The Heisenberg group and $\mathrm{SL}_2(\mathbb{R})$

a survival pack

Vladimir V. Kisil

School of Mathematics
University of Leeds (England)

email: kisilv@maths.leeds.ac.uk
Web: http://www.maths.leeds.ac.uk/~kisilv

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Motivation

various origins of the Heisenberg(–Weyl) group

In 1980 paper[1] Roger Howe wrote:

> *An investigator might be able to get what he wanted out of a situation while overlooking the extra structure imposed by the Heisenberg group, structure which might enable him to get much more.*

Howe's suggestion is still valuable today.

As an illustration: the basic analytic operators of differentiation $\frac{d}{dx}$ and multiplication by x in satisfy to the same Heisenberg commutation relations $[Q, P] = i\hbar$ as observables of momentum and coordinate in quantum mechanics.

We shall start from the general properties the Heisenberg group and its representations. Many important applications will follow.

[1] Howe, "On the Role of the Heisenberg Group in Harmonic Analysis", 1980.

The Symplectic Form

The following notion is central for Hamiltonian mechanics.

Definition 1.

The *symplectic form* ω on $\mathbb{R}^{2n}$ is a function of two vectors such that:

$$\omega(x, y; x', y') = xy' - x'y, \qquad \text{where } (x, y), (x', y') \in \mathbb{R}^{2n}, \tag{1}$$

Exercise 2.

Check the following properties:

1. ω is anti-symmetric $\omega(x, y; x', y') = -\omega(x', y'; x, y)$.
2. ω is bilinear:

$$\begin{aligned} \omega(x, y; \alpha x', \alpha y') &= \alpha\omega(x, y; x', y'), \\ \omega(x, y; x' + x'', y' + y'') &= \omega(x, y; x', y') + \omega(x, y; x'', y''). \end{aligned}$$

For complex vectors z, $w \in \mathbb{C}^n$, $n \geqslant 1$ complex inner product is:

$$z\bar{w} = z_1\bar{w}_1 + z_2\bar{w}_2 + \cdots + z_n\bar{w}_n, \tag{2}$$

where $z = (z_1, z_2, \ldots, z_n)$, $w = (w_1, w_2, \ldots, w_n)$.

Exercise 3.

1. Let $z = x + iy$ and $w = x' + iy'$ then ω can be expressed through the complex inner product (2) as $\omega(x, y; x', y') = -\Im(z\bar{w})$.
2. The symplectic form on $\mathbb{R}^2$ is equal to $\det \begin{pmatrix} x & x' \\ y & y' \end{pmatrix}$. Consequently, it vanishes if and only if (x, y) and (x', y') are collinear.
3. Let $A \in \mathrm{SL}_2(\mathbb{R})$ be a real 2×2 matrix with the unit determinant. Define:
$$\begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} \tilde{x}' \\ \tilde{y}' \end{pmatrix} = A \begin{pmatrix} x' \\ y' \end{pmatrix}. \tag{3}$$
Then, $\omega(\tilde{x}, \tilde{y}; \tilde{x}', \tilde{y}') = \omega(x, y; x', y')$. Moreover, the *symplectic group* $\mathrm{Sp}(2)$—the set of all linear transformations of $\mathbb{R}^2$ preserving ω—coincides with $\mathrm{SL}_2(\mathbb{R})$.

Now we define the main object of our consideration.

Definition 4.

An element of the n-dimensional *Heisenberg group* $\mathbb{H}^n$ is $(s, x, y) \in \mathbb{R}^{2n+1}$, where $s \in \mathbb{R}$ and x, $y \in \mathbb{R}^n$. The group law on $\mathbb{H}^n$ is given as follows:

$$(s, x, y) \cdot (s', x', y') = (s + s' + \tfrac{1}{2}\omega(x, y; x', y'), x + x', y + y'), \tag{4}$$

where ω the symplectic form.

Exercise 5.

For the Heisenberg group $\mathbb{H}^n$, check that:

1. The unit is $(0, 0, 0)$ and the inverse $(s, x, y)^{-1} = (-s, -x, -y)$.
2. It is a non-commutative, the *centre* of $\mathbb{H}^n$ is:
$$Z = \{(s, 0, 0) \in \mathbb{H}^n, \ s \in \mathbb{R}\}. \tag{5}$$
3. The group law is continuous in the topology of $\mathbb{R}^{2n+1}$, so we have a Lie group.

Alternative group laws I

for the Heisenberg group

It is convenient to have several alternative forms to parameterise the Heisenberg group or express its group law.

1. Introduce complexified coordinates (s, z) on $\mathbb{H}^1$ with $z = x + iy$. Then the group law can be written as:

$$(s, z) \cdot (s', z') = (s + s' + \tfrac{1}{2}\Im(z'\bar{z}), z + z').$$

2. Show that the set $\mathbb{R}^3$ with the group law

$$(s, x, y) * (s', x', y') = (s + s' + xy', x + x', y + y') \tag{6}$$

is isomorphic to the Heisenberg group $\mathbb{H}^1$. It is called the *polarised Heisenberg group* $\mathbb{H}^1_p$ (aka *canonical coordinates* on $\mathbb{H}^1$).
Hint: Use the explicit form of the homomorphism $\mathbb{H}^1 \to \mathbb{H}^1_p$ as $(s, x, y) \mapsto (s + \frac{1}{2}xy, x, y)$.$\diamond$ Check that inverse element on polarised Heisenberg group is $(s, x, y)^{-1} = (-s + xy, -x, -y)$.

Alternative group laws II

for the Heisenberg group

3. Define the map $\phi : \mathbb{H}^1 \to M_3(\mathbb{R})$ by

$$\phi(s, x, y) = \begin{pmatrix} 1 & x & s + \frac{1}{2}xy \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}. \tag{7}$$

   This is a group homomorphism from $\mathbb{H}^1$ to the group of 3×3 matrices with the unit determinant and the matrix multiplication as the group operation. Write also a group homomorphism from the polarised Heisenberg group to $M_3(\mathbb{R})$.

4. Expand the above items from this Exercise to $\mathbb{H}^n$.

The Weyl algebra

The key idea of analysis is a linearization of complicated object in small neighbourhoods. Applied to Lie groups it leads to the Lie algebras. The Lie algebra of the Heisenberg group $\mathfrak{h}_1$ is also called *Weyl algebra*. From the general theory we know, that $\mathfrak{h}_1$ is a three-dimensional real vector space, thus, it can be identified as a set with the group $\mathbb{H}^1 \sim \mathbb{R}^3$ itself.

There are several standard possibilities to realise $\mathfrak{h}_1$:

1. *Generators* X of one-parameter subgroups: $x(t) = \exp(Xt)$, $t \in \mathbb{R}$.
2. *Tangent vectors* to the group at the group unit.
3. *Invariant vector fields* (first-order differential operators) on the group.

There is the important exponential map between a Lie algebra and respective Lie group. The exponent function can be defined in any topological algebra as the sum of the following series:

$$\exp(tX) = \sum_{n=0}^{\infty} \frac{(tX)^n}{n!}. \tag{8}$$

1. Matrices from (7) are created by the following exponential map (8):

$$\exp\begin{pmatrix}0 & x & s\\ 0 & 0 & y\\ 0 & 0 & 0\end{pmatrix} = \begin{pmatrix}1 & x & s+\frac{1}{2}xy\\ 0 & 1 & y\\ 0 & 0 & 1\end{pmatrix}. \tag{9}$$

   Thus $\mathfrak{h}_1$ isomorphic to the vector space of matrices in the left-hand side. We can define the explicit identification $\exp : \mathfrak{h}_1 \to \mathbb{H}^1$ by (9), which is also known as the exponential coordinates on $\mathbb{H}^1$.

2. Define the basis of $\mathfrak{h}_1$:

$$S = \begin{pmatrix}0 & 0 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}, \quad X = \begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}, \quad Y = \begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{pmatrix}. \tag{10}$$

   Write the one-parameter subgroups of $\mathbb{H}^1$ generated by S, X and Y.

Invariant vector fields

A (continuous) *one-parameter subgroup* is a continuous group homomorphism F from $(\mathbb{R}, +)$ to $\mathbb{H}^n$:

$$F(t + t') = F(t) \cdot F(t').$$

We can calculate the left and right derived action at any point $g \in \mathbb{H}^n$:

$$\left. \frac{d(F(-t) \cdot g)}{dt} \right|_{t=0} \quad \text{and} \quad \left. \frac{d(g \cdot F(t))}{dt} \right|_{t=0}. \tag{11}$$

1. Check that the following vector fields on $\mathbb{H}^1$ are left (right) invariant:
$$S^{l(r)} = \pm\partial_s, \quad X^{l(r)} = \pm\partial_x - \tfrac{1}{2}y\partial_s, \quad Y^{l(r)} = \pm\partial_y + \tfrac{1}{2}x\partial_s. \tag{12}$$
Show, that they are linearly independent and, thus, are bases of the Lie algebra $\mathfrak{h}_1$ (in two different realizations).
2. Find one-parameter groups of right (left) shifts on $\mathbb{H}^1$ generated by these vector fields.

Commutator I

The principal operation on a Lie algebra, besides the linear structure, is the Lie bracket—a bi-linear, anti-symmetric form with the Jacoby identity:

$$[[A, B], C] + [[B, C], A] + [[C, A], B] = 0.$$

In the above exercises, as in any algebra, we can define the Lie bracket as the commutator $[A, B] = AB - BA$, e.g. for matrices and vector fields through the corresponding algebraic operations in these algebras.

1. Check that bases from (10) and (12) satisfy the Heisenberg *commutator relation*

$$[X^{l(r)}, Y^{l(r)}] = S^{l(r)} \tag{13}$$

and all other commutators vanishing. More generally:

$$[A, A'] = \omega(x, y; x', y')S, \qquad \text{where } A^{(\prime)} = s^{(\prime)}S + x^{(\prime)}X + y^{(\prime)}Y, \tag{14}$$

and ω is the symplectic form.

Commutator II

2. Show that any second (and, thus, any higher) commutator $[[A, B], C]$ on $\mathfrak{h}_1$ vanishes. This property is encoded in the statement "the Heisenberg group is a step 2 nilpotent Lie group".

3. Check the formula

$$\exp(A)\exp(B) = \exp(A + B + \tfrac{1}{2}[A, B]), \qquad \text{where } A, B \in \mathfrak{h}_1. \tag{15}$$

   The formula is also true for any step 2 nilpotent Lie group and is a particular case of the Baker–Campbell–Hausdorff formula. Hint: In the case of $\mathbb{H}^1$ you can use the explicit form of the exponential map (9).$\diamond$

4. Define the vector space decomposition of the Lie algebra:

$$\mathfrak{h}_1 = V_0 \oplus V_1, \qquad \text{such that } V_0 = [V_1, V_1] = [\mathfrak{h}_1, \mathfrak{h}_1]. \tag{16}$$

   Explicitly: $V_1 = \{(0, x, y) : \ x, y \in \mathbb{R}\}$ and $V_0 = Z = \{(s, 0, 0), \ s \in \mathbb{R}\}$.

Erlangen programme suggests investigation of invariants under group action. This recipe can be applied recursively to groups themselves. Transformations of a group which preserve its structure are called *group automorphisms*. Automorphisms of $\mathbb{H}^1$ are compositions of the following:

1. *Inner automorphisms* or *conjugation* with $(s, x, y) \in \mathbb{H}^1$:

$$\begin{aligned} (s', x', y') &\mapsto (s, x, y) \cdot (s', x', y') \cdot (s, x, y)^{-1} \qquad (17) \\ &= (s' + \omega(x, y; x', y'), x', y') = (s' + xy' - x'y, x', y'). \end{aligned}$$

2. *Symplectic maps* $(s, x, y) \mapsto (s, \tilde{x}, \tilde{y})$, where $\begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}$ with A from the symplectic group $\mathrm{Sp}(2) \sim \mathrm{SL}_2(\mathbb{R})$, see Exercise 3.3.
3. *Dilations*: $(s, x, y) \mapsto (r^2 s, rx, ry)$ for a positive real r.
4. *Inversion*: $(s, x, y) \mapsto (-s, y, x)$.

The last three types of transformations are *outer automorphisms*.

The Scrödinger group

For future use we will need $\widetilde{\mathrm{Sp}}(2)$ which is the double cover of $\mathrm{Sp}(2)$.
Recall, the last group isomorphic to $\mathrm{SL}_2(\mathbb{R})$.
We can build the semidirect product $G = \mathbb{H}^1 \rtimes \widetilde{\mathrm{Sp}}(2)$ with the standard group law for semidirect products:

$$(h, g) * (h', g') = (h * g(h'), g * g'), \tag{18}$$

where h, $h' \in \mathbb{H}^1$, g,$g' \in \widetilde{\mathrm{Sp}}(2)$. Here the stars denote the respective group operations while the action $g(h')$ is defined as the composition of the projection map $\widetilde{\mathrm{Sp}}(2) \to \mathrm{Sp}(2)$ and the action (3).
This group is sometimes called the *Schrödinger group* (or *Jacoby group*[2]) and it is the maximal kinematical invariance group of both the free Schrödinger equation and the quantum harmonic oscillator. It is also of interest for paraxial beams and quantum optics.
It is also the full group of symmetries of the θ-function with applications in number theory.[3]

[2] Berndt, *Representations of linear groups*, 2007.
[3] Ibid., 2007.

Let G be a Lie group and H be its closed subgroup. The *homogeneous space* G/H from the equivalence relation: $\mathsf{g}' \sim \mathsf{g}$ iff $\mathsf{g}' = \mathsf{g}\mathsf{h}$, $\mathsf{h} \in \mathsf{H}$. The *natural projection* $\mathsf{p} : \mathsf{G} \to \mathsf{G}/\mathsf{H}$ puts $\mathsf{g} \in \mathsf{G}$ into its equivalence class.

A section $\mathsf{s} : \mathsf{G}/\mathsf{H} \to \mathsf{G}$ is a right inverse of p, i.e. $\mathsf{p} \circ \mathsf{s}$ is an identity map on G/H. Then, the *left action* of G on itself $\Lambda(\mathsf{g}) : \mathsf{g}' \mapsto \mathsf{g}^{-1} * \mathsf{g}'$ generates the action on G/H:

$$\mathsf{g} : x \mapsto \mathsf{p}(\mathsf{g}^{-1} * \mathsf{s}(x)), \quad \text{or graphically} \quad \begin{tikzcd} \mathsf{G} \arrow[r, "g^{-1}*"] \arrow[d, shift left, "p"] & \mathsf{G} \arrow[d, shift left, "p"] \\ \mathsf{G}/\mathsf{H} \arrow[r, "g^{-1}\cdot"] \arrow[u, shift left, "s"] & \mathsf{G}/\mathsf{H} \arrow[u, shift left, "s"] \end{tikzcd} \tag{19}$$

We want to classify up to certain equivalences all possible $\mathbb{H}^1$–homogeneous spaces. According to the diagram we will look subgroups of $\mathbb{H}^1$, staring from continuous and commutative ones.

One-dimensional continuous subgroups of $\mathbb{H}^1$ can be classified up to group automorphism. Two one-dimensional subgroups of $\mathbb{H}^1$ are the centre Z (5) and

$$H_x = \{(0, t, 0) \in \mathbb{H}^1,\ t \in \mathbb{R}\}. \tag{20}$$

Exercise 6.

Show that:

1. There is no an automorphism which maps Z to H_x.
2. For any one-parameter continuous subgroup of $\mathbb{H}^1$ there is an automorphism which maps it either to Z or H_x.
3. The classification of one-parameter subgroups can be based on their infinitesimal generators from the Weyl algebra.

Next, we wish to describe the respective homogeneous spaces and actions of $\mathbb{H}^1$ on them.

Exercise 7.

Check that:

1. The $\mathbb{H}^1$-action on $\mathbb{H}^1/Z \sim \{(x,y) : \ x,y \in \mathbb{R}\}$ is:
$$(s,x,y) : (x',y') \mapsto (x+x', y+y'). \tag{21}$$
Hint: The decomposition $(s,x,y) = (0,x,y) * (s,0,0)$ defines maps: $\mathsf{p} : (s',x',y') \mapsto (x',y')$ and $\mathsf{s} : (x',y') \mapsto (0,x',y')$.$\diamond$
2. The $\mathbb{H}^1$-action on $\mathbb{H}^1/H_x \sim \{(s,y) : \ s,y \in \mathbb{R}\}$ is:
$$(s,x,y) : (s',y') \mapsto (s+s'+xy'+\tfrac{1}{2}xy, y+y'). \tag{22}$$
Hint: The decomposition $(s,x,y) = (s+\frac{1}{2}xy,0,y) * (0,x,0)$ defines maps: $\mathsf{p} : (s',x',y') \mapsto (s'+\frac{1}{2}x'y',y')$ and $\mathsf{s} : (s',y') \mapsto (s',0,y')$.$\diamond$
3. Calculate the derived action similar to (11).

The classification of two-dimensional subgroups is as follows:

Exercise 8.

Show that:

1. For any two-dimensional continuous subgroup of $\mathbb{H}^1$ there is an automorphism of $\mathbb{H}^1$ which maps the subgroup to
$$H'_x = \{(s, 0, y) \in \mathbb{H}^1,\ s, y \in \mathbb{R}\}.$$
2. $\mathbb{H}^1$-action on $\mathbb{H}^1/H'_x$ is
$$(s, x, y) : x' \mapsto x + x'. \tag{23}$$
HINT: The decomposition $(s, x, y) = (0, x, 0) * (s - \frac{1}{2}xy, 0, y)$ defines maps $\mathsf{p} : (s', x', y') \mapsto x'$ and $\mathsf{s} : x' \mapsto (0, x', 0)$.$\diamond$
3. Calculate the derived action similar to (11).

Actions (21) and (23) are simple shifts. Nevertheless, the associated representations of the Heisenberg group will be much more interesting.

A discrete subgroup of $\mathbb{H}^1$

The above subgroups were commutative and continuous. There is a remarkable non-commutative discrete subgroup of $\mathbb{H}^1$. The discreteness hides its non-commutativity in certain cases.

Exercise 9.

1. Show that the following set is a non-commutative discrete subgroup of $\mathbb{H}^1$ or the polarised Heisenberg group $\mathbb{H}^1_p$:
$$H_d = \{(s, n, k) : \; s \in \mathbb{R}, \; n, k \in \mathbb{Z}\}. \tag{24}$$
2. The homogeneous space $\mathbb{H}^1_p / H_d$ can be identified with the torus $\mathbb{T}^2 = \{(u, v) : \; u, v \in [0, 1)\}$ through the following decomposition:
$$(s, x, y) = (0, \{x\}, \{y\}) * (s - \{x\}[y], [x], [y]), \tag{25}$$
where $[x]$ and $\{x\}$ are the integer and fractional parts of x.
3. The $\mathbb{H}^1$-action on $\mathbb{H}^1_p / H_d \sim \mathbb{T}^2$ is a "periodic" shift:
$$(s, x, y) : (u, v) \mapsto (\{u + x\}, \{v + y\}), \tag{26}$$
for the $p : (s, x, y) \mapsto (\{x\}, \{y\})$ and $s : (u, v) \mapsto (0, u, v)$.

Definition 10 (traditional).

A (linear) *representation* ρ of a group G is a group homomorphism $\rho : G \to B(V)$ from G to (bounded) linear operators on a space V:

$$\rho(gg') = \rho(g)\rho(g').$$

Informally: A representation of G is an introduction of an operation of addition on G, which is compatible with group multiplication.

Exercise 11.

Check that the following are group representations:

1. Let $G = (\mathbb{R}, +)$, $V = \mathbb{C}$, $\rho(x) = e^{iax}$, $a \in \mathbb{R}$. It is a 1D-representation called a *character*.
2. Let $G = (\mathbb{R}, +)$, $V = L_2(\mathbb{R})$, representation by shifts: $[\rho(x)f](t) = f(x + t)$ is infinite-dimensional.
3. For any group G shifts $f(g') \mapsto f(g^{-1}g')$ and $f(g') \mapsto f(g'g)$ are the left and right *regular* representations respectively.

Continuous Representations of Topological Groups

A representation is a map which respects the group structure. If we have a topological group, it is natural to consider representations respecting topology as well, that is representation which are continuous in some topology. It is most common (and convenient!) to use the following type.

Definition 12.

A representation ρ of G in a vector space V is *strong continuous* if for any convergent sequence $(g_n) \to g \in G$ and for any $x \in V$ we have $\|\rho(g_n)x - \rho(g)x\| \to 0$.

Exercise 13.

Which representations from the previous Exercise 11 are strongly continuous in a suitable topology?

From now, we consider strongly continuous representations only.

Decomposition of Representations

Definition 14.

A subspace $U \subset V$ is called *invariant* if $\rho(g)U \subset U$ for all $g \in G$. We can always consider a restriction of ρ to any its invariant subspace. Such a restriction is called *subrepresentation*.

Definition 15.

A representation is *irreducible* if the only closed invariant subspaces are trivial (the whole V and $\{0\}$). Otherwise it is *reducible*.

The regular representation of $(\mathbb{R}, +)$ on $V = L_2(\mathbb{R})$ by shifts has closed invariant subspaces, e.g. the Hardy space—space of all functions having an analytic extension to the upper half-plane. So it is reducible. A character (and any 1D-representation) is an irreducible representation.

Definition 16.

A representation is *decomposable* if $V = V_1 \oplus V_2$, where V_1 and V_2 are invariant.

Equivalence of Representations

Let ρ be a representation of a group G in a linear normed space E and there is an isometric isomorphism $U : E \to F$.

Exercise 17.

Check that the map $\rho_1 : g \mapsto U\rho(g)U^{-1}$, $g \in G$ is a representation of G in F. Show that, if ρ is continuous then the new representation is continuous as well.

Obviously, the representations ρ and ρ_1 are not essentially different. Two such representations $\rho_1(g) = U\rho(g)U^{-1}$ (ditto $\rho_1(g)U = U\rho(g)$), $g \in G$ are called *equivalent* with an *intertwining operator* U.

Exercise 18 (Revising Exercise 11).

Let $G = (\mathbb{R}, +)$, $E = F = L_2(\mathbb{R})$ and $[\rho_1(x)f](t) = e^{2\pi i t x} f(t)$ and $[\rho_2(x)f](t) = f(x + t)$. Check that, the Fourier transform:

$$[\mathcal{F}f](\lambda) = \int_{\mathbb{R}} f(t)\, e^{-2\pi i \lambda t}\, dt$$

intertwines ρ_1 and ρ_2, that is $\mathcal{F}\rho_2(x) = \rho_1(x)\mathcal{F}$ for all $x \in \mathbb{R}$. Thus, ρ_1 and ρ_2 are equivalent.

Unitary Representations

The representation theory is much simpler if representing operators belong to a nice class.

Definition 19.
A strongly continuous representation ρ of G in V is *unitary* if V is a Hilbert space and all $\rho(g)$, $g \in G$ are unitary operators.

Exercise 20.
Define Hilbert spaces such that representations from the Exercise 11 become unitary.

One of the important properties of unitary representations is complete reducibility. Namely, a representation in a normed space can be reducible but indecomposable. However, any reducible unitary representation in a Hilbert space H is decomposable: for any closed invariant subspace $F \subset H$, the orthogonal complement $F^\perp \subset H$ is again a closed invariant subspace. Thus, H is a sum of two invariant subspaces: $H = F \oplus F^\perp$. We will consider unitary representations of $\mathbb{H}^n$ only.

Induced Representations

version on the group

Let G be a group, H its closed subgroup, χ be a character of H. A space $L^\chi(G)$ of functions with the H-covariance property

$$F(gh) = \chi(h)F(g), \qquad g \in G,\ h \in H \tag{27}$$

is invariant under left G-shifts

$$\Lambda(g) : F(g') \mapsto F(g^{-1}g'), \qquad g, g' \in G. \tag{28}$$

The restriction of the left regular representation (28) to $L^\chi(G)$ is called an *induced representation* from the character χ of the subgroup H.
In fact, an induction can be done from any representation of H, not only a character. However, we do not need this generality in our course.
The covariance property (27) allows to recover the value $F(g)$ from $F(g')$ for any g' in the equivalence class of g. Thus, we may push the representation to the homogeneous space G/H.

Induced Representations

version on the homogeneous space

For the natural projection $p : G \to X = G/H$ and its right inverse $s : X \to G$, we define the map $r : G \to H$ by:

$$r(g) = s(x)^{-1}g \quad \text{for } x = p(g), \qquad \text{thus } g = s(p(g))r(g). \tag{29}$$

Define the *lifting* $\mathcal{L}_\chi$ of a function $f(x)$ on $X = G/H$ to $F(g) \in L^\chi(G)$ and its left inverse—the *pulling* $\mathcal{P}_H : L^\chi(G) \to L(X)$:

$$F(g) = [\mathcal{L}_\chi f](g) = \chi(r(g))f(p(g)), \qquad f(x) = [\mathcal{P}_H F] = F(s(x)) \tag{30}$$

since $[\mathcal{L}_\chi f](g)$ satisfying (27) and $\mathcal{P}_H \circ \mathcal{L}_\chi = I$. The lifting transforms the left regular representation (28) on $L^\chi(G)$ to the following action:

$$[\rho^\chi(g)f](x) = \chi(r(g^{-1} * s(x)))f(g^{-1} \cdot x), \text{ from } \begin{array}{ccc} L^\chi(G) & \xrightarrow{\Lambda(g)} & L^\chi(G) \\ \mathcal{L}_\chi \uparrow\downarrow \mathcal{P}_H & & \mathcal{L}_\chi \uparrow\downarrow \mathcal{P}_H \\ L(X) & \xrightarrow{\rho^\chi(g)} & L(X) \end{array} \tag{31}$$

and using (19) to define $x \mapsto g^{-1} \cdot x$.

Informally: In the "decomposition" $G \sim (G/H) \times H$ the part G/H acts in functions' domain, and H—in the range. Everything is used somewhere!

Induced Representations of $\mathbb{H}^1$ I

Here is some examples of induced representations. The choice of characters will be discussed later.

1. For $H = Z$, the decomposition $(s,x,y) = (0,x,y) * (s,0,0)$ defines the map $r : \mathbb{H}^1 \to Z$ is $r(s,x,y) = (s,0,0)$. For the character $\chi_\hbar(s,0,0) = e^{2\pi i\hbar s}$, the representation of $\mathbb{H}^1$ on $L_2(\mathbb{R}^2)$ is, cf. (21):
$$[\rho_\hbar(s,x,y)f](x',y') = e^{2\pi i\hbar(-s-\frac{1}{2}\omega(x,y;x'y'))}f(x'-x,y'-y). \quad (32)$$
This is the *Fock–Segal–Bargmann (FSB) representation*.
2. For $H = H_x$, the decomposition $(s,x,y) = (s+\frac{1}{2}xy,0,y) * (0,x,0)$ defines the map $r(s,x,y) = (0,x,0)$. For the character $\chi(0,x,0) = e^{2\pi iqx}$, $q \in \mathbb{R}$ the representation $\mathbb{H}^1$ on $L_2(\mathbb{R}^2)$ as, cf. (22):
$$[\rho_q(s,x,y)f](s',y') = e^{-2\pi iqx}f(s'-s-xy'+\frac{1}{2}xy,y'-y). \quad (33)$$

Induced Representations of $\mathbb{H}^1$ II

3. For $H = H'_x = \{(s, 0, y) \in \mathbb{H}^1\}$, the decomposition $(s, x, y) = (0, x, 0) * (s - \frac{1}{2}xy, 0, y)$ defines the map $r : \mathbb{H}^1 \to H'_x$ as $r(s, x, y) = (s - \frac{1}{2}xy, 0, y)$.
   For the character $\chi_\hbar(s, 0, y) = e^{2\pi i \hbar s}$, the representation of $\mathbb{H}^1$ on $L_2(\mathbb{R})$ is, cf. (23):
   $$[\rho_\hbar(s, x, y)f](x') = \exp(-\pi i\hbar(2s - 2yx' + xy)))\, f(x' - x). \tag{34}$$
   This is one of forms of the *Schrödinger representation* of $\mathbb{H}^1$.
4. The above representations are intertwined with the left regular representation as in (31) by:
   $$[\mathcal{L}_\chi f](s, x, y) = e^{2\pi i\hbar s'} f(x, y), \qquad [\mathcal{P}_Z F](x, y) = F(0, x, y)\,, \tag{35}$$
   $$[\mathcal{L}_\chi f](s, x, y) = e^{2\pi i q x} f(s, y), \qquad [\mathcal{P}_{H_x} F](s, y) = F(s, 0, y)\,, \tag{36}$$
   $$[\mathcal{L}_\chi f](s, x, y) = e^{2\pi i\hbar(s - xy/2)} f(x), \qquad [\mathcal{P}_{H'_x} F](x) = F(0, x, 0)\,. \tag{37}$$
   for subgroups Z, H_x and H'_x, respectively. Check (27) for the image spaces of corresponding lifting $\mathcal{L}_\chi$.

Induction from the Discrete Subgroup

For convenience, we use the polarised Heisenberg group $\mathbb{H}^1_p$ with the group law (6). For the above discrete subgroup $H = H_d = \{(s', n, k) : \ s' \in \mathbb{R}, \ n, k \in \mathbb{Z}\}$ take the character $\chi_m(s', n, k) = e^{2\pi m i s'}$, $m \in \mathbb{Z}$, which *kills the commutator of* H_d. Then, H-covariance $F(gh) = \chi_m(h)F(g)$ (27) for $g = (s, x, y)$ and $h = (-xk, n, k) \in H_d$ implies:

$$F(s, x + n, y + k) = e^{-2\pi m i x k} F(s, x, y). \tag{38}$$

That is, functions in $L^{\chi_m}(G)$ are periodic in x and *quasi-periodic* in y. The respective map $r : \mathbb{H}^1_p \to H_d : (s, x, y) \mapsto (s - \{x\}[y], [x], [y])$, cf. (25). Since $m \in \mathbb{Z}$ we note

$$\chi_m(r(s, x, y)) = e^{2\pi m i(s - \{x\}[y])} = e^{2\pi m i(s - (\{x\} + [x])[y])} = e^{2\pi m i(s - x[y])}.$$

Thus, using the lifting $\mathcal{L}_{\chi_m}$ we transfer the left group action to $L(\mathbb{T}^2)$:

$$\begin{aligned} [\rho^m_d(s, x, y)f](u, v) &= e^{-2\pi m i(s + x\{v - y\} + u[v - y])} f(\{u - x\}, \{v - y\}) \quad (39)\\ &= e^{-2\pi m i(s + x(v - y) + u[v - y])} f(u - x, v - y). \end{aligned}$$

The last form treats f on $L(\mathbb{T}^2)$ as double quasi-periodic on $\mathbb{R}^2$, cf. (38).

Equivalence of Induced Representations

We built many (but still not all possible!) representations of $\mathbb{H}^1$.
Are they essentially different?
We will show that any two above representations $\rho_\hbar$ and $\tilde{\rho}_\hbar$ with the same value of the parameter $\hbar$ are unitary equivalent, namely there is a unitary operator U intertwining them:

$$\rho_\hbar(g)U = U\tilde{\rho}_\hbar(g), \qquad \text{for all } g \in G.$$

Specifically, for induced representations: recall the inner automorphisms $\tilde{g} : g \mapsto \tilde{g}^{-1} g \tilde{g}$ of any non-commutative group G by the adjoint action.
For a representation ρ of G the correspondence $g \mapsto \tilde{\rho}(g) := \rho(\tilde{g}^{-1} g \tilde{g})$ is also a representation of G. It is easy to see that $\tilde{\rho}$ and ρ are equivalent with the intertwining operator $U = \rho(\tilde{g})$, that is $\rho(g)U = U\tilde{\rho}(g)$.
For the representations induced from a character, this observation leads to the *orbit method* of Kirillov, which we are considering now.

Adjoint Representation

for a Matrix Group

Let G be a *matrix group*, i.e. subgroup *and* a smooth submanifold of $GL_n(\mathbb{R})$.

$\mathfrak{g} = \mathrm{Lie}(G)$—the Lie algebra of G, the tangent space $T_e(G)$ at unit $e \in G$.

$A(g) : x \mapsto gxg^{-1}$—$G$-action on itself by inner automorphisms. It fixes the group unit e, thus generates a linear transformation of the tangent space $T_e(G)$, which is identified with $\mathfrak{g}$.

$A_*(g) : \mathfrak{g} \mapsto \mathfrak{g}$—the above derived map which is usually denoted by $Ad(g)$.

$g \mapsto Ad(g)$—is called the *adjoint representation* of G.

Luckily, this construction can greatly simplified for matrix groups: the adjoint representation is *matrix conjugation*:

$$Ad(g)B = gBg^{-1}, \qquad \text{where} \quad B \in \mathfrak{g}, \quad g \in G.$$

Co-Adjoint Representation

Dual to Adjoint One

$\mathfrak{g}^*$—dual space to the Lie algebra $\mathfrak{g}$. It produces characters on one-dimensional subgroups of G by:

$$\chi_F(e^{tX}) = e^{2\pi i t \langle X, F\rangle}, \qquad X \in \mathfrak{g}, \quad F \in \mathfrak{g}^*. \tag{40}$$

$\langle A, B\rangle = \operatorname{tr}(AB)$—a bilinear form on $Mat_n(\mathbb{R})$ invariant under matrix conjugation $A \to C^{-1}AC$.

$\mathfrak{g}^\perp$—the orthogonal complement of $\mathfrak{g}^*$ in $Mat_n(\mathbb{R})$ with respect to $\langle \cdot, \cdot \rangle$. Then $Mat_n(\mathbb{R})/\mathfrak{g}^\perp$ serves as a model for $\mathfrak{g}^*$.

p— the orthogonal projection of $Mat_n(\mathbb{R})$ on $\mathfrak{g}^*$ parallel to $\mathfrak{g}^\perp$.

Then the *co-adjoint representation* K of G, which is dual to the adjoint representation defined above, can be written in the simple form

$$K(g) : F \mapsto p(gFg^{-1}), \qquad \text{where} \quad F \in \mathfrak{g}^*, \; g \in G.$$

Under the co-adjoint representation $\mathfrak{g}^*$ is split into a family of disjoint orbits, giving the name *orbit method* by Kirillov. *For F_1 and F_2 in one orbit characters χ_{F_1} and χ_{F_2} (40) induce equivalent representations of G.*

Co-Adjoint Representation

For the Heisenberg group

Realising $\mathbb{H}^1$ as a matrix group, we calculate the matrix conjugation:

$$g = \begin{pmatrix} 1 & x & s+\frac{1}{2}xy \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \in \mathbb{H}^1, \quad B = \begin{pmatrix} 0 & x' & s' \\ 0 & 0 & y' \\ 0 & 0 & 0 \end{pmatrix} \in \mathfrak{h}^1,$$

$$\mathrm{Ad}(g)B = \begin{pmatrix} 0 & x' & -x'y+xy'+s' \\ 0 & 0 & y' \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} s'' \\ x'' \\ y'' \end{pmatrix} = \begin{pmatrix} 1 & -y & x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} s' \\ x' \\ y' \end{pmatrix}$$

We introduce coordinates $(\hbar, q, p)$ in $\mathfrak{h}^*_n \sim \mathbb{R}^{2n+1}$ in bi-orthonormal coordinates to the exponential ones (s, x, y) on $\mathfrak{h}^n$. Then the co-adjoint representation $\mathrm{Ad}^* : \mathfrak{h}^*_n \to \mathfrak{h}^*_n$ becomes:

$$\mathrm{Ad}^*(s,x,y) : (\hbar, q, p) \mapsto (\hbar, q - \hbar y, p + \hbar x), \quad \text{where } (s,x,y) \in \mathbb{H}^n \tag{41}$$

Note, that every $(0, q, p)$ is fixed. Also all hyperplanes $\hbar = \text{const} \neq 0$ are orbits, thus characters $(\hbar, q, p)$ induce representation equivalent to (32), (33) and (34) induced from $(\hbar, 0, 0)$.

Orbit Space

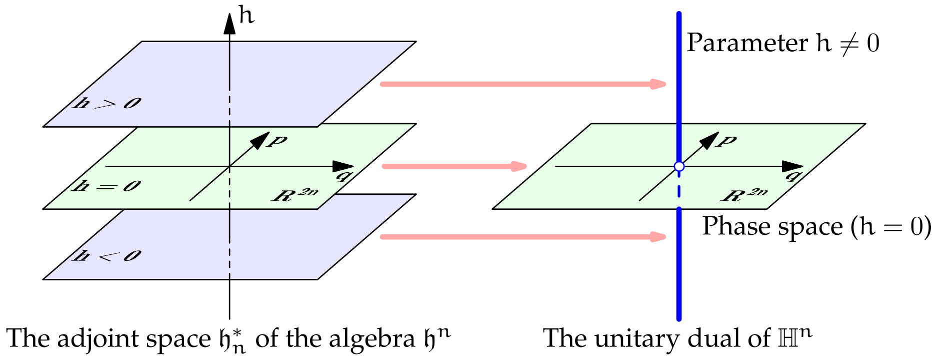

Figure: The structure of unitary dual to $\mathbb{H}^n$ from the method of orbits. The space $\mathfrak{h}_n^*$ is sliced into "horizontal" hyperplanes. Planes with $\hbar \neq 0$ form single orbits and correspond to different classes of UIR. The plane $\hbar = 0$ is a family of one-point orbits $(0, q, p)$, which produce one-dimensional representations. The topology on the dual object is the factor topology inherited from the $\mathfrak{h}_n^*$.

Let M be a unit of mass, L—of length, T—of time. We adopt the following

Convention 21.

1. Only physical quantities of the *same dimension* can be added or subtracted. However, any quantities can be multiplied/divided.
2. Therefore, mathematical functions, e.g. $\exp(u) = 1 + u + u^2/2! + \ldots$ or $\sin(u)$, can be constructed out of a dimensionless number u only. Thus, Fourier dual variables, say x and q, should posses reciprocal dimensions to enter the expression $e^{2\pi i x q}$.
3. For physical reasons being seen later, we assign to x and y components of (s, x, y) physical units $1/L$ and $T/(LM)$ respectively.

Consequently, the parameter s should be measured in $T/(L^2M)$—the product of units of x and y. The coordinates $\hbar$, q, p should have units of an *action* ML^2/T, *coordinates* L, and *momenta* LM/T, respectively.

We now build induced representations generated by the coadjoint orbits. Starting from the action (41) on an orbit $\hbar \neq 0$ and the character $e^{-2\pi i \hbar s}$ of the centre we obtain the representation:

$$\rho_\hbar(s, x, y) : f(q, p) \mapsto e^{-\pi i(2\hbar s + qx + py)} f(q - \hbar y, p + \hbar x). \tag{42}$$

The same formula is obtained if we use the Fourier transform $(x', y') \to (q, p)$ for the representation (32). Note that the representation (42) obeys our agreement on physical units, if (q, p) is treated as a point of the phase space.

Similarly, we can apply the Fourier transform $x' \to q$ for the representation (34) and obtain another form of the *Schrödinger representation* :

$$[\rho_\hbar(s, x, y) f](q) = e^{\pi i \hbar(-2s + xy) - 2\pi i x q} f(q + \hbar y). \tag{43}$$

The variable q is treated here as the coordinate on the configurational space of a particle.

Derivation of Representations

Let ρ be a representation of a Lie group G with the Lie algebra $\mathfrak{g}$. For any $X \in \mathfrak{g}$ and real t we have $\exp(tX) \in G$. Recall, these elements form a semigroup: $\exp((t+s)X) = \exp(tX)\exp(sX)$.
For a representation ρ of G in a space V we obtain one-parameter semigroup of operators $\rho(\exp(tX))$ on V. Its generato is:

$$d\rho^X := \left.\frac{d\,\rho(e^{tX})}{dt}\right|_{t=0}. \tag{44}$$

Even for a bounded representation ρ the above operator may be unbounded and we need to define its domain as a proper subspace $U \subset V$. In this way obtain the *derived representation* of the Lie algebra $\mathfrak{g}$.

Example 1.

1. Let $G = (\mathbb{R}, +)$, $V = \mathbb{C}$, $\rho(x) = e^{iax}$, $a \in \mathbb{R}$. The derived representation is $d\rho^T = iaT$ for $T \in \mathfrak{r} \sim \mathbb{R}$.
2. Let $G = (\mathbb{R}, +)$, $V = L_2(\mathbb{R})$, $[\rho(x)f](t) = f(x+t)$ then $d\rho^T = T\frac{d}{dx}$. As the domain we can take the Schwartz space $S(\mathbb{R})$.

Definition 22.
Let ρ be a representation of a group G in a space V and F be an operator from V to a space U. We define a *covariant transform* (CT) $\mathcal{W}$ from V to the space $L(G, U)$ of U-valued functions on G by the formula:

$$\mathcal{W} : v \mapsto \tilde{v}(g) = F(\rho(g^{-1})v), \qquad v \in V,\ g \in G. \tag{45}$$

Operator F will be called *fiducial operator* in this context.

1. We do not require that fiducial operator F shall be linear. Sometimes the positive homogeneity, i.e. $F(tv) = tF(v)$ for $t > 0$, alone can be already sufficient.
2. Usefulness of the covariant transform is in the reverse proportion to the dimensionality of the space U. The simplest situation (unattainable sometimes) is $\dim U = 1$.

Theorem 23.
The covariant transform intertwines ρ *and the left regular representation* Λ*:*

$$\mathcal{W}\rho(g) = \Lambda(g)\mathcal{W}. \tag{46}$$

Here Λ *on* $L(G, U)$ *is defined as usual by:* $\Lambda(g) : f(h) \mapsto f(g^{-1}h)$.

Proof.
This is a simple calculation:

$$\begin{aligned}
[\mathcal{W}(\rho(g)v)](h) &= F(\rho(h^{-1})\rho(g)v) \\
&= F(\rho((g^{-1}h)^{-1})v) \\
&= [\mathcal{W}v](g^{-1}h) \\
&= \Lambda(g)[\mathcal{W}v](h).
\end{aligned}$$

The following example represents the most developed case of covariant transform with many important realisation.

Example 2.

Let V be a Hilbert space and ρ be a unitary representation of a group G in the space V. Let $F : V \to \mathbb{C}$ be a functional $v \mapsto \langle v, v_0 \rangle$ defined by a vector $v_0 \in V$.
Then the CT is the well-known expression for a *wavelet transform*:

$$\mathcal{W} : v \mapsto \tilde{v}(g) = \left\langle \rho(g^{-1})v, v_0 \right\rangle = \langle v, \rho(g)v_0 \rangle\,, \qquad v \in V,\ g \in G. \tag{47}$$

The wavelet transforms maps abstract vectors to scalar-valued functions on the group G.
The family of vectors $v_g = \rho(g)v_0$ is called *wavelets* or *coherent states*. In this case we obtain scalar valued functions on G.

Example 3 (The affine group wavelets).

Let $G = \mathrm{Aff}$ be the "$ax + b$" (or *affine*) group: the set of points (a, b), $a \in \mathbb{R}_+$, $b \in \mathbb{R}$ in the upper half-plane with the group law:

$$(a, b) * (a', b') = (aa', ab' + b) \tag{48}$$

Its isometric representation on $V = L_p(\mathbb{R})$ is given by the formula:

$$[\rho_p(g)\, f](x) = a^{\frac{1}{p}} f\,(ax + b)\,, \qquad \text{where } g^{-1} = (a, b). \tag{49}$$

For a mother wavelet $\phi(x)$ the wavelet transform is:

$$[\mathcal{W}f](a, b) = a^{\frac{1}{2}} \int_{\mathbb{R}} f(x)\, \phi\,(ax + b)\; dx.$$

Various oscillating mother wavelets are considered in signal processing, oftenly with a compact support.

Also, for the (inadmissible) mother wavelet $\phi(x) = 1/(x + i)$ the covariant transform (45) is the Cauchy integral on the real line.

Consider $G = \mathbb{H}^1$ and its Schrödinger representation (34). Then for a mother wavelet v_0 the wavelet transform becomes:

$$[\mathcal{W}f](s,x,y) = \int_{\mathbb{R}} f(x')\,\exp(-2\pi i\hbar(-s + yx' - \tfrac{1}{2}xy))\,\bar{v}_0(x'-x)\,dx'. \quad (50)$$

The very important example of a mother wavelet is the Gaussian $v_0(x) = e^{-\pi\hbar x^2}$. The choice will become clear later due to the connection with the harmonic oscillator and its analytic properties. Then the wavelet transform (50) becomes the celebrated Fock–Segal–Bargmann transform:

$$[\mathcal{W}f](s,x,y) = \int_{\mathbb{R}} f(x')\,e^{-\pi\hbar(i(-2s+2yx'-xy)+(x'-x)^2)}\,dx'. \quad (51)$$

As we can see the influence of the group centre is very trivial here, this can be addressed as follows.

Induced Covariant Transform

from eigenvectors

The choice of a mother wavelet or fiducial operator F for the covariant transform (45) can significantly influence the behaviour of the covariant transform. Let G be a group and $\tilde{H}$ be its closed subgroup with the corresponding homogeneous space $X = G/\tilde{H}$.

Definition 24.
Let χ be a representation of the subgroup $\tilde{H}$ in a space U and $F : V \to U$ be an intertwining operator between χ and the representation ρ:

$$F(\rho(h)v) = F(v)\chi(h), \qquad \text{for all } h \in \tilde{H},\ v \in V. \tag{52}$$

Then the covariant transform (45) generated by F is called the *induced covariant transform.*

The special case of an induced covariant transform, which is explained below, is known as *Gilmore–Perelomov coherent states.*

Example 4.

Consider the traditional wavelet transform as outlined in Ex. 2. Chose a vacuum vector v_0 to be a joint eigenvector for all operators $\rho(h)$, $h \in \tilde{H}$, that is $\rho(h)v_0 = \chi(h)v_0$, where $\chi(h)$ is a complex number depending of h. Then χ is obviously a character of $\tilde{H}$.
The image of wavelet transform (47) with such a mother wavelet will have a property:

$$\tilde{v}(gh) = \langle v, \rho(gh)v_0\rangle = \langle v, \rho(g)\chi(h)v_0\rangle = \bar{\chi}(h)\tilde{v}(g). \qquad (53)$$

Proposition 25.

1. *The image of induced wavelet transform $\mathcal{W}$ consist of functions with the property $\tilde{v}(gh) = \chi(h)\tilde{v}(g)$.*
2. *Thus, $\mathcal{W}$ intertwines ρ with a representation induced by the character χ of $\tilde{H}$.*

Induced Wavelet Transform

on the Heisenberg group

The induced wavelet transform is uniquely defined by cosets on the homogeneous space $G/\tilde{H}$. Thus for a fixed section $s : G/\tilde{H} \to G$ it is enough to take wavelets $v_x = \rho(x)v_0$ parametrised by points of the homogeneous space $x \in G/\tilde{H}$:

$$\mathcal{W} : v \mapsto \tilde{v}(x) = [\mathcal{W}v](x) = \left\langle \rho(x^{-1})v, w_0 \right\rangle = \left\langle v, \rho(x)w_0 \right\rangle . \tag{54}$$

We consider the representation $\rho_\hbar$ induced by the character $\chi_\hbar(s, 0, 0) = e^{2\pi i\hbar s}$ of the centre Z. Any vector is eigenvector for $\rho_\hbar(s, 0, 0)$ since it is a multiplication operator. The associated section $s(x, y) = (0, x, y)$.
Explicitly, in the induced form of (51) gives the Fock–Segal–Bargmann transform:

$$[\mathcal{W}f](x, y) = \int_{\mathbb{R}} f(x')\, e^{-\pi\hbar(i(2yx'-xy)+(x'-x)^2)}\, dx'. \tag{55}$$

Contravariant Transform

Definition 26.

Let G, H, $X = G/H$ and ρ on B be as before. For $b_0 \in B$, called the *reconstruction vector*, the *contravariant transform* $\mathcal{M} = \mathcal{M}_{b_0}$ from $F(X)$ to B is defined by:

$$\mathcal{M}_{b_0}[f(x)] = \int_X f(x) b_x \, d\mu(x) = \int_X f(x)\rho(x)\, d\mu(x) b_0 = \rho(f) b_0. \tag{56}$$

Proposition 27.

$\mathcal{M}$ intertwines representations Λ on $F(X)$ and ρ on B: $\mathcal{M}\Lambda(g) = \rho(g)\mathcal{M}$. Thus the image $\mathcal{M}(F(X)) \subset B$ of a left-invariant subspace $F(X)$ under $\mathcal{M}$ is invariant under the representation ρ.

Theorem 28.

The operator $\mathcal{P} = \mathcal{M}\mathcal{W} : B \to B$ is a projection of B to its linear subspace for which b_0 is cyclic. Particularly, if ρ is an irreducible representation then the contravariant transform $\mathcal{M}$ is a left inverse operator on B for the wavelet transform $\mathcal{W}$: $\mathcal{M}\mathcal{W} = cI$ (up to a constant c).

Avoiding a talk on the experimental framework and interpretations, we adopt the following set-up of quantum mechanics (QM):

- A physical system is described by a *state*, which can vary over the time. In QM states are provided by vectors in a Hilbert space H with the unit norm.
- Our measurements on a state of physical system are never precise, even for a classical system. A particular quantum *observable* is associated with a self-adjoint operator on H. For a state $\phi \in H$ an observable A produces a probability distribution with the *expectation value* $\langle A\phi, \phi \rangle$, which is a real number.
- Let $\bar{A} = \langle A\phi, \phi \rangle$ be the expectation value. The *dispersion* of A on ϕ:

$$\Delta^2_\phi(A) = \left\langle (A - \bar{A})^2 \phi, \phi \right\rangle = \left\langle (A - \bar{A})\phi, (A - \bar{A})\phi \right\rangle = \left\| (A - \bar{A})\phi \right\|^2 . \quad (57)$$

Example 5 (Eigenvectors and eigenvalues).

Let ϕ be an eigenvector of A with the eigenvalue λ. Then λ is the expectation value of A on ϕ with zero dispersion.

Since all separable infinite-dimensional Hilbert spaces are isometrically isomorphic, all of them provides essentially the same realisation of QM. However some models have definite advantages.

Example 6 (The coordinate observable).

Consider the case of $H = L_2(\mathbb{R})$ as QM model of a particle on a line. Assuming that $|f(q)|^2$ provides a probability distribution to find a particle at certain location, then the observable $M = qI$ produces

$$\langle Mf, f\rangle = \int_{\mathbb{R}} qf(q)\,\bar{f}(q)\,dq = \int_{\mathbb{R}} q\,|f(q)|^2\,dq \tag{58}$$

the average value of such distribution. Consider the action (43) of the subgroup $\{(0, x, 0)\}$ of $\mathbb{H}^n$ and the respective derived representation:

$$[\rho_\hbar(0, x, 0)f](q) = e^{-2\pi i x q}\,f(q), \qquad [d\rho_\hbar^X f](q) = -2\pi i q f(q). \tag{59}$$

Thus $M = \frac{i}{2\pi} d\rho_\hbar^X$.

In classical mechanics a particle is completely characterised by a point of phase space (q, p), where q corresponds to a position in the configuration space and p—the momentum. The momentum is proportional to velocity—the rate of change of the coordinate q.

Example 7 (momentum observable).

Consider again the Schrödinger representation of the subgroup $\{(0, 0, y)\}$ in $\mathbb{H}^n$ and its derived form:

$$[\rho_\hbar(0, 0, y)f](q) = f(q + \hbar y), \qquad [d\rho_\hbar^Y f](q) = \hbar \frac{d}{dq} f(q). \tag{60}$$

So it produces the shift in configuration space and we associate the self-adjoint operator $D = i\hbar \frac{d}{dq} = i d\rho_\hbar^Y$ produces the expectation of momentum observables. Two operators M and D comes from the derived representation of the Lie algebra $\mathfrak{h}_n$ and

$$[M, D] = -\frac{1}{2\pi}[d\rho_\hbar^X, d\rho_\hbar^Y] = -\frac{1}{2\pi} d\rho_\hbar^S = -\frac{-2\pi i\hbar}{2\pi} I = i\hbar I. \tag{61}$$

Theorem 29 (The Uncertainty relation).
If A *and* B *are self-adjoint operators on* H*, then*

$$\|(A-a)u\| \, \|(B-b)u\| \geqslant \tfrac{1}{2} \left|\langle (AB-BA)u, u\rangle\right|, \tag{62}$$

for any $u \in H$ *from the domains of* AB *and* BA *and* $a, b \in \mathbb{R}$*. Equality holds precisely when* u *is a solution of* $((A-a)+ir(B-b))u = 0$, $r \in \mathbb{R}$*. So, only commuting observables have exact simultaneous measurements.*

Proof.

$$\begin{aligned} \langle (AB-BA)u, u\rangle &= \langle ((A-a)(B-b)-(B-b)(A-a))u, u\rangle \\ &= \langle (B-b)u, (A-a)u\rangle - \langle (A-a)u, (B-b)u\rangle \\ &= 2i\Im \langle (B-b)u, (A-a)u\rangle \end{aligned} \tag{63}$$

Then by the Cauchy–Schwartz inequality:
$\frac{1}{2}\langle (AB-BA)u, u\rangle \leqslant |\langle (B-b)u, (A-a)u\rangle| \leqslant \|(B-b)u\| \, \|(A-a)u\|.$ ∎

Corollary 30 (Heisenberg–Kennard uncertainty relation). *For the coordinate M and momentum D observables we have the Heisenberg–Kennard uncertainty relation:*

$$\Delta_\phi(M) \cdot \Delta_\phi(D) \geqslant \frac{\hbar}{2}. \tag{64}$$

The equality holds iff $\phi(q) = e^{-cq^2}$, $c \in \mathbb{R}_+$ in the Schrödinger model.

Proof.
The relations follows from the commutator $[M, D] = i\hbar I$, which, in turn, is the representation of the Lie algebra commutator in $\mathfrak{h}_n$. The minimal uncertainty state in the Schrodinger representation is a solution of the differential equation: $(M - irD)\phi = 0$, or, explicitly:

$$(M - irD)\phi = \left(q + r\hbar\frac{d}{dq}\right)\phi(q) = 0. \tag{65}$$

The solution is $\phi(q) = e^{-cq^2}$, $c = \frac{1}{2r\hbar}$. For $c > 0$ it is in $L_2(\mathbb{R})$.

The Gaussian
and the related coherent states

The Gaussian $\phi(q) = e^{-\pi q^2}$, which minimises the uncertainty of the coordinate and momentum is a perfect mother wavelet. The respective *coherent states*, very important in quantum optics, cf. (43):

$$\phi_{x,y}(q) = [\rho_\hbar(x,y)\phi](q) = e^{-\pi i\hbar xy - 2\pi i x q}\, e^{-\pi(q+\hbar y)^2}. \tag{66}$$

The expectation value of coordinates is $\hbar y$ and the expectation value of the moment is x. The corresponding induced wavelet transform is the Fock–Segal–Bargmann transform (51).

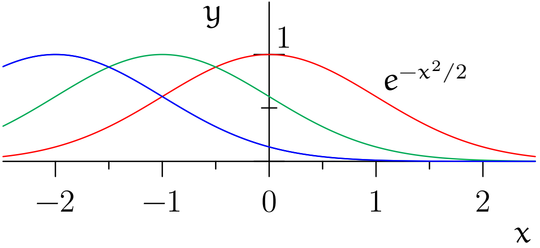

Figure: Shifted Gaussian

The larger set of wavelets, known as *Gabor functions*, is obtained if we add scaling (the automorphism of $\mathbb{H}^n$) and apply the action of the full Schrödinger group (18):

$$\phi_{\delta,x,y}(q) = [\rho_\hbar(x,y)\phi](q) = e^{-\pi i\hbar xy - 2\pi i x q}\, e^{-\delta\pi(q+\hbar y)^2} \qquad (67)$$

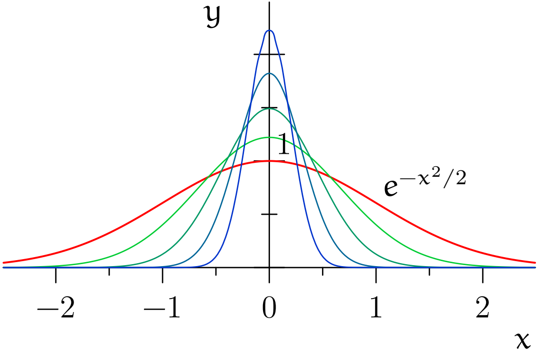

Figure: Scaled Gaussian

The Right Regular Representation

and the action on the mother wavelet

Proposition 31.

Let G *be a Lie group and* ρ *be a representation of* G *in a space* V*. Let* $[\mathcal{W}f](g) = F(\rho(g^{-1})f)$ *be a covariant transform defined by the fiducial operator* $F : V \to U$*. Then the right shift* $[\mathcal{W}f](gg')$ *by* g' *is the covariant transform* $[\mathcal{W}'f](g) = F'(\rho(g^{-1})f)]$ *defined by the fiducial operator* $F' = F \circ \rho(g^{-1})$*.*

In other words the covariant transform intertwines right shifts on the group G with the associated action ρ_B on fiducial operators.

Example 8 (Wavelet transform).

Let the fiducial operator $F(f) = \langle f, \phi \rangle$ is defined by the mother wavelet ϕ and the wavelet transform is $[\mathcal{W}_\phi f](g) = \langle f, \rho(g)\phi \rangle$. Then the proposition tells that the right shifts R on the group G are intertwined with the representation ρ on the mother wavelets:

$$R(g) \circ \mathcal{W}_\phi = \mathcal{W}_{\rho(g)\phi}. \tag{68}$$

Corollary 32 (Analyticity of the wavelet transform).
Let $d\rho_B$ *be the derived representation of a Lie algebra* $\mathfrak{g}$ *on fiducial operators. Let a fiducial operator* F *be a null-solution, i.e.* $AF = 0$, *for the operator* $A = \sum_j a_j d\rho_B^{X_j}$, *where* $X_j \in \mathfrak{g}$ *and* a_j *are constants. Then the covariant transform* $\tilde{f}(g) = F(\rho(g^{-1})f)$ *for any* f *satisfies:*

$$D\tilde{f}(g) = 0, \qquad \textit{where} \quad D = \sum_j \bar{a}_j \mathfrak{L}^{X_j}.$$

$\mathfrak{L}^{X_j}$ *are the left invariant fields (Lie derivatives) on* G *produced by* X_j.

Example 9 (Gaussian and Fock–Segal–Bargmann trans.).
The Gaussian $\phi(x) = e^{-\pi x^2}$ is a null-solution of the operator $M - \frac{i}{h}D$, which insures, that it minimises the uncertainty for operator M and D, see Thm. 29. Therefore, the Fock–Segal–Bargmann transform generated by ϕ as mother wavelet consists of null-solutions of the operator $\mathfrak{L}^M + \frac{i}{h}\mathfrak{L}^D$, which is related to the Cauchy–Riemann operator.

Proposition 33.

The Schrödinger representation (43) is unitary on $L_2(\mathbb{R}^n)$*:*

$$[\rho_\hbar(s,x,y)f](q) = e^{-\pi i\hbar(2s+xy)-2\pi i x q}\, f(q+\hbar y).$$

Proof.

It is obvious for shifts and multiplications separately. □

Proposition 34.

The Schrödinger representation is irreducible.

Proof.

We show that the only operators commuting with all $\rho_\hbar(s,x,y)$ are multiples of the identity. Operators of multiplication $\rho_\hbar(0,0,y)$ separate points, thus any operator commuting with them is an operator of multiplication by a function. Shifts $\rho_\hbar(0,x,0)$ act transitively on the real line, thus only multiplications by a constant commute with all shifts.

Fourier Transform

and the Schrödinger representation

Recall, the symplectic transform $\iota : (s, x, y) \mapsto (s, y, -x)$ is an outer automorphism of the Heisenberg group. For a group G, composition of a representation of G with an automorphism of G is again a representation of G. In the case of Schrödinger representation (34) the composition with ι produces:

$$[\rho_\hbar^\iota(s, x, y)f](y') = e^{2\pi i\hbar(-s-xy'+\frac{1}{2}xy)}\, f(y'-y). \tag{69}$$

Let $\mathcal{F}$ be the Fourier transform:

$$\mathcal{F} : f(x) \mapsto \hat{f}(y) = \int_{\mathbb{R}^n} e^{-2\pi i\hbar xy}\, f(x)\, dx. \tag{70}$$

Checking the action of the Fourier transform on operators of shifts $\rho_\hbar(0, x, 0)$ and multiplication $\rho_\hbar(0, 0, y)$, we see that $\mathcal{F}$ intertwine two representations: $\mathcal{F}\rho_\hbar(s, x, y) = \rho_\hbar^\iota(s, x, y)\mathcal{F}$.
Furthermore, because $\iota^2 : (s, x, y) \mapsto (s, -x, -y)$ we obtain that $\mathcal{F}^2 = R : f(x) \mapsto f(-x)$, thus $\mathcal{F}^{-1} = R \circ \mathcal{F}$.

Fock–Segal–Bargmann Transform

the induced wavelet transform and the Gaussian

Consider $G = \mathbb{H}^1$ and its Schrödinger representation ρ (43). Since the centre Z of $\mathbb{H}^1$ acts with multiplication by scalars we consider the *induced* wavelet transform for $\mathbb{H}^1/Z \sim \mathbb{R}^2$. The very important example of a mother wavelet is the Gaussian $\phi(q) = 2^{\frac{1}{4}} e^{-\pi q^2}$, which produces the celebrated Fock–Segal–Bargmann (FSB) transform $L_2(\mathbb{R}) \to L_2(\mathbb{R}^2)$:

$$[\mathcal{W}f](x,y) = \langle f, \rho(x,y)\phi \rangle = 2^{\frac{1}{4}} \int_{\mathbb{R}} f(x')\, e^{\pi i y(x-2x') - \pi(x'-x)^2}\, dx'. \qquad (71)$$

It intertwines ρ with the following action of $\mathbb{H}^1$ on $L_2(\mathbb{R}^2)$, cf. (32):

$$[\tilde{\rho}(s',x',y')f](x,y) = e^{2\pi i(-s' - \frac{1}{2}\omega(x',y';x,y))} f(x-x', y-y'). \qquad (72)$$

The derived action is:

$$d\tilde{\rho}^S = -2\pi i, \quad d\tilde{\rho}^X = -\pi i y - \partial_x, \quad d\tilde{\rho}^Y = \pi i x - \partial_y. \qquad (73)$$

A representation of the Heisenberg commutator: $[d\tilde{\rho}^X, d\tilde{\rho}^Y] = d\tilde{\rho}^S$.

The Gaussian $\phi(x) = e^{-\pi x^2}$ is a null-solution of the operator $d\rho^X + i d\rho^Y = -\partial_x - 2\pi x$, which insures, that it minimises the uncertainty for operator M and D, see Thm. 29. Therefore by Cor. 32, the Fock–Segal–Bargmann transform generated by ϕ as mother wavelet consists of null-solutions of the operator

$$\mathfrak{L}^X - i\mathfrak{L}^Y = -\pi i y + \partial_x - i(\pi i x + \partial_y) = (\partial_x - i\partial_y) + \pi(x - iy) = \partial_z + \pi\bar{z}, \quad (74)$$

where $z = x + iy$.

If we define the *peeling* map $\mathcal{P} : f(x, y) \mapsto e^{(\pi/2)(x^2+y^2)} f(x, y)$, then image of $\mathcal{P}\mathcal{W}$—wavelet transform and peeled—is the null solution of the operator ∂_z. By a move to the complex conjugate coordinates we can get the Cauchy–Riemann operator. This is Fock–Segal–Bargmann transform in the classic sense and its image consists of analytic functions on the complex plane which are square-integrable with respect to the measure $e^{-\pi|z|^2}\, dz$.

The Fourier–Wigner Transform

the duality in the wavelet transform

For the Schrödinger representation of the Heisenberg group the wavelet transform (47) can be written as follows:

$$\begin{aligned}
[\mathcal{W}_\phi f](x,y) &= \langle f, \rho_\hbar(x,y)\phi\rangle \\
&= \int_{\mathbb{R}} f(x')\, e^{-2\pi i\hbar(yx'-\frac{1}{2}xy)}\, \bar{\phi}(x'-x)\, dx' \\
&= \int_{\mathbb{R}} e^{-2\pi i\hbar y(x'-\frac{1}{2}x)}\, f(x')\, \bar{\phi}(x'-x)\, dx' \\
&= \int_{\mathbb{R}} e^{-2\pi i\hbar y x''}\, f(x''+\tfrac{1}{2}x)\, \bar{\phi}(x''-\tfrac{1}{2}x)\, dx''.
\end{aligned} \tag{75}$$

It is known as the *Fourier–Wigner transform*. It is a composition of measure preserving change of variables $(x'',x) \mapsto (x''+\frac{1}{2}x, x''-\frac{1}{2}x)$ and the Fourier transform $x'' \mapsto y$. Thus, it is unitary on $L_2(\mathbb{R}^2)$ space and preserves the Schwartz space.

Corollary 35.

For the wavelet transform $\mathcal{W}_\phi$ *with the mother wavelet* $\phi \in L_2(\mathbb{R})$ *and the contravariant transform* $\mathcal{M}_\psi \in L_2(\mathbb{R})$ *with the reconstruction vector* ψ *we have the relation:*

$$\mathcal{M}_\psi(\mathcal{W}_\phi f) = \langle \psi, \phi \rangle f, \tag{76}$$

for all $f \in L_2(\mathbb{R})$*. In other words:* $\mathcal{M}_\psi \circ \mathcal{W}_\phi = \langle \psi, \phi \rangle I$*.*

Proof.

For an arbitrary $g \in L_2(\mathbb{R})$ take the inner product:

$$\begin{aligned} \left\langle \mathcal{M}_\psi(\mathcal{W}_\phi f), g \right\rangle &= \left\langle \int_{\mathbb{R}^2} \mathcal{W}_\phi f(x,y) \rho_\hbar(x,y)\, dx\, dy\, \psi, g \right\rangle \\ &= \int_{\mathbb{R}^2} \mathcal{W}_\phi f(x,y) \left\langle \rho_\hbar(x,y)\psi, g \right\rangle\, dx\, dy \\ &= \left\langle \mathcal{W}_\phi f, \mathcal{W}_\psi g \right\rangle = \langle f, g \rangle \langle \psi, \phi \rangle \ . \end{aligned}$$

whence the result is immediate.

Ladder Operators

and the Hermite functions

Consider complexification of the Weyl algebra $\mathfrak{h}_1$ and define operators:

$$a^{\pm} = \frac{1}{2}(X \mp iY), \quad \text{then} \quad [a^+, a^-] = \frac{1}{4}[X - iY, X + iY] = \frac{i}{2}S. \tag{77}$$

Thus in the Schrödinger representation $[d\rho^{a^+}, d\rho^{a^-}] = \pi I$. As we already know $d\rho^{a^-}\phi = 0$ and we define $u_m = (d\rho^{a^+})^m\phi$. Then:

1. $(d\rho^{a^-})^* = d\rho^{a^+}$ on $L_2(\mathbb{R})$.
2. $d\rho^{a^-}u_m = -\pi m u_{m-1}$, because $d\rho^{a^-}u_m = d\rho^{a^-}d\rho^{a^+}u_{m-1} = \pi u_{m-1} + d\rho^{a^+}d\rho^{a^-}u_{m-1} = -\pi(1 + (m-1))u_{m-1}$ (induction!).
3. $\langle u_n, u_m\rangle = \left\langle (d\rho^{a^+})^n\phi, (d\rho^{a^+})^m\phi \right\rangle = \left\langle (d\rho^{a^-})^m(d\rho^{a^+})^n\phi, \phi \right\rangle = 0$, $n < m$. Also $\langle u_m, u_m\rangle = \pi^m m!$.
4. The *Hermite functions* $\phi_m = (\pi^m m!)^{-1/2}(d\rho^{a^+})^m\phi$ make an orthonormal system.
5. $(d\rho^{a^-})\phi_m = -(\pi m)^{1/2}\phi_{m-1}, \quad (d\rho^{a^+})\phi_m = (\pi(m+1))^{1/2}\phi_{m+1}$, $d\rho^{a^-}d\rho^{a^+}\phi_m = -\pi(m+1)\phi_m, \quad d\rho^{a^+}d\rho^{a^-}\phi_m = -\pi m\phi_m$.

Ladder Operators

in FSB space

From the derived form (73) we calculate

$$d\tilde{\rho}^{a^\pm} = \frac{1}{2}(d\tilde{\rho}^X \mp i d\tilde{\rho}^Y) = \frac{1}{2}(-\pi i y - \partial_x \mp i(\pi i x - \partial_y)). \qquad (78)$$

Thus $d\tilde{\rho}^{a^+} = \frac{1}{2}(-\partial_z + \pi\bar{z})$ and $d\tilde{\rho}^{a^+} = -\frac{1}{2}(\partial_{\bar{z}} + \pi z)$.

Because $\mathcal{W}$ intertwines ρ with $\tilde{\rho}$ and $d\rho^{a^-}\phi = 0$, the function $\Phi = \mathcal{W}_\phi \phi$ is the null solution of $d\tilde{\rho}^{a^-}$ or $\partial_{\bar{z}} + \pi z$. Since, it is also the null solution of $\mathfrak{L}^{a^-} = \partial_z + \pi\bar{z}$ it is a multiple of $e^{-\pi z\bar{z}}$.

Define $\Phi_m = (\pi^m m!)^{-1/2}(d\tilde{\rho}^{a^+})^m \Phi = \mathcal{W}_\phi \phi_m$, then we will have the above orthonormality identities for ϕ_m by the intertwining property of $\mathcal{W}$.

Moreover, $\Phi_m(z) = (\pi^m/m!)^{1/2}\bar{z}^m e^{-\pi|z|^2}$.

The name *ladder* operators is explained by the diagram:

$$\Phi_0 \underset{a^-}{\overset{a^+}{\rightleftarrows}} \Phi_1 \underset{a^-}{\overset{a^+}{\rightleftarrows}} \Phi_2 \underset{a^-}{\overset{a^+}{\rightleftarrows}} \Phi_3 \underset{a^-}{\overset{a^+}{\rightleftarrows}} \dots$$

An existence of vacuum Φ_0 implies the *Stone-von Neumann theorem*.

We obtain the Hermite operator as

$$H = a^+a^- + a^-a^+ = 2a^+a^- - \frac{i}{2}S = \frac{1}{2}(X^2 + Y^2) \qquad (79)$$

It is also the Hamiltonian of the harmonic oscillator.
Using identities in 5 we see

$$d\rho^H \phi_m = -\pi(2m+1)\phi_m, \qquad d\tilde{\rho}^H \Phi_m = -\pi(2m+1)\Phi_m.$$

1. The spectrum of the harmonic oscillator is discrete.
2. The eigenfunctions are provided by the Hermite functions in the Schrödinger model or by the powers of $\bar{z}$ in the FSB spaces.
3. The ladder operators acts on the spectrum due to the following commutation relations $[H, a^\pm] = 2a^\pm$:

$$\begin{aligned} H(a^+\phi_k) &= (a^+H + 2a^+)\phi_k = a^+(H\phi_k) + 2a^+\phi_k \\ &= (2k+1)a^+\phi_k + 2a^+\phi_k = (2k+3)a^+\phi_k. \end{aligned}$$

4. The *harmonic oscillator's dynamics* in FSB space is geometric rotation: $A_t : f(z) \mapsto e^{it} f(e^{2it}z)$.

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