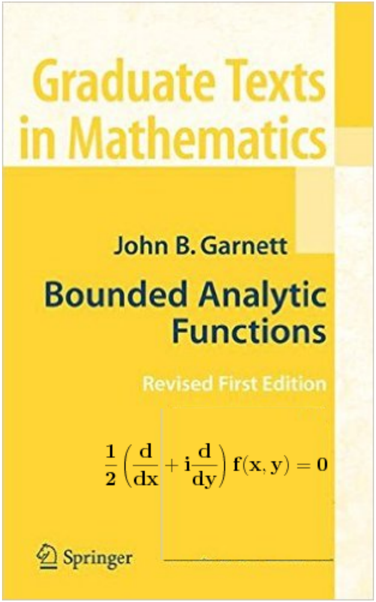
1. does not require an introduction of the imaginary unit for a study of real-valued harmonic functions of a real variable (Occam's Razor);
2. allows a straightforward generalisation to several dimensions.

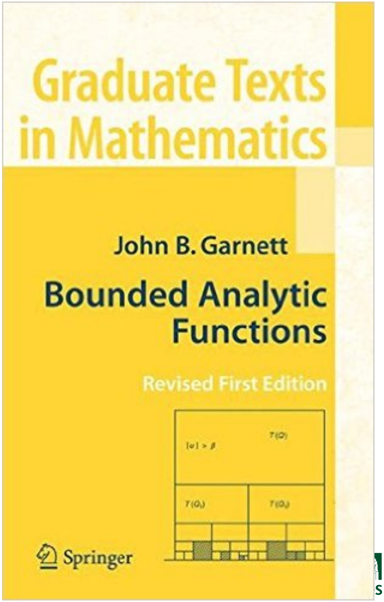
By contrast, an access to the beauty and mighty of analytic functions (e.g., Möbius transformations, factorisation of zeroes, etc.) is the main reason to use the complex variable technique.

Complex vs real elucidated



Complex vs real elucidated





Complex Methods

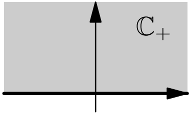
in the upper half-plane

1. The Cauchy and Poisson integrals:

$$f(t) \mapsto f(z) = \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{f(t)\,dt}{t - z}, \qquad f(t) \mapsto f(z) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{f(t) y\,dt}{(t-x)^2 + y^2}.$$

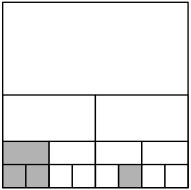
2. The upper half-plane of complex numbers $\mathbb{C}_+ = \{z \mid \Im z > 0\}$

to



3. The Cauchy–Riemann and Laplace equations on $\mathbb{C}_+$.
4. Boundary limit values $\lim\limits_{y \to 0+} f(x + iy)$ on the real line.
5. The Hardy space H_p, that is $L_p(\mathbb{R})$ functions with analytic extension to $\mathbb{C}_+$.
6. Sokhotski–Plemelj formula, singular integrals (SIO), e.g. the Szegö projection $L_2(\mathbb{R}) \to H_2$.

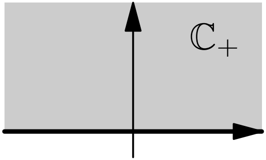
The techniques which do not use neither analytic or harmonic functions:

1. The Hardy–Littlewood maximal function
$$f(t) \mapsto f^*(x) = \sup_{I \ni x} \frac{1}{|I|} \int_I |f(t)|\, dt;$$
2. Only work on the real line?
Note the hidden two-parameter family parametrising intervals I!
3. The dyadic squares techniques: 
4. Non-tangential maximal function.
5. The Hardy space of functions with atomic decomposition;
6. Singular integrals.

Complex and Real Methods

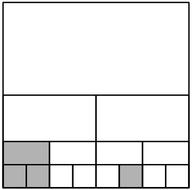
the comparison

1. The Cauchy integral
   $f(t) \mapsto f(z) = \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{f(t)\,dt}{t-z}$;
2. The upper half-plane



3. The Cauchy–Riemann equations;
4. Limit values $\lim_{y\to 0+} f(x+iy)$;
5. The Hardy space (analytic);
6. Singular Integrals.

1. The HL maximal function
   $f^*(x) = \sup_{I\ni x} \frac{1}{|I|} \int_I |f(t)|\,dt$;
2. Only work on the real line?
3. The dyadic squares

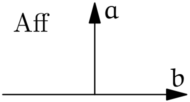


4. NT maximal function;
5. The Hardy space (atomic);
6. Singular Integrals.

Affine Group

Dilations and Shifts

Combining two operations—dilations and shits—on $\mathbb{R}$ we obtain the *affine* or $ax+b$ group $G = \mathrm{Aff}$:



$$(a, b)\cdot(a', b') = (aa', ab' + b), \quad a \in \mathbb{R}_+,\ b \in \mathbb{R}$$

Identity is $(1, 0)$ and the inverse $(a, b)^{-1} = (\frac{1}{a}, -\frac{b}{a})$.
Outer automorphism: $J : (a, b) \mapsto (a, -b)$.
A left invariant (Haar) measure on Aff is $a^{-2}\,da\,db$.
An isometric representation of Aff on $V = L_p(\mathbb{R})$ is given by the formula:

$$[\rho_p(a, b)\, f](x) = a^{-\frac{1}{p}} f\left(\frac{x-b}{a}\right). \qquad (1)$$

It is called the *quasi-regular representation* and it is reducible.

An alternative *co-adjoint representation* acts isometrically on $L_p(\mathbb{R})$:

$$[\hat{\rho}_p(a, b)\, f](\lambda) = a^{\frac{1}{p}}\, e^{-2\pi i b\lambda} f(a\lambda). \tag{2}$$

Since $a > 0$, there is a decomposition into invariant subspaces of $\hat{\rho}_p$:

$$L_p(\mathbb{R}) = L_p(-\infty, 0) \oplus L_p(0, \infty). \tag{3}$$

Denote by $\hat{\rho}_p^-$ and $\hat{\rho}_p^+$ restrictions of $\hat{\rho}_p$ to these subspaces, the subrepresentations are irreducible and not equivalent to each other. Gelfand&Naimark showed that any unitary irreducible representation of Aff is equivalent to either $\hat{\rho}_2^-$ or $\hat{\rho}_2^+$. There is a unitary intertwining operator between the quasi-regular and co-adjoint representation—the Fourier transform:

$$\mathcal{F}f(\lambda) = \int_{\mathbb{R}} f(x) e^{-2\pi i x\lambda}\, dx.$$

The outer automorphism J swaps $\hat{\rho}_2^-$ or $\hat{\rho}_2^+$.

Definition 1.
The Hardy space H_p is an irreducible component of $L_p(\mathbb{R})$ of the quasi-regular representation of Aff.

It is easy to see, that this coincides with one of the traditional definitions of H_2 as the space of L_2 functions such that $\mathcal{F}v(\lambda) = 0$ for $\lambda < 0$.
In relation to Aff, the Fourier transform

- intertwines shifts in the quasi-regular representation to operators of multiplication in the co-adjoint representation;
- intertwines dilations in the quasi-regular representation to dilations in the co-adjoint representation;
- maps the decomposition $L_2(\mathbb{R}) = H_2 \oplus H_2^{\perp}$ into spatially separated spaces with disjoint supports;
- anticommutes with J, which interchanges ρ_2^{+} and ρ_2^{-}.

For a group $G = \mathrm{Aff}$, its representation ρ_2 in $L_2(\mathbb{R})$ and a given vector $v \in L_2(\mathbb{R})$ there is the wavelet transform:

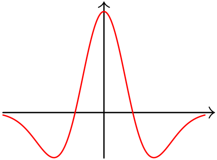
$$\mathcal{W}_v : f \mapsto \hat{f}(g) = \left\langle \rho(g^{-1})f, v \right\rangle = \langle f, \rho(g)v \rangle , \qquad f \in L_2(\mathbb{R}),\ g \in G. \tag{4}$$

The inverse wavelet transform $L_2(G) \to L_2(\mathbb{R})$ with the reconstructing vector $w \in L_2(\mathbb{R})$ is done by the integration over the Haar measure:

$$\mathcal{M}_w h = \int_G h(g)\, w(g)\, dg, \qquad \text{where } w(g) = \rho(g)w. \tag{5}$$

To have unitary operators we need to put the admissibility condition:

$$\int_0^\infty \frac{|\widehat{v}(\xi)|^2}{\xi}\, d\xi < \infty \quad \text{or} \quad \int_{\mathbb{R}} v(x)\, dx = 0. \tag{6}$$



For two (possibly different) admissible v and w we still have $\mathcal{M}_w \mathcal{W}_v = kI$.

If we could drop the admissibility, then we can incorporate into covariant transform many important familiar examples from the harmonic analysis.

Example 1 (Complex variable technique).

Take $v(x) = \frac{1}{x+i}$, then wavelets coincide with the Cauchy kernel:

$$[\rho_p(a,b)v](x) = a^{-\frac{1}{p}} \frac{1}{\left(\frac{x-b}{a}\right)+i} = a^{-\frac{1}{q}} \frac{1}{x-(b-ia)}.$$

Thus the Cauchy integral (the wavelet transform) maps functions on $\mathbb{R}$ to functions on $G = \mathrm{Aff}$, which is commonly confused with the upper half-plane in $\mathbb{C}$.

Example 2 (Real variable technique).

Similarly $\frac{1}{x^2+1}$ produces the Poisson kernel and the corresponding wavelet transform is the Poisson integral. Again, with extension to $G = \mathrm{Aff}$, rather than the upper half-plane in $\mathbb{R}^2$.

For contravariant transform we can drop admissibility if, instead of the Haar integration, we use the left invariant Hardy-type pairing on $G = \mathrm{Aff}$:

$$\langle f_1, f_2 \rangle_H = \lim_{a \to 0} \int_{-\infty}^{\infty} f_1(a,b)\, f_2(a,b)\, \frac{db}{a}. \tag{7}$$

Example 3 (The boundary limit).

Take $w(t) = \frac{1}{2}\chi_{[-1,1]}(t)$, then the reconstruction formula becomes:

$$\begin{aligned} [\mathcal{M}_w f](x) &= \lim_{a \to 0} \int_{-\infty}^{\infty} f(a,b)\, a^{-\frac{1}{p}}\, \frac{\chi_{[-1,1]}}{2} \left(\frac{x-b}{a} \right) \frac{db}{a} \\ &= \lim_{a \to 0} a^{-\frac{1}{p}} \frac{1}{2a} \int_{x-a}^{x+a} f(a,b)\, db. \end{aligned}$$

Thus we obtained the “boundary” value of a function on Aff.

Recall the left $\Lambda(g) : f(h) \mapsto f(g^{-1}h)$ and the right $R(g) : f(h) \mapsto f(hg)$ actions of G on $L(G)$. The well-known intertwining properties of the wavelet transform and the left action:

$$\mathcal{W}_v \rho(g) = \Lambda(g) \mathcal{W}_v \quad \text{and} \quad \mathcal{M}_w \Lambda(g) = \rho(g) \mathcal{M}_w .$$

It is less-known that the right action is intertwined with the action on mother wavelet (for $\mathcal{M}$ with a possible modulation):

$$R(g) \circ \mathcal{W}_v = \mathcal{W}_{\rho(g)v} \quad \text{and} \quad \mathcal{M}_w \circ R(g) = \mathcal{M}_{\rho(g^{-1})w} .$$

Corollary 2.

If the mother wavelet v *is annihilated by an operator* $A = \sum_J a_j d\rho_B^{X_j}$, *the wavelet transform* $\mathcal{W}_v f(g) = \langle f, \rho(g) v \rangle$ *is in the kernel of the operator* $D = \sum_j \bar{a}_j \mathfrak{L}^{X_j}$.

Examples of Analyticity

For the affine group the derived and Lie actions are:

$$[d\rho^A f](x) = -f(x) - xf'(x), \ [d\rho^N f](x) = -f'(x), \quad \mathfrak{L}^A = a\partial_a, \ \mathfrak{L}^N = a\partial_b.$$

Example 4 (Cauchy–Riemann and Laplace operators).

- The mother wavelet $\frac{1}{x+i}$ is a null solution of the operator

  $$-d\rho^A - i d\rho^N = I + (x+i)\frac{d}{dx}.$$

  The wavelet transform are the null solutions to the operator $-\mathfrak{L}^A + i\mathfrak{L}^N = ia(\partial_b + i\partial_a)$—the Cauchy–Riemann operator.
- The function $\frac{1}{\pi}\frac{1}{x^2+1}$ is a null solution of the operator:

  $$(d\rho^A)^2 - d\rho^A + (d\rho^N)^2 = 2I + 4x\frac{d}{dx} + (1+x^2)\frac{d^2}{dx^2}.$$

  The Poisson integral produces null solutions of $(\mathfrak{L}^A)^2 - \mathfrak{L}^A + (\mathfrak{L}^N)^2 = a^2(\partial_b^2 + \partial_a^2)$—the Laplacian.

Laplace, Fourier and Cauchy transforms

from the co-adjoint representation

A mother wavelet v_0 satisfying $(-d\hat{\rho}^A - id\hat{\rho}^N)v_0 = -\lambda(2\pi + \partial_\lambda)v_0 = 0$ in the co-adjoint representation (like $\frac{1}{x+i}$ in the quasi-regular) is $v_0(\lambda) = e^{-2\pi\lambda}$. The respective coherent states are $\hat{\rho}(a, b)v_0(\lambda) = e^{-2\pi(a+ib)\lambda}$ and the corresponding wavelet transform:

$$[\widehat{\mathcal{W}}f](a, b) = \langle f, \hat{\rho}(a, b)v_0\rangle = \int_{\mathbb{R}_+} f(\lambda)\, e^{-2\pi i(a-ib)\lambda}\, \frac{d\lambda}{\lambda}$$

is effectively the *Laplace transform*. Since $\widehat{\mathcal{W}} \circ \mathcal{F}$ and $\mathcal{W}$ intertwine the same pair of representations (ρ and Λ), they are different by a factor.

Corollary 3.

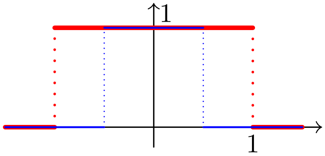
Up to some constant factors:

1. *The Fourier transform of* $\frac{1}{x+i}$ *is* $e^{-2\pi\lambda}\chi_{[0,\infty]}$.
2. *The composition of the Fourier and Laplace transforms is the Cauchy integral.*
3. $\lim_{a\to 0} [\widehat{\mathcal{W}}f](a, b) = \hat{f}(b)$, *if* $\hat{f} \in L_1(\mathbb{R})$..

Example 5 (Dyadic squares).

The function $\chi_{[-1,1]}$ is a null solution of the following functional equation:

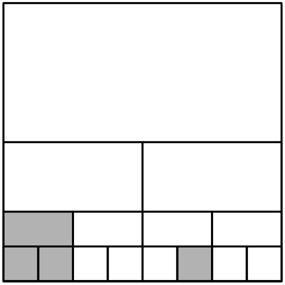
$$\left(I - \rho_\infty(\tfrac{1}{2}, \tfrac{1}{2}) - \rho_\infty(\tfrac{1}{2}, -\tfrac{1}{2})\right)\chi_{[-1,1]} = 0.$$



Consequently, the image of wavelet transform $\mathcal{W}_p^m$ solves the equation:

$$(I - R(\tfrac{1}{2}, \tfrac{1}{2}) - R(\tfrac{1}{2}, -\tfrac{1}{2}))f = 0 \quad \text{or}$$

$$f(a, b) = f(\tfrac{1}{2}a, b + \tfrac{1}{2}a) + f(\tfrac{1}{2}a, b - \tfrac{1}{2}a).$$



The last relation is the key to the dyadic cubes technique.

Composing Co- and Contra- Transforms

For square-integrable representations and admissible vectors we know that $\mathcal{M}_v \mathcal{W}_w = kI$ (from Schur's Lemma).
Let the mother wavelet $v(x) = \delta(x)$ be the Dirac delta function, then the wavelet transform $\mathcal{W}_\delta$ is $[\mathcal{W}_\delta f](a, b) = f(b)$.
Take the reconstruction vector $w(t) = (1 - \chi_{[-1,1]}(t))/(t\pi)$, consider $\mathcal{M}_w$ produced by the Hardy pairing. The composition of both maps is:

$$\begin{aligned}
[\mathcal{M}_w \circ \mathcal{W}_\delta f](t) &= \lim_{a\to 0} \frac{1}{\pi} \int_{-\infty}^{\infty} f(b)\, \rho_\infty(a, b) w(t) \,\frac{db}{a} \\
&= \lim_{a\to 0} \frac{1}{\pi} \int_{-\infty}^{\infty} f(b)\, \frac{1 - \chi_{[-a,a]}(t-b)}{t-b}\, db \\
&= \lim_{a\to 0} \frac{1}{\pi} \int_{|b|>a} \frac{f(b)}{t-b}\, db. \qquad (8)
\end{aligned}$$

Composing Co- and Contra- Transforms

Thus, we obtained:

$$[\mathcal{M}_w \circ \mathcal{W}_\delta f](t) = \lim_{a\to 0} \frac{1}{\pi} \int_{|b|>a} \frac{f(b)}{t-b}\, db.$$

This is the *Hilbert transform* $\mathcal{H}$, an example of SIO defined through the principal value in the sense of Cauchy.

Schur's Lemma tells that $\mathcal{H} = k_1 I_{H_2} \oplus k_2 I_{H_2^\perp}$ for some constants k_1, $k_2 \in \mathbb{C}$.

Furthermore, we can directly check that $\mathcal{H}J = -J\mathcal{H}$, thus $k_1 = -k_2$.

An evaluation of $\mathcal{H}$ on a simple function from H_2 (say, the Cauchy kernel $\frac{1}{x+i}$) gives the value of the constant $k_1 = -i$. Thus, $\mathcal{H} = (-iI_{H_2}) \oplus (iI_{H_2^\perp})$.

Proposition 4 (Uniqueness of SIO on the real line).

Any bounded linear operator on $L_2(\mathbb{R})$ commuting with quasi-regular representation ρ_2 and anticommuting with reflection J is a constant multiple of the Hilbert transform (??).

Composing Co- and Contra- Transforms

Let $w(t) = \chi_{[-1,1]}(t)$ and $w_{(a,b)} = \rho_\infty(a,b)w$. Obviously, $w_{(a,b)}(0) = w(-\frac{b}{a})$ is an eigenfunction for operators $\Lambda(a',0)$, $a' \in \mathbb{R}_+$ of the left regular representation of Aff, i.e. $\Lambda(a',0)w_{(a,b)}(0) = w_{(a,b)}(0)$.
Then:

$$\begin{aligned}[\mathcal{M}^H_w \circ \mathcal{W}_\nu f](0) &= [\mathcal{M}^H_{\Lambda(1/a,0)w} \circ \Lambda(1/a,0) \circ \mathcal{W}_\nu f](0)\\ &= [\mathcal{M}^H_w \circ \Lambda(1/a,0) \circ \mathcal{W}_\nu f](0)\\ &= [\mathcal{M}^H_w \circ \mathcal{W}_\nu \circ \rho_\infty(1/a,0)f](0).\end{aligned}$$

For $\nu \in L_1(\mathbb{R})$ and a continuous f such that $f(0) = 0$, the expression $\mathcal{W}_\nu \rho_\infty(1/a,0)f$ tenses to 0 as $a \to 0$. By the linearity, for any continuous function f the above expression is $cf(0)$, where $c = \int_{\mathbb{R}} \nu\, dt \neq 0$ (cf. the admissibility condition $\int_{\mathbb{R}} \nu\, dt = 0$).
For uniformly continuous functions, we can transfer to any point $x \in \mathbb{R}$ by the commutation of $\mathcal{M}^H_{w^*} \circ \mathcal{W}_\nu$ with the shifts $\rho_\infty(1,x)$.
Thus, $\mathcal{M}^H_{w^*} \circ \mathcal{W}_\nu = cI$, e.g. the boundary behaviour of the Poisson integral.

Summary

- Complex and real methods as well as wavelets are closely related branches of the same construction, which uses the affine group.
- The Cauchy and Poisson integrals, maximal functions are build by the same method as wavelet transform.
- Boundary values of analytic and harmonic functions, non-tangential maximal functions are relatives of the inverse wavelet transform.
- The Cauchy–Riemann and Laplace equations, together with dyadic squares follows from the intertwining properties of the wavelet transform.
- The Hardy space is an irreducible component of the affine group representation.
- Boundary values of analytic extensions, maximal functions and SIO are examples of composition of the wavelet transform and inverse wavelet transform, possibly with different analysing and reconstructing vectors.

Definition 5.

A *Lie algebra* is a vector space $\mathfrak{g}$ together with the *Lie bracket*: a non-associative bilinear map $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g} : (X, Y) \mapsto [X, Y]$ such that:

1. It is alternating: $[X, Y] = -[Y, X]$.
2. Satisfy the Jacobi identity

$$[x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0, \qquad \text{for all } x, y, z \in \mathfrak{g}.$$

Definition 6.

A real *Lie group* is a group that is also a finite-dimensional real smooth manifold, in which the group operations of multiplication and inversion are smooth maps. That is, the mapping

$$(x, y) \mapsto x^{-1}y$$

be a smooth mapping of the product manifold $G \times G$ into G.

Connecting Lie groups and Lie algebras

The key idea of analysis is a linearization of complicated object in small neighbourhoods. Applied to Lie groups it leads to the Lie algebras. There are several standard possibilities to realise the Lie algebra $\mathfrak{g}$ associated to a Lie group G:

1. *Generators* X of one-parameter subgroups: $x(t) = \exp(Xt)$, $t \in \mathbb{R}$.
2. *Tangent vectors* to the group at the group unit.
3. *Invariant vector fields* (first-order differential operators) on G.

To work in the opposite direction—from a Lie algebra $\mathfrak{g}$ to the Lie group G—we use the important *exponential map* between a Lie algebra and respective Lie group. The exponent function can be defined in any topological algebra as the sum of the following series:

$$\exp(tX) = \sum_{n=0}^{\infty} \frac{(tX)^n}{n!}. \tag{9}$$

This correspondence is one-to-many since it does not distinguish locally isomorphic Lie groups, say $\mathbb{R}$ and $\mathbb{T}$.

One- and two-dimensional Lie algebra

We now list the Lie algebras of small dimensions over $\mathbb{R}$.

- $\dim L = 1$. There is only one one-dimensional Lie algebra, the line $\mathbb{R}$ with null commutator.
  The corresponding Lie group is either $\mathbb{R}$ or $\mathbb{T}$, which are locally isomorphic but have different global properties, e.g. compactness..
- $\dim L = 2$. Let x and y be a basis in L.
  1. If $[x, y] = 0$, then the commutator of any two elements is equal to 0. The corresponding Lie groups are $\mathbb{R}^2$, $\mathbb{T}^2$, $\mathbb{R} \times \mathbb{T}$.
  2. If $[x, y] = z \neq 0$, then the commutator of any two vectors is proportional to z. There exists in particular a vector a such that $[a, z] = z$.
     The corresponding Lie group is the *affine* $ax + b$ *group*, denoted here by Aff. This is the smallest possible non-commutative Lie group. It is already very prominent and will be considered in details.

Thus there exist two nonisomorphic Lie algebras of dimension 2.

$\dim L = 3$. We consider the space $L_1 \subset L$ generated by all commutators.

1. If $\dim L_1 = 0$ then all commutators are equal to 0. Thus we have again abelian case.
   The corresponding Lie groups are $\mathbb{R}^3$, $\mathbb{T}^3$, $\mathbb{R}^2 \times \mathbb{T}$, ...
2. If $\dim L_1 = 1$, then $[x, y] = B(x, y)z$, where z is a fixed vector and $B(x, y)$ is a skew-symmetric bilinear form in L. Two cases are possible:
   a) $B(x, z) = 0$ for all $x \in L$; then one can choose a basis x, y, z in L with the commutation relations $[x.y] = z$, $[x, z] = [y, z] = 0$.
   The corresponding group is the *Heisenberg group* $\mathbb{H}$—one of two main topics here.
   b) There exists a vector $x \in L$ such that $B(x, z) = 1$; in this case there exists a basis x, y, z with the commutation relations $[x, y] = [y, z] = 0$, $[x, z] = z$.
   The corresponding group is $\mathbb{R} \times \mathrm{Aff}$ (not original).

Suppose that $\dim L_1 = 2$. Note that the subspace L_1 is itself a Lie algebra, since L_1 contains all commutators. We already know that there are exactly two two-dimensional Lie algebras, with the commutation relations $[x, y] = 0$ and $[x, y] = y$, respectively.

Let z be a vector which with x and y forms a basis in L. The operator $\operatorname{ad} z : x \mapsto [z, x]$ is a differentiation of the algebra $L_1 \to L_1$.

- If $[x, y] = y$, then the Jacobi identity implies that the operator $\operatorname{ad} z$ acts by $x \mapsto ay$, $y \mapsto by$, which contradicts the condition $\dim L_1 = 2$.
- If $[x, y] = 0$, then $\operatorname{ad} z$ can be an arbitrary matrix A of the second order. We note however that the condition $\dim L_1 = 2$ implies that A is nonsingular—we have obtained an entire family of Lie algebras, parametrized by nonsingular matrices of the second order, up to the conjugation.

  The corresponding Lie group is a semidirect product of $\mathbb{R}^2$ with a one-parameter group of its automorphisms, e.g. rotations.

Plane with automorphisms

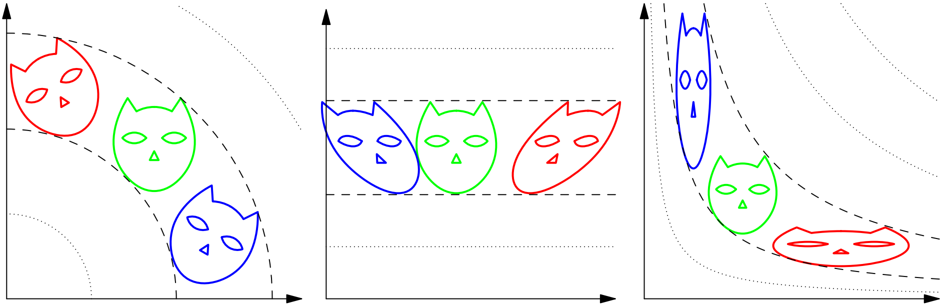


Figure: Linear automorphisms of the Euclidean plane. Unimodular matrices preserve the area, that is the symplectic form. They can be treated as rotations of elliptic, parabolic and hyperbolic type.

This type of Lie groups will appear as subgroups of the Schrödinger group considered later.

Three-dimensional Lie algebra I

Semi-simple case

Finally we consider the most non-commutative case $\dim L_1 = 3$, that is, $L_1 = L$. In short, there are two nonisomorphic 3D Lie algebras:

1. The commutation relations have the forms

$$[x, y] = z, \quad [y, z] = x, \quad [z, x] = y\,.$$

Mnemonically: cyclic permutation of three generators.
The algebra is the Lie algebra of skew-symmetric matrices of order 3:

$$ax + by + cz \mapsto \begin{pmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{pmatrix}.$$

The Lie group is $SO(3)$. It is compact—the first example that a Lie algebra dictates this condition. Surprisingly, this group will not be considered here at all, but this does not diminish its importance.

Three-dimensional Lie algebra II

Semi-simple case

2. Another possibility is:

$$[x, y] = 2y, [y, z] = x, [x, z] = -2z.$$

This algebra is isomorphic to the Lie algebra of matrices of the second order with zero trace:

$$x \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad y \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad z \mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

The corresponding group $SL_2(\mathbb{R})$ is non-compact and will play the major role in our consideration.
We also note that vectors $\frac{1}{2}x$ and y span a Lie algebra isomorphic to the Lie algebra of the $ax + b$ group.
Summary: All lower dimensional Lie groups (except $SO(3)$) will appear in our study and will interact with each others.