# The Many Faces of Elastica

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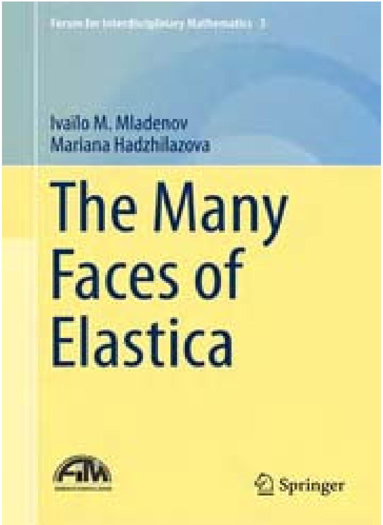
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The Many Faces of Elastica
Springer

# Chapter 4
# Surface Tension and Equilibrium

# Mechanical Equilibrium
# Laplace-Young Equation

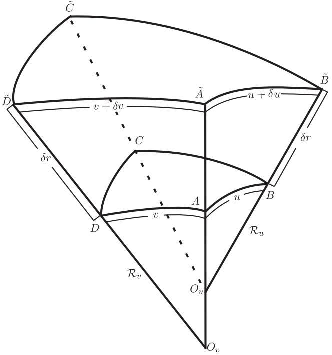


Figure: An infinitesimal local expansion of the surface of the membrane under increasing of the pressure.

Let us consider the infinitesimal curvilinear quadrangle $ABCD$ of the membrane. If the point $A$ coincides with the origin of the orthogonal coordinate lines on the membrane with $\mathcal{R}_u$ and $\mathcal{R}_v$ being their curvature radii and $p_{\rm in}$, $p_{\rm out}$ are the inner, respectively the outer pressures, the work $W$ needed for the infinitesimal expansion of $ABCD$ with sides $u$ and $v$ to the quadrangle $\tilde{A}\tilde{B}\tilde{C}\tilde{D}$ with sides $u + \delta u$ and $v + \delta v$ ($\delta u$ and $\delta v$ are their respective infinitesimal increments) is given

# Mechanical Equilibrium
# Laplace-Young Equation

by the expression

$$\begin{aligned} W &= (p_{\rm in} - p_{\rm out})S\delta r = \Delta p S \delta r = \sigma \Delta S \\ &= \sigma[u(1 + \frac{\delta r}{\mathcal{R}_u})v(1 + \frac{\delta r}{\mathcal{R}_v}) - uv] = \sigma(\frac{1}{\mathcal{R}_u} + \frac{1}{\mathcal{R}_v})uv\delta r \\ &= \sigma(\frac{1}{\mathcal{R}_u} + \frac{1}{\mathcal{R}_v})S\delta r \end{aligned}$$

and therefore

$$\Delta p = 2\sigma H \qquad (1)$$

where $\delta r$ is the infinitesimal displacement of the membrane under the pressure difference $\Delta p = p_{\rm in} - p_{\rm out}$, $\sigma$ is the surface tension, and the equation (2) bears the name Laplace-Young equation.

# Mechanical Equilibrium
# Laplace-Young Equation

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# Tensions and Geometry
## Membrane Geometry

Our membrane will be modeled by a surface of revolution about the $z$-axis generated by a profile curve $(r(s), z(s))$ in the (first quadrant of the) $xz$-plane (where $s$ is the arclength parameter and we take $z(s)$ to increase with increasing $s$: $z(s)$ rising from the $x$-axis and meeting the $z$-axis orthogonally). This surface has a parameterization

$$\mathbf{x}(s, v) = (r(s)\cos v, r(s)\sin v, z(s)) = r(s)\mathbf{e}_1(v) + z(s)\mathbf{e}_3(v)$$

where the unit radial vector is $\mathbf{e}_1(v) = \cos v\,\mathbf{i} + \sin v\,\mathbf{j}$ and $\mathbf{e}_3(v) = \mathbf{k}$.

# Tensions and Geometry
## Membrane Geometry

We also take $\mathbf{e}_2(v) = \mathbf{k} \times \mathbf{e}_1 = -\sin v\,\mathbf{i} + \cos v\,\mathbf{j}$, the unit vector along the parallels of revolution. A *meridian* $r(s)\mathbf{e}_1(\mathring{v}) + z(s)\mathbf{k}$ (i.e., fixed $\mathring{v}$) has tangent vector $\mathbf{t} = r'(s)\mathbf{e}_1(\mathring{v}) + z'(s)\mathbf{k}$, where the primes denote differentiation with respect to $s$. Because we parametrize the meridian by arclength, the tangent vector has unit length, i.e., $r'(s)^2 + z'(s)^2 = 1$. Hence, we can define

$$r'(s) = -\sin\theta(s), \qquad z'(s) = \cos\theta(s) \tag{3}$$

where $\theta(s)$ is the angle between $\mathbf{t}$ and $\mathbf{k}$, and write $\mathbf{t} = -\sin\theta\,\mathbf{e}_1 + \cos\theta\,\mathbf{k}$. Note that, since we assume $z$ increases with $s$, $\mathbf{t}$ has an upward component. Hence, the angle $\theta$ is positive to the "left" of $\mathbf{k}$ in the plane of the profile curve.

# Tensions and Geometry
## Membrane Geometry

There is also a unit normal $\mathbf{n}(s, v)$ to the surface $\mathbf{x}(s, v)$ determined as follows: the unit vector $\mathbf{t}$ and the vector $\mathbf{x}_v = r(s)\mathbf{e}_2$ give a basis for the tangent plane to $\mathbf{x}(s, v)$, so

$$\mathbf{n}(s, v) = \frac{\mathbf{t} \times \mathbf{x}_v}{|\mathbf{t} \times \mathbf{x}_v|} = -\cos\theta(s)\,\mathbf{e}_1(v) - \sin\theta(s)\,\mathbf{k}$$

is the desired unit normal. For a surface of revolution parameterized in the form $(h(s)\cos v, h(s)\sin v, g(s))$ and unit normal specified above, we know (see Oprea [2007, Section 3.3.3] that the principal curvatures are given by

$$k_\mu = \frac{g''h' - g'h''}{(g'^2 + h'^2)^{3/2}}, \qquad k_\pi = \frac{g'}{h(g'^2 + h'^2)^{1/2}} \cdot \tag{4}$$

# Tensions and Geometry
## Membrane Geometry

The subscript $\mu$ denotes that $k_\mu$ is the curvature of the meridian (given by the intersection of the plane determined by **t** and **n** at any point and **x**). The subscript $\pi$ denotes that $k_\pi$ is the curvature given by the intersection of the plane determined by $\mathbf{x}_v$ and **n** at any point and **x**. For our surface $\mathbf{x}(s, v)$,we have $g = z$ and $h = r$, so we obtain

$$k_\mu = \frac{z''r' - z'r''}{1} = \left(\frac{-r'r''}{z'}\right) r' - z'r'' = \frac{-r''}{z'} = \frac{\cos\theta\,\theta'}{\cos\theta} = \theta'$$

where we have used $r'(s)^2 + z'(s)^2 = 1$ and $r'r'' + z'z'' = 0$ (by differentiating the first equation). We also obtain

$$k_\pi = \frac{z'}{r} = \frac{\cos\theta}{r}.$$

These principal curvatures will help us later on to understand the crucial interactions between tensions and geometry.

# Tensions and Geometry
## Tensions

There are three possible tensions to consider: the *meridian stress* $\sigma_m$ in the direction $\mathbf{t}$, the *circumferential* (or *hoop*) stress $\sigma_c$ in the direction $\mathbf{e}_2$ and the shear stress. As argued in Irvine [1981], for membranes that are surfaces of revolution, shear stresses are zero due essentially to symmetry about an axis. These internal tensions are given in units of force per unit length. An inflated membrane has an external pressure $p(s)\bar{\mathbf{n}}(s,v) - w(s)\mathbf{k}$, where $\bar{\mathbf{n}} = -\mathbf{n}$ is the outward normal, the pressure $p(s)$ depends only on the meridian parameter $s$ by symmetry about the $z$-axis and $w(s)$ is a weight density associated to the membrane itself. Note that pressure pushes the membrane outward normally while weight is directed downward as usual. Consider a patch on the membrane (see Fig. 2) with parameter bounds $\mathring{s} \leq s$ and $\mathring{v} \leq v$. The patch is in equilibrium, so the total force acting on it is zero.

# Tensions and Geometry
## Tensions

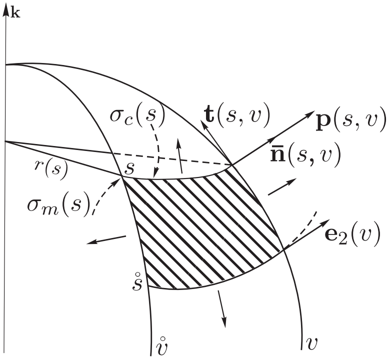


Figure: A patch on an axisymmetric membrane which is in equilibrium under various forces acting on it.

# Tensions and Geometry
## Tensions

Instead of writing things componentwise, we use vector notation and write

$$
\begin{aligned}
0 \;=\; & \int_{\mathring{v}}^{v} \sigma_m(s) r(s) \mathbf{t}(s,u)\,\mathrm{d}u - \int_{\mathring{v}}^{v} \sigma_m(\mathring{s}) r(\mathring{s}) \mathbf{t}(\mathring{s},u)\,\mathrm{d}u \\
& + \int_{\mathring{s}}^{s} \sigma_c(t) \mathbf{e}_2(v)\,\mathrm{d}t - \int_{\mathring{s}}^{s} \sigma_c(t) \mathbf{e}_2(\mathring{v})\,\mathrm{d}t \\
& + \int_{\mathring{s}}^{s} \int_{\mathring{v}}^{v} p(t) r(t) \bar{\mathbf{n}}(t,u)\,\mathrm{d}u\,\mathrm{d}t - \int_{\mathring{s}}^{s} \int_{\mathring{v}}^{v} w(t) r(t) \mathbf{k}\,\mathrm{d}u\,\mathrm{d}t.
\end{aligned}
$$

Now take $\partial/\partial s$ on both sides to obtain

$$
\begin{aligned}
0 \;=\; & \int_{\mathring{v}}^{v} \frac{\partial}{\partial s}(\sigma_m(s) r(s) \mathbf{t}(s,u))\,\mathrm{d}u + \sigma_c(u) \mathbf{e}_2(v) - \sigma_c(u) \mathbf{e}_2(\mathring{v}) \\
& + \int_{\mathring{v}}^{v} p(u) r(u) \bar{\mathbf{n}}(s,u)\,\mathrm{d}u - \int_{\mathring{v}}^{v} w(s) r(s) \mathbf{k}\,\mathrm{d}u.
\end{aligned}
$$

# Tensions and Geometry
## Tensions

Take $\partial/\partial v$ on both sides of this equation to obtain

$$0 = \frac{\partial}{\partial s}(\sigma_m(s)r(s)\mathbf{t}(s,v)) - \sigma_c(s)\mathbf{e}_1(v) + p(s)r(s)\bar{\mathbf{n}}(s,v) - w(s)r(s)\mathbf{k}. \tag{5}$$

Now we can project onto $\mathbf{t}$ and $\bar{\mathbf{n}}$ by dotting with $\mathbf{t}$ and $\bar{\mathbf{n}}$ respectively. We use several facts: $\mathbf{t}\cdot\partial\mathbf{t}/\partial s = 0$, $\mathbf{e}_1\cdot\mathbf{t} = -\sin\theta$ and

$$\begin{aligned}\frac{\partial}{\partial s}(\sigma_m r\mathbf{t})\cdot\mathbf{t} + (\sigma_m r\mathbf{t})\cdot\frac{\partial\mathbf{t}}{\partial s} &= \frac{\partial}{\partial s}(\sigma_m r\mathbf{t}\cdot\mathbf{t})\\ &= \frac{\partial}{\partial s}(\sigma_m r) + 2(\sigma_m r)\frac{\partial\mathbf{t}}{\partial s}\cdot\mathbf{t}\end{aligned}$$

$$\frac{\partial}{\partial s}(\sigma_m r\mathbf{t})\cdot\mathbf{t} = \frac{\partial}{\partial s}(\sigma_m r).$$

# Tensions and Geometry
## Tensions

Therefore we have the equations

$$\begin{aligned} 0 &= \frac{\partial}{\partial s}(\sigma_m r) + \sigma_c \sin\theta - wr\cos\theta \\ \frac{\partial}{\partial s}(\sigma_m r) &= -\sigma_c \sin\theta + wr\cos\theta. \end{aligned} \tag{6}$$

Dotting with $\bar{\mathbf{n}}$ gives (using $\partial \mathbf{t}/\partial s = -\theta' \bar{\mathbf{n}}$)

$$\begin{aligned} 0 &= \frac{\partial}{\partial s}(\sigma_m r\mathbf{t}) \cdot \bar{\mathbf{n}} - \sigma_c \mathbf{e}_1 \cdot \bar{\mathbf{n}} + pr\bar{\mathbf{n}} \cdot \bar{\mathbf{n}} - wr\mathbf{k} \cdot \bar{\mathbf{n}} \\ 0 &= -\sigma_m r\theta' - \sigma_c \cos\theta + pr - wr\sin\theta \\ \sigma_m r\theta' &= pr - \sigma_c \cos\theta - wr\sin\theta. \end{aligned} \tag{7}$$

# Tensions and Geometry
## The case $w = 0$

Let us consider the case where the weight of the membrane is negligible, that is, $w \equiv 0$. Recall that $k_\mu = \theta'$ and $k_\pi = \cos\theta / r$. From (7), we get

$$\sigma_m \theta' + \frac{\sigma_c}{r} \cos\theta = p$$
$$\sigma_m k_\mu + \sigma_c k_\pi = p.$$

If we define the radii of curvature by $r_\mu = 1/k_\mu$ and $r_\pi = 1/k_\pi$, then we have a version of the *Laplace-Young equation* (see for instance Oprea [2000])

$$\frac{\sigma_m}{r_\mu} + \frac{\sigma_c}{r_\pi} = p. \tag{8}$$

# Tensions and Geometry
## The case $w = 0$

**Remark.** Of course, when $w \neq 0$, we then have

$$\frac{\sigma_m}{r_\mu} + \frac{\sigma_c}{r_\pi} = p - \frac{w}{r}\sin\theta. \tag{9}$$

Now, when $w = 0$, the equation (6) becomes $\frac{\partial}{\partial s}(\sigma_m r) = -\sigma_c \sin\theta$. Since $r' = -\sin\theta$, a solution is given by

$$\sigma_m = \sigma_c = \sigma = \text{ constant}.$$

Put this in the Laplace-Young equation (8) to get

$$\sigma\left(\theta' + \frac{\cos\theta}{r}\right) = p, \qquad \frac{1}{2}(k_\mu + k_\pi) = \frac{1}{2}\frac{p}{\sigma}, \qquad H = \frac{1}{2}\frac{p}{\sigma}$$

where $H$ is the *mean curvature* of the membrane. If the pressure $p$ is constant, then $H$ is constant as well and the membrane is a *surface of Delaunay* (see Oprea [2000]).

# Tensions and Geometry
## The case $w = 0$

It is easy to conclude also that when $p$ is still constant, but $\sigma_m \neq \sigma_c$ the equation (8) defines the so called *anisotropic Delaunay surfaces*, and the reader is referred for more details on their subject to Koiso & Palmer [2008].
Finally, if $p = 0$ and $\sigma_m \neq \sigma_c$, one ends with the quite interesting class of the *linear Weingarten surfaces* Mladenov & Oprea [2003] and Lopez [2008].
Consider (6) again (when $w = 0$) and suppose $\sigma_c = 0$ (this situation is known in the literature as the natural shape of the ballon - see Baginski & Winker [2004] and Baginski [2005]), $p = \alpha$, constant. Then $\sigma_m r = \beta$ is a constant as well and $\sigma_m = \beta / r$. From equation (7), we get (using $r' = -\sin\theta$)

# Tensions and Geometry
## The case $w = 0$

$$\frac{\beta}{r} r\theta' = \alpha r$$
$$\beta\theta' = \alpha r$$
$$\theta' = \frac{\alpha}{\beta} r$$
$$2r'\theta' = 2\frac{\alpha}{\beta} rr'$$
$$-2\sin\theta\theta' = 2\frac{\alpha}{\beta} rr'$$
$$2(\cos\theta)' = \frac{\alpha}{\beta}(r^2)'$$
$$2\cos\theta = \frac{\alpha}{\beta} r^2 + d.$$

# Tensions and Geometry
## The case $w = 0$

From our assumptions that the profile curve rises from the $OX$-axis and goes to the $OZ$-axis orthogonally, we see that $\theta = \pi/2$ exactly when $r = 0$. Hence, $d = 0$. Therefore, we have

$$\begin{aligned} 2\frac{\cos\theta}{r} = \frac{\alpha}{\beta} r = \theta' \\ 2k_\pi = k_\mu. \end{aligned} \tag{10}$$

This condition will be explored in the next section. It describes the Mylar balloon. We can thus say the following.

# Tensions and Geometry
## The case $w = 0$

### Theorem

*If a membrane with $w = 0$, constant pressure $p$ and hoop stress $\sigma_c = 0$ is a surface of revolution*

$$\mathbf{x}(s, v) = (r(s)\cos v, r(s)\sin v, z(s)) = r(s)\mathbf{e}_1(v) + z(s)\mathbf{e}_3(v)$$

*then $2k_\pi = k_\mu$, that is, the membrane is a Mylar balloon* (see Section **??**).

# Tensions and Geometry
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# Tensions and Geometry
## Shapes and the Corresponding Surfaces

In the period between 1960 and 1970 J. Smalley did an extensive work on axisymmetric ballon shapes and implement these models on a digital computer.

As most of Smalley's considerations were of numerical origin it deserve to look for those models possesing analytical solutions.

Despite that the system governing these shapes is highly nonlinear we have been successful in finding a few exact solutions which are presented below (see Popova *et al* [2006]).

These solutions have been found neglecting some parameters in the equilibrium equations.

# Tensions and Geometry
## Shapes and the Corresponding Surfaces

Let us start with the case when we can neglect the film weight contribution, i.e., we suppose that $w(s) \equiv 0$ and hence in such a case we have instead the equations (6) and (7) the system

$$\frac{\partial(\sigma_m r)}{\partial s} = -\sigma_c \sin\theta \tag{11}$$

$$(\sigma_m r)\theta' = pr - \sigma_c \cos\theta. \tag{12}$$

In order to be coherent with the geometrical relation (3), the first equation in the system implies that the meridional and circumferential stresses are constant and of the same magnitude, i.e., $\sigma_m = \sigma_c = \sigma = \mathrm{constant}$, while (12) specifies the mean curvature of $\mathcal{S}$, namely

$$H = \frac{p}{2\sigma}\cdot \tag{13}$$

# Tensions and Geometry
## Delaunay Surfaces

If we can arrange that the hydrostatic pressure is also a constant, i.e., $p(u) = p_o = \mathrm{const}$, then we end up with a surface of constant mean curvature

$$H = \frac{p_o}{2\sigma} = \mathrm{const}. \tag{14}$$

Delaunay [1841], has isolated this class of surfaces guided by a genuine geometrical argument - all they are just the traces of the foci of the non-degenerate conics when they roll along a straight line in a plane (*roulettes* in French).

The complete list of Delaunay's surfaces includes cylinders of radius $R$ and mean curvature $H = 1/2R$, spheres of radius $R$ and mean curvature $H = 1/R$, catenoids of mean curvature $H = 0$, and nodoids and unduloids of constant non-zero mean curvatures. Their profile curves are shown in Fig. 3.

# Tensions and Geometry
## Delaunay Surfaces

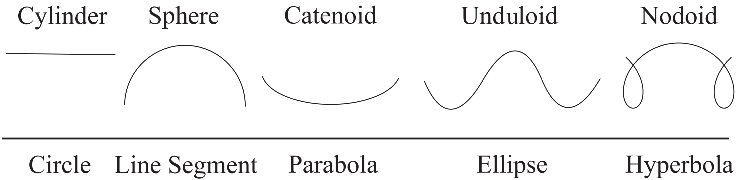


Figure: The profile curves of Delaunay's surfaces obtained by rolling the conics listed below the horizontal line on it.

# Tensions and Geometry
## Nodoids and Unduloids

In this section we will derive the analytical description of the last two and most interesting cases from the Delaunay's list. We start with the system formed by the equations (11) and (12) which ensures the geometrical relation

$$\cos\theta = \frac{\mathring{p} r}{2} + \frac{C}{r} \tag{15}$$

where $C$ is some integration constant. Combined with (3) this leads to the equation

$$r' = -\sin\theta = -\frac{1}{2r}\sqrt{-\mathring{p}^2 r^4 + 4(1-\mathring{p}C)r^2 - 4C^2} \tag{16}$$

in which the variables can be separated, i.e.,

$$\frac{2r\mathrm{d}r}{\sqrt{-\mathring{p}^2 r^4 + 4(1-\mathring{p}C)r^2 - 4C^2}} = -\mathrm{d}s. \tag{17}$$

# Tensions and Geometry
## Nodoids and Unduloids

The unpleasant sign on the right hand side can be eliminated by going to a new variable, say $u = -s$ as we can measure the distance along the curve just in two ways. Introducing additionally as a new variable $\xi = r^2$ we end up with the task for the evaluation of the elementary integral (on the left) written below

$$\int \frac{\mathrm{d}\xi}{\sqrt{(c^2 - \xi)(\xi - a^2)}} = \int \mathrm{d}u = u + \phi \tag{18}$$

where

$$a = \frac{1 - \sqrt{1 - 2\mathring{p}C}}{\mathring{p}}, \quad c = \frac{1 + \sqrt{1 - 2\mathring{p}C}}{\mathring{p}} \quad \text{and} \quad \phi \in \mathbb{R} \tag{19}$$

is some integration constant.

# Tensions and Geometry
# Nodoids and Unduloids

After some calculations the result of integration can be written in the form

$$\xi(u) = r^2(u) = [(c^2 - a^2)\sin u + (c^2 + a^2)]/2 \qquad (20)$$

in which the integration constant is omitted as it is inessential for our further considerations.
In order to find the generating curve we have to solve also the second equation in (3), which in view of the above notation reads

$$\frac{\mathrm{d}z}{\mathrm{d}r} = -\tan\theta = -\frac{\mathring{p}r^2 + 2C}{\sqrt{(c^2 - r^2)(r^2 - a^2)}} \qquad (21)$$

and, therefore

$$z(r) = -\int \frac{(\mathring{p}r^2 + 2C)\mathrm{d}r}{\sqrt{(c^2 - r^2)(r^2 - a^2)}}. \qquad (22)$$

# Tensions and Geometry
## Nodoids and Unduloids

The integral on the right hand side can be uniformized by performing the change

$$r(t) = c\,\mathrm{dn}(t, k) \tag{23}$$

where $\mathrm{dn}(t, k)$ is one of the Jacobian elliptic function of the argument $t$ and the elliptic module $k$. Choosing $k$ to be $\sqrt{c^2 - a^2}/c$ we get

$$z(t) = c\mathring{p} \int \mathrm{dn}^2(t, k)\mathrm{d}t + \frac{2C}{c} \int \mathrm{d}t \tag{24}$$

and consequently

$$z(t) = c\mathring{p}E\left(\mathrm{am}(t, k), k\right) + \frac{2C}{c}F\left(\mathrm{am}(t, k), k\right). \tag{25}$$

# Tensions and Geometry
## Nodoids and Unduloids

Finally, equations (20) and (23) will be compatible if the natural parameter $u$ and the uniformizing parameter $t$ are related by the equation

$$\sin u = 1 - 2\mathrm{sn}^2(t, k). \tag{26}$$

Let us mention also that another pair of formulas in place of (23) and (25) used for drawing unduloid and nodoid in Fig. 4 has been derived following the variational approach in Mladenov & Oprea [2003a].

# Tensions and Geometry
## Intrinsic Equation of the Profile Curves of Delaunay Surfaces

For some reasons (including typographical ones) it will be useful to exchange slightly the notation as follows

$$H = \frac{p_o}{2\sigma} = \lambda = \mathrm{constant} \qquad (27)$$

and $C = \mu$, so that equation (15) takes the form

$$\cos\theta = \lambda r + \frac{\mu}{r}. \qquad (28)$$

The later can be recognized as the Gauss map of the Delaunay surfaces (see Eells [1987]). Without any loss of generality we can assume that the constant $\lambda$ is a strictly positive number relying either to physical experiments with membranes and balloons or taking into account the mathematical fact that $r \equiv r(s)$ is always positive and that we can measure $\theta \equiv \theta(s)$ only in two ways - clockwise or counterclockwise. The case when $\lambda \equiv 0$ will be treated separately below.

# Tensions and Geometry
## Intrinsic Equation of the Profile Curves of Delaunay Surfaces

Differentiating consecutively (28) with respect to $s$ we get

$$\theta' = \lambda - \frac{\mu}{r^2} \tag{29}$$

and

$$\theta'' = -\frac{2\mu}{r^3}\sin\theta. \tag{30}$$

Taking into account that $\theta'(s)$ coincides with the curvature $\kappa \equiv \kappa(s) = \kappa_\mu(s)$ of the profile curve of the surface in the $XOZ$ plane the equation (30) can be rewritten into the form

$$\kappa' = -2(\lambda - \kappa)\sqrt{\frac{\lambda - \kappa}{\mu} - (2\lambda - \kappa)^2} \tag{31}$$

which is just the intrinsic equation (Mladenov *et al* [2008])of the meridional curve we have sought.

# Tensions and Geometry
## Intrinsic Equation of the Profile Curves of Delaunay Surfaces

As before, the minus sign in front of (31) suggests again to change the independent variable, i.e., $s = -u$. Respectively, the solution to equation (31) is

$$\kappa(u) = \lambda \frac{1 - 4\lambda\mu + \sqrt{1 - 4\lambda\mu}\,\sin(2\lambda u)}{1 - 2\lambda\mu + \sqrt{1 - 4\lambda\mu}\,\sin(2\lambda u)}, \quad -\infty \leq \mu \leq \frac{1}{4\lambda} \qquad (32)$$

which further implies (via (29)) that

$$r(u) = \frac{\sqrt{1 - 2\lambda\mu + \sqrt{1 - 4\lambda\mu}\,\sin(2\lambda u)}}{\lambda\sqrt{2}}. \qquad (33)$$

# Tensions and Geometry
## Intrinsic Equation of the Profile Curves of Delaunay Surfaces

Integrating the second equation in (3) (in conjunction with (28) and (33)) we obtain immediately

$$z(u) = \frac{\mu}{m(\lambda,\mu)} F(\lambda u - \frac{\pi}{4}, k) + \frac{m(\lambda,\mu)}{\lambda} E(\lambda u - \frac{\pi}{4}, k) \qquad (34)$$

where the numerical factors are given by the expressions

$$m(\lambda,\mu) = \frac{\sqrt{1 - 2\lambda\mu + \sqrt{1 - 4\lambda\mu}}}{\sqrt{2}}, \quad k = \sqrt{\frac{2\sqrt{1-4\lambda\mu}}{1 - 2\lambda\mu + \sqrt{1 - 4\lambda\mu}}} \qquad (35)$$

while $F(\varphi, k)$ and $E(\varphi, k)$ denote the so called incomplete elliptic integrals of the first, respectively second kind which are functions of their argument $\varphi$ and the parameter $k$ is known as an elliptic modulus.

# Tensions and Geometry
## Intrinsic Equation of the Profile Curves of Delaunay Surfaces

Let us consider also the case when the differential hydrostatic pressure across the membrane vanishes, i.e., $\lambda \equiv 0$. In that case the intrinsic equation (31) reduces to

$$\tilde{\kappa}' = 2\tilde{\kappa}\sqrt{-\frac{\tilde{\kappa}}{\mu} - \tilde{\kappa}^2} \tag{36}$$

and its solution is

$$\tilde{\kappa}(u) = -\frac{\mu}{u^2 + \mu^2} \,. \tag{37}$$

This time (29) and (28) produce the parameterization of the catenoid

$$\tilde{r}(u) = \sqrt{u^2 + \mu^2}, \qquad \tilde{z}(u) = \mu \mathrm{Ln}\left(u + \sqrt{u^2 + \mu^2}\right) \tag{38}$$

drawn in Fig. 4.

# Tensions and Geometry
## Intrinsic Equation of the Profile Curves of Delaunay Surfaces

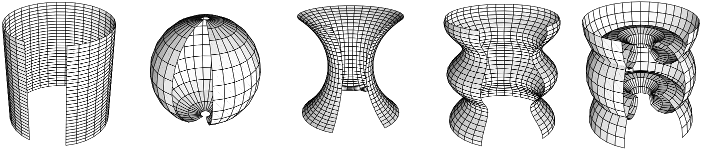

Figure: The open parts of the cylinder, sphere, catenoid, unduloid and nodoid shown here are drawn via the profile curves (33) and (34) or (38) and various combinations of the parameters $\lambda$ and $\mu$.

# Tensions and Geometry
## Some Useful Formulas

Having the explicit form of the parameterization (33) and (34) or (38) one can easily find any other geometrical characteristic of the surface $\mathcal{S}$. It is a general theorem in the classical differential geometry that for such a purpose one needs to know only the first and the second fundamental forms of the surface under consideration. Actually, we have already derived the corresponding formulas for $E, F, G, L$ and $M$ and further direct computations (in $\lambda \neq 0$ case) produce

$$\begin{aligned} N &= \frac{1 + \sqrt{1 - 4\lambda\mu}\sin(2\lambda u)}{2\lambda} \\ \kappa_\pi(s) &= \lambda \frac{1 + \sqrt{1 - 4\lambda\mu}\sin(2\lambda u)}{1 - 2\lambda\mu + \sqrt{1 - 4\lambda\mu}\sin(2\lambda u)} . \end{aligned} \qquad (39)$$

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The formulas mentioned above allows an easy check that by taking $\mu = \dfrac{1}{4\lambda}$ one indeed gets a cylinder, and that $\mu = 0$ corresponds to a sphere. The intermediate cases when $0 < \mu < \dfrac{1}{4\lambda}$ generate unduloids and those ones with $\mu < 0$ lead to nodoids. If one is interested in the solution of the *inverse problem*, i.e., how to find the corresponding parameters $\lambda, \mu$ if the maximal $r_{\max}$ and minimal $r_{\min}$ distances from the symmetry axis are given, one easily ends with the conclusion that in the case of the unduloid these are

$$\lambda = \frac{1}{r_{\max} + r_{\min}}, \qquad \mu = \frac{r_{\max} r_{\min}}{r_{\max} + r_{\min}} \tag{40}$$

$$r_{\max} = \frac{\sqrt{1 - 2\lambda\mu + \sqrt{1 - 4\lambda\mu}}}{\lambda\sqrt{2}}, \qquad r_{\min} = \frac{\sqrt{1 - 2\lambda\mu - \sqrt{1 - 4\lambda\mu}}}{\lambda\sqrt{2}}$$

# Tensions and Geometry
## Some Useful Formulas

and the respective nodoid with the same geometrical data can be built with

$$\lambda = \frac{1}{r_{\max} - r_{\min}}, \qquad \mu = -\frac{r_{\max} r_{\min}}{r_{\max} - r_{\min}} \cdot \tag{41}$$

The parameters for the cylinders and spheres are recovered directly via (41) taking into account their geometry is specified respectively by $r_{\max} = r_{\min}$ in the first and $r_{\max} \in \mathbb{R}^+$, $r_{\min} = 0$ in the second case.

For the catenoids ($\lambda = 0$) one has respectively $E = 1$, $F = 0$, $G = u^2 + \mu^2$ for the first, and $L = -\frac{\mu}{u^2 + \mu^2}$, $M = 0$, $N = \mu$ for the second fundamental form, while by $\mu = \pm r_{\min}$ one recovers the explicit parameterization (38) of the surface.

# Tensions and Geometry
## Delaunay Construction

It was already mentioned that Delaunay starts with finding the evolute $\tilde{\mathcal{C}}$ of the sought profile curve $\mathcal{C}$. In what follows we will keep his notation as much as possible.
If $\rho$ is the radius of the curvature of $\mathcal{C}$ and $\tilde{s}$ is the natural parameter on its evolute $\tilde{\mathcal{C}}$ by its very definition one has

$$\rho = \mathring{c} - \tilde{s} \tag{42}$$

where $\mathring{c}$ is an arbitrary real parameter. Let $\mathfrak{n}$ denotes the part of the tangent $\tilde{\mathbf{T}}$ to $\tilde{\mathcal{C}}$ between $M$ and its intersection with the symmetry axis $OX$.The condition that the surface $\mathcal{S}$ obtained by revolving $\mathcal{C}$ about $OX$ has a constant mean curvature $\frac{1}{2a}$in which $a$ is another real parameter can be written as

$$\frac{1}{\rho} + \frac{1}{\mathfrak{n}} = \frac{1}{a}\,. \tag{43}$$

# Tensions and Geometry
# Delaunay Construction

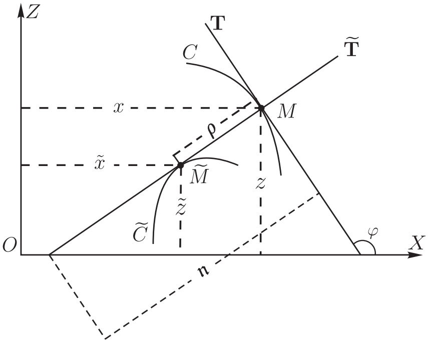


Figure: The curve $\mathcal{C}$ and it's evolute $\tilde{\mathcal{C}}$.

# Tensions and Geometry
# Delaunay Construction

By expecting Fig. 5 one easily finds also that

$$\mathfrak{n} = \tilde{z}\frac{\mathrm{d}\tilde{s}}{\mathrm{d}\tilde{z}} + \mathring{c} - \tilde{s} \qquad (44)$$

and therefore

$$\frac{1}{\mathring{c} - \tilde{s}} + \frac{1}{\tilde{z}\frac{\mathrm{d}\tilde{s}}{\mathrm{d}\tilde{z}} + \mathring{c} - \tilde{s}} = \frac{1}{a} \cdot \qquad (45)$$

Integrating the last equation one gets

$$\tilde{z}^2 = \alpha(\mathring{c} - \tilde{s})(2a - \mathring{c} + \tilde{s}) \qquad (46)$$

where $\alpha$ is the integration constant.

# Tensions and Geometry. Delaunay Construction

This equation can be solved for $\tilde{s}$ and after that the result differentiated with respect to $\tilde{z}$ in order to obtain

$$\frac{\mathrm{d}\tilde{z}}{\mathrm{d}\tilde{s}} = -\frac{\alpha\sqrt{a^2 - \tilde{z}^2/\alpha}}{\tilde{z}} \tag{47}$$

along with

$$\frac{\mathrm{d}\tilde{x}}{\mathrm{d}\tilde{s}} = \sqrt{1 - (\frac{\mathrm{d}\tilde{z}}{\mathrm{d}\tilde{s}})^2} = \frac{\sqrt{(1+\alpha)\tilde{z}^2 - a^2\alpha^2}}{\tilde{z}} \cdot \tag{48}$$

An inspection of (47) and (48) leads to the conclusion that in the above expressions the constant $\alpha$ can take all positive values and that in this case $\tilde{z}$ will vary in the interval $[\frac{a\alpha}{\sqrt{1+\alpha}}\,,\ a\sqrt{\alpha}]$. If $\alpha$ is negative it can take values between $-1$ and 0 while $|\tilde{z}|$ can take any value greater than $-\frac{a\alpha}{\sqrt{1+\alpha}}\cdot$ This means that in the last case the evolute will have infinite branches. The two alternatives just described will be considered below separately. They are described in detail in Hadzhilazova & Mladenov [2009]

# Tensions and Geometry
## Nodary

Assuming that $0 < \alpha \leq \infty$ and introducing

$$m^2 = \frac{a^2\alpha^2}{1+\alpha}, \qquad n^2 = a^2\alpha, \qquad n^2 > m^2 \tag{49}$$

equations (47) and (48) can be combined into the form

$$\frac{\mathrm{d}\tilde{x}}{\mathrm{d}\tilde{z}} = -\frac{\sqrt{(1+\alpha)\tilde{z}^2 - a^2\alpha^2}}{\alpha\sqrt{a^2 - \tilde{z}^2/\alpha}} = -\sqrt{\frac{1+\alpha}{\alpha}}\frac{\sqrt{\tilde{z}^2 - m^2}}{\sqrt{n^2 - \tilde{z}^2}} \cdot \tag{50}$$

The last expression suggests that it can be uniformized via

$$\tilde{z} = \frac{m}{\mathrm{dn}(u,k)}, \qquad m = \frac{a\alpha}{\sqrt{1+\alpha}} = a\alpha k, \qquad k = \frac{1}{\sqrt{1+\alpha}} \tag{51}$$

where $\mathrm{dn}(u,k)$ is one of the three Jacobian elliptic functions, $u$ is its argument and the parameter $k$ is known as an elliptic modulus.

# Tensions and Geometry
## Nodary

Using relations (51) and equation (50) we have

$$\tilde{x} = m(u - E(\mathrm{am}(u,k),k)) \tag{52}$$

where $\mathrm{am}(u,k)$ is Jacobi's amplitude function, $E(\psi,k)$ denotes the so called incomplete elliptic integral of the second kind and the integration constant is omitted. Taken together (51) and (52) provide the explicit parameterization of the evolute $\tilde{\mathcal{C}}$. Its involute, i.e., the profile curve $\mathcal{C}$ of the Delaunay surface of constant mean curvature $\dfrac{1}{2a}$ can be found relying on direct geometrical relations (or consulting some of the textbook on classical differential geometry as Gray [1998] or Oprea [2000])

$$x = \tilde{x} + \rho\frac{\mathrm{d}\tilde{x}}{\mathrm{d}\tilde{s}}, \qquad z = \tilde{z} + \rho\frac{\mathrm{d}\tilde{z}}{\mathrm{d}\tilde{s}} \cdot \tag{53}$$

# Tensions and Geometry
## Nodary

By (42), (46-48) and (51) one easily find

$$\begin{aligned}\rho &= a - \sqrt{a^2 - \frac{\tilde{z}^2}{\alpha}} = a(1 - k\frac{\mathrm{cn}(u,k)}{\mathrm{dn}(u,k)}) \\ \frac{\mathrm{d}\tilde{x}}{\mathrm{d}\tilde{s}} &= \mathrm{sn}(u,k), \qquad \frac{\mathrm{d}\tilde{z}}{\mathrm{d}\tilde{s}} = -\mathrm{cn}(u,k)\end{aligned} \tag{54}$$

which taken together give the parameterization of the nodary

$$\begin{aligned}x[u] &= m(u - E(\mathrm{am}(u,k),k)) + a(1 - k\frac{\mathrm{cn}(u,k)}{\mathrm{dn}(u,k)})\mathrm{sn}(u,k) \\ z[u] &= \frac{m}{\mathrm{dn}(u,k)} - a(1 - k\frac{\mathrm{cn}(u,k)}{\mathrm{dn}(u,k)})\mathrm{cn}(u,k).\end{aligned} \tag{55}$$

Both, the *nodary* $\mathcal{C}$ and its evolute $\tilde{\mathcal{C}}$ are depicted in Fig. 6 for a concrete values of the parameters $\alpha$ and $a$.

# Tensions and Geometry
## Nodary

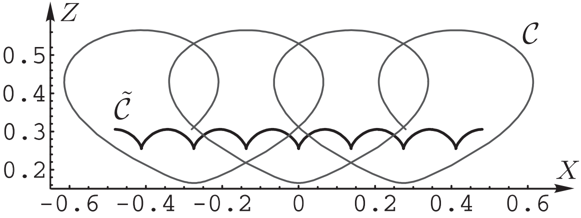


Figure: The evolute $\tilde{C}$ of the *nodary* $C$ generated with $\alpha = 2.333$ and $a = 0.2$ by (51), (52) and (55).

# Tensions and Geometry. Undulary

Following the plan announced above we will consider in this section the case when $\alpha$ belongs to the interval $-1 < \alpha < 0$. Because the treatment will be quite parallel to that of nodary the details will be just outlined but in order to distinguish the cases we use in respective formulas the bars which will reminiscent that we are dealing with negative $\alpha$. In this setting we will have

$$\frac{\mathrm{d}\tilde{\bar{x}}}{\mathrm{d}\tilde{s}} = \frac{\sqrt{(1+\alpha)\tilde{\bar{z}}^2 - a^2\alpha^2}}{\tilde{\bar{z}}}, \qquad \frac{\mathrm{d}\tilde{\bar{z}}}{\mathrm{d}\tilde{s}} = \frac{\sqrt{-\alpha}\sqrt{\tilde{\bar{z}}^2 - a^2\alpha}}{\tilde{\bar{z}}} \tag{56}$$

and respectively

$$\frac{\mathrm{d}\tilde{\bar{x}}}{\mathrm{d}\tilde{\bar{z}}} = \frac{1}{\sqrt{-\alpha}}\sqrt{\frac{(1+\alpha)\tilde{\bar{z}}^2 - a^2\alpha^2}{\tilde{\bar{z}}^2 - a^2\alpha}} = \sqrt{\frac{1+\alpha}{-\alpha}}\sqrt{\frac{\tilde{\bar{z}}^2 - \bar{m}^2}{\tilde{\bar{z}}^2 + \bar{n}^2}} \tag{57}$$

where

$$\bar{m}^2 = \frac{a^2\alpha^2}{1+\alpha}, \qquad \bar{n}^2 = -a^2\alpha, \qquad \bar{m}^2 + \bar{n}^2 = -\frac{a^2\alpha}{1+\alpha}. \tag{58}$$

# Tensions and Geometry
## Undulary

This time the uniformization can be accomplished by dn and cn, i.e.,

$$\begin{aligned} \tilde{\bar{z}} &= \sqrt{\bar{m}^2 + \bar{n}^2}\, \frac{\mathrm{dn}(u, \bar{k})}{\mathrm{cn}(u, \bar{k})} = \sqrt{\frac{-\alpha}{1+\alpha}}\, \frac{\mathrm{dn}(u, \bar{k})}{\mathrm{cn}(u, \bar{k})} \\ \bar{k}^2 &= \frac{\bar{n}^2}{\bar{m}^2 + \bar{n}^2} = 1 + \alpha. \end{aligned} \tag{59}$$

Doing this, we obtain

$$\mathrm{d}\tilde{\bar{x}} = -a \frac{\mathrm{cn}^2(u, \bar{k})}{\mathrm{sn}^2(u, \bar{k})}\, \mathrm{d}u \tag{60}$$

and consequently

$$\tilde{\bar{x}} = a \left( E(\mathrm{am}(u, \bar{k}), \bar{k}) + \frac{\mathrm{cn}(u, \bar{k})\mathrm{dn}(u, \bar{k})}{\mathrm{sn}(u, \bar{k})} \right) \tag{61}$$

in which case as before, the integration constant is omitted.

# Tensions and Geometry
## Undulary

Further on, it is easy to find also that

$$\bar{\rho} = a(1 - \frac{1}{\bar{k}} \frac{1}{\mathrm{sn}(u, \bar{k})}), \qquad \frac{\mathrm{d}\tilde{\tilde{x}}}{\mathrm{d}\tilde{s}} = \bar{k}\, \frac{\mathrm{cn}(u, \bar{k})}{\mathrm{dn}(u, \bar{k})}, \qquad \frac{\mathrm{d}\tilde{\tilde{z}}}{\mathrm{d}\tilde{s}} = \frac{\sqrt{-\alpha}}{\mathrm{dn}(u, \bar{k})} \tag{62}$$

which immediately gives

$$\bar{x}[u] = \tilde{\tilde{x}} + \bar{\rho}\frac{\mathrm{d}\tilde{\tilde{x}}}{\mathrm{d}\tilde{s}} = a(\bar{k} - \frac{1}{\mathrm{sn}(u, \bar{k})})\frac{\mathrm{cn}(u, \bar{k})}{\mathrm{dn}(u, \bar{k})}$$

$$\bar{z}[u] = \tilde{\tilde{z}} + \bar{\rho}\frac{\mathrm{d}\tilde{\tilde{z}}}{\mathrm{d}\tilde{s}} = \frac{a\sqrt{-\alpha}}{\bar{k}}\left(\frac{\mathrm{dn}(u, \bar{k})}{\mathrm{sn}(u, \bar{k})} + a(\bar{k} - \frac{1}{\mathrm{sn}(u, \bar{k})})\frac{1}{\mathrm{dn}(u, \bar{k})}\right). \tag{63}$$

# Tensions and Geometry
## Nodary

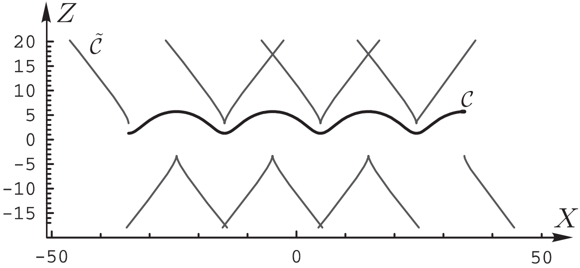


Figure: The evolute $\tilde{C}$ of the *undulary* $C$ generated with $\alpha = 3.5$ and $a = -0.6$ by (59), (61) and (63).

# Tensions and Geometry. Nodary

**Remarks** The generating curves of the nodoids and the unduloids called by Eells [1987] *nodary*, respectively *undulary* and their evolutes, have been found following the original Delaunay construction. These curves are periodic along the symmetry axis and have one local minimum and one local maximum in each period and do not depend on the chosen point on the evolutes as the constant $\mathring{c}$ disappears from all formulae. The parameterizations (up to integration) found by Delaunay himself (cf Delaunay [1841]) are given below for a comparison with those derived here and elsewhere (Mladenov [2002] and Hadzhilazova *et al* [2007a]), i.e.,

$$\tilde{x} = -\frac{a\alpha \tan\varphi}{\sqrt{1+\alpha-\sin^2\varphi}} + \int_0^{\varphi} \frac{a\alpha \mathrm{d}\varphi}{\cos^2\varphi\sqrt{1+\alpha-\sin^2\varphi}} \tag{64}$$

$$\tilde{z} = \frac{a\alpha}{\sqrt{1+\alpha-\sin^2\varphi}}, \qquad \varphi \in \mathbb{R}$$

# Tensions and Geometry. Nodary

**Remarks** parameterize the evolutes, and

$$x[\varphi] = a\sin\varphi - a\tan\varphi\sqrt{1+\alpha-\sin^2\varphi} + \int_0^{\varphi}\frac{a\alpha\mathrm{d}\varphi}{\cos^2\varphi\sqrt{1+\alpha-\sin^2\varphi}} \quad (65)$$

$$z[\varphi] = -a\cos\varphi + a\sqrt{1+\alpha-\sin^2\varphi}$$

do the same for the nodary and undulary. The parameters $\alpha$ and $a$ have the same meaning as specified before.
Finally, it can be easily realized that the integrals which appear in Delaunay formulas exist only on restricted intervals on which the evolute can be found.

*Thank you for attention!*