On rotationally invariant (super)integrability with magnetic fields in 3D

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What is (Liouville) integrability?

Let us consider a N-dimensional Hamiltonian system,

$$H = \frac{1}{2}\sum_{i=1}^{N} g^{ii}(\vec{x})(p_i^A)^2 + W(\vec{x}),$$

which takes value on a $2N$-dimensional phase space $(\vec{x}, \vec{p})$. For this Hamiltonian system to be said (Liouville) integrable, there must exist $N-1$ integrals of motion X_i (in addition to the Hamiltonian) that are in involution, i.e.

$$\frac{dX_i}{dt} = \{X_i, H\} = 0, \qquad \{X_i, X_j\} = 0$$

and such that H and all X_i are functionally independent.
The Poisson bracket is defined as

$$\{a, b\} = \sum_{i=1}^{N}\left(\frac{\partial a}{\partial x_i}\frac{\partial b}{\partial p_i} - \frac{\partial b}{\partial x_i}\frac{\partial a}{\partial p_i}\right).$$

What is superintegrability?

For the same integrable Hamiltonian system to be called superintegrable, there must exist M (where $1 \leq M \leq N-1$) additional integrals of motion Y_j, i.e.

$$\{Y_j, H\} = 0$$

and H, all X_i and all Y_j must be functionally independent, i.e.

$$\text{Rank}\left[\frac{\partial(H, X_i, Y_j)}{\partial(\vec{x}, \vec{p})}\right] = N + M.$$

Note: Y_{j+1} is not required to be in involution with (X_i, Y_j).
When $M = 1$, the system is minimally superintegrable.
When $M = N - 1$, the system is maximally superintegrable.

Quantum integrability and superintegrability

The quantum version is defined in a similar way:

1. The classical phase space coordinates must be replaced by their associated quantum operators.
2. The Hamiltonian and the integrals of motion must be well-defined Hermitian operators.
3. The Poisson bracket must be replaced by the commutator.

In the following results, no purely quantum system exists, i.e. the quantum results are equivalent to the classical ones. Hence, we will only consider the classical version from now on, unless specified otherwise.

Properties

For the classical case:

- Separation of the Hamilton–Jacobi equation in one (or more) coordinate system.
- For superintegrable systems, the trajectories are restrained to a $N - M$ subspace.
- For maximally superintegrable systems, finite trajectories are closed and periodic.
- A resilience to perturbations.

For the quantum case:

- Separation of the Schrödinger equation in one (or more) coordinate system.
- Degeneration of the energy levels.
- Conjecture that all maximally superintegrable systems are exactly solvable.

Leading order terms

For a quadratic integral in the 3D Cartesian coordinates,

$$X = \sum h_i(\vec{x})(p_i^A)^2 + \frac{1}{2}\sum |\epsilon_{ijk}| n_i(\vec{x}) p_j^A p_k^A + \sum s_i(\vec{x}) p_i^A + m(\vec{x}),$$

where $p_k^A = p_k + A_k(\vec{x})$, with or without a magnetic field, the (ten) third order equations are the same, i.e.

$$\partial_i h_i = \vec{\nabla}\cdot\vec{n} = 0, \qquad i = 1,2,3,$$
$$\partial_i h_j + \partial_j n_k = 0, \qquad i \neq j \neq k \neq i, \qquad i,j,k = 1,2,3.$$

Hence, the leading order terms are given by

$$\sum_{1\le i\le j\le 3} a_{ij} p_i^A p_j^A + \sum_{1\le i\le j\le 3} b_{ij} L_i^A L_j^A + \sum_{i,j} c_{ij} p_i^A L_j^A,$$

where $L_i^A = \epsilon_{ijk} x_j p_k^A$.

Systems of coordinates allowing separation of variables of the Hamilton–Jacobi equation and the leading order terms of the second order integrals of motion.

	Coordinate systems	X_1	X_2
1	Cartesian	$(p_x)^2$	$(p_y)^2$
2	Cylindrical	$(L_z)^2$	$(p_z)^2$
3	Elliptic cylindrical	$(p_z)^2$	$(L_z)^2 + a(p_x)^2$
4	Parabolic cylindrical	$(p_z)^2$	$p_y L_z$
5	Spherical	$(L_z)^2$	L^2
6	Prolate spheroidal	$(L_z)^2$	$L^2 - a^2(p_x)^2 - a^2(p_y)^2$
7	Oblate spheroidal	$(L_z)^2$	$L^2 + a^2(p_x)^2 + a^2(p_y)^2$
8	Circular parabolic	$(L_z)^2$	$p_y L_x - p_x L_y$
9	Conical	L^2	$b^2(L_x)^2 + c^2(L_y)^2$
10	Ellipsoidal	...	...
11	Paraboloidal	...	...

$$L^2 = (L_x)^2 + (L_y)^2 + (L_z)^2$$

Electromagnetic field

For a 3D Hamiltonian with an electromagnetic field, one must consider the scalar potential $W(\vec{x})$ and the vector potential $A(\vec{x})$ that can be written as a 1-form, i.e.

$$A = A_x(\vec{x})dx + A_y(\vec{x})dy + A_z(\vec{x})dz$$

and it is linked to the magnetic field

$$B = B_x(\vec{x})dy \wedge dz + B_y(\vec{x})dz \wedge dx + B_z(\vec{x})dz \wedge dy$$

by the relations

$$B_i = \epsilon_{ijk}\partial_j A_k,$$

where B is invariant under the transformation $\tilde{A} = A + \nabla\chi$.

Determining equations of $\{H, X\} = 0$

Second order determining equations:

$$\partial_x s_1 = n_2 B_2 - n_3 B_3, \quad \partial_y s_2 = n_3 B_3 - n_1 B_1, \quad \partial_z s_3 = n_1 B_1 - n_2 B_2,$$
$$\partial_y s_1 + \partial_x s_2 = n_1 B_2 - n_2 B_1 + 2(h_1 - h_2) B_3,$$
$$\partial_z s_1 + \partial_x s_3 = n_3 B_1 - n_1 B_3 + 2(h_3 - h_1) B_2,$$
$$\partial_y s_3 + \partial_z s_2 = n_2 B_3 - n_3 B_2 + 2(h_2 - h_3) B_1.$$

First order determining equations:

$$\partial_x m = 2h_1 \partial_x W + n_3 \partial_y W + n_2 \partial_z W + s_3 B_2 - s_2 B_3,$$
$$\partial_y m = n_3 \partial_x W + 2h_2 \partial_y W + n_1 \partial_z W + s_1 B_3 - s_3 B_1,$$
$$\partial_z m = n_2 \partial_x W + n_1 \partial_y W + 2h_3 \partial_z W + s_2 B_1 - s_1 B_2.$$

Zeroth order determining equation:

$$\vec{s} \cdot \nabla W + \frac{\hbar^2}{4} \left(\partial_z n_1 \partial_z B_1 - \partial_y n_1 \partial_y B_1 + \partial_x n_2 \partial_x B_2 - \partial_z n_2 \partial_z B_2 \right.$$
$$\left. + \partial_y n_3 \partial_y B_3 - \partial_x n_3 \partial_x B_3 + \partial_x n_1 \partial_y B_2 - \partial_y n_2 \partial_x B_1 \right) = 0.$$

Integrability for the circular parabolic case

$$X_1 = L_x^A p_y^A - L_y^A p_x^A + \dots \qquad X_2 = (L_z^A)^2 + \dots$$

(Laplace–Runge–Lenz component)

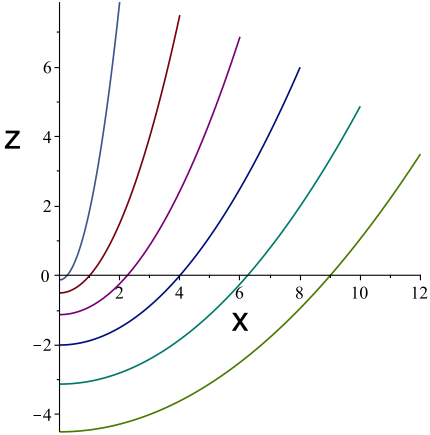
The circular parabolic coordinates

The circular parabolic coordinates are related to the 3D Cartesian coordinates by the transformation

$$x = \xi\eta\cos\phi,$$
$$y = \xi\eta\sin\phi,$$
$$z = \frac{1}{2}\left(\xi^2 - \eta^2\right),$$

metric:

$$g_{11} = \xi^2 + \eta^2,$$
$$g_{22} = \xi^2 + \eta^2,$$
$$g_{33} = \xi^2\eta^2,$$
$$g_{ij} = 0, \qquad i \neq j.$$



The associated Hamiltonian and the integrals of motions

In the circular parabolic coordinates, the Hamiltonian is

$$H = \frac{1}{2}\left(\frac{(p_\xi^A)^2}{\xi^2+\eta^2} + \frac{(p_\eta^A)^2}{\xi^2+\eta^2} + \frac{(p_\phi^A)^2}{\xi^2\eta^2}\right) + W(\xi,\eta,\phi)$$

and the integrals of motion are

$$X_1 = \frac{\eta^2}{2(\xi^2+\eta^2)}(p_\xi^A)^2 - \frac{\xi^2}{2(\xi^2+\eta^2)}(p_\eta^A)^2 + \frac{1}{2}\left(\frac{1}{\xi^2} - \frac{1}{\eta^2}\right)(p_\phi^A)^2 + \ldots$$

$$X_2 = (p_\phi^A)^2 + \ldots$$

Solution to the 18 second order equations

The first order coefficients

$$s_1^\xi = \frac{c_1\,\xi}{\xi^2+\eta^2}, \qquad s_1^\eta = \frac{-c_1\,\eta}{\xi^2+\eta^2}, \qquad s_1^\phi = \frac{f\left(\eta^2\right)-g\left(\xi^2\right)}{\xi^2+\eta^2},$$

$$s_2^\xi = s_2^\eta = 0, \qquad s_2^\phi = 2\frac{\xi^2 f\left(\eta^2\right)+\eta^2 g\left(\xi^2\right)}{\eta^2+\xi^2},$$

and the magnetic field components

$$B_\xi = \xi^2\partial_\eta\left(\frac{g\left(\xi^2\right)-f\left(\eta^2\right)}{\eta^2+\xi^2}\right), \qquad B_\eta = \eta^2\partial_\xi\left(\frac{g\left(\xi^2\right)-f\left(\eta^2\right)}{\eta^2+\xi^2}\right),$$

$$B_\phi = 0, \qquad A = -\frac{\xi^2 f\left(\eta^2\right)+\eta^2 g\left(\xi^2\right)}{\eta^2+\xi^2}\mathrm{d}\phi.$$

Solution to the remaining equations

The scalar potential is

$$W(\xi,\eta,\phi)=\frac{\eta^2\beta\left(\xi^2\right)-\xi^2\alpha\left(\eta^2\right)}{\xi^2\eta^2\left(\eta^2+\xi^2\right)}+\frac{1}{2}\left(\frac{f\left(\eta^2\right)-g\left(\xi^2\right)}{\eta^2+\xi^2}\right)^2$$

and the integrals of motion are

$$\begin{aligned} X_1 = & \frac{\eta^2(p_\xi^A)^2-\xi^2(p_\eta^A)^2}{2\left(\eta^2+\xi^2\right)}+\frac{1}{2}\left(\frac{1}{\xi^2}-\frac{1}{\eta^2}\right)(p_\phi^A)^2 \\ & +\left(\frac{f\left(\eta^2\right)-g\left(\xi^2\right)}{\eta^2+\xi^2}\right)p_\phi^A+\frac{\xi^4\alpha\left(\eta^2\right)+\eta^4\beta\left(\xi^2\right)}{\eta^2\xi^2(\eta^2+\xi^2)}, \\ X_2 = & \left(p_\phi^A+\frac{\xi^2 f\left(\eta^2\right)+\eta^2 g\left(\xi^2\right)}{\eta^2+\xi^2}\right)^2. \end{aligned}$$

The magnetic field remains unchanged.

Integrability for the prolate spheroidal case

$$X_1 = (L^A)^2 - a^2\,(p_x^A)^2 - a^2\,(p_y^A)^2 + \ldots \qquad X_2 = (L_z^A)^2 + \ldots$$

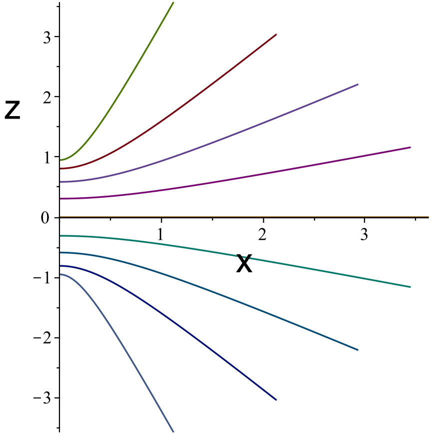
The prolate spheroidal coordinates

The prolate spheroidal coordinates are related to the 3D Cartesian coordinates by the transformation

$$x = a\sinh(\xi)\sin(\eta)\cos(\phi),$$
$$y = a\sinh(\xi)\sin(\eta)\sin(\phi),$$
$$z = a\cosh(\xi)\cos(\eta),$$

metric:

$$g_{11} = a^2(\sinh^2(\xi) + \sin^2(\eta))$$
$$g_{22} = a^2(\sinh^2(\xi) + \sin^2(\eta))$$
$$g_{33} = a^2\sinh^2(\xi)\sin^2(\eta),$$
$$g_{ij} = 0, \qquad i \neq j.$$



Associated integrable physical system

The scalar potential is

$$W = \frac{\alpha(\eta) + \beta(\xi)}{2a^2(\sinh^2(\xi) + \sin^2(\eta))} + \frac{1}{8}\left(\frac{f(\eta) - g(\xi)}{a(\sinh^2(\xi) + \sin^2(\eta))}\right)^2.$$

The potential vector can be chosen as

$$A_\xi = A_\eta = 0, \qquad A_\phi = -\frac{\sin^2(\eta)g(\xi) + \sinh^2(\xi)f(\eta)}{2(\sinh^2(\xi) + \sin^2(\eta))},$$

such that the magnetic field is

$$B_\xi = \partial_\eta A_\phi, \qquad B_\eta = -\partial_\xi A_\phi, \qquad B_\phi = 0.$$

Associated integrals of motion

$$X_1 = \frac{\sinh^2(\xi)(p_\eta^A)^2 - \sin^2(\eta)(p_\xi^A)^2}{\sinh^2(\xi) + \sin^2(\eta)} + \frac{\sinh^2(\xi) - \sin^2(\eta)}{\sinh^2(\xi)\sin^2(\eta)}(p_\phi^A)^2$$
$$+ \left(\frac{g(\xi) - f(\eta)}{\sinh^2(\xi) + \sin^2(\eta)}\right) p_\phi^A + \frac{\sinh^2(\xi)\alpha(\eta) - \sin^2(\eta)\beta(\xi)}{\sinh^2(\xi) + \sin^2(\eta)},$$

$$X_2 = \left(p_\phi^A + \frac{\sinh^2(\xi) f(\eta) + \sin^2(\eta) g(\xi)}{2(\sinh^2(\xi) + \sin^2(\eta))}\right)^2 .$$

Integrability for the oblate spheroidal case

$$X_1 = (L^A)^2 + a^2\,(p_x^A)^2 + a^2\,(p_y^A)^2 + \ldots \qquad X_2 = (L_z^A)^2 + \ldots$$

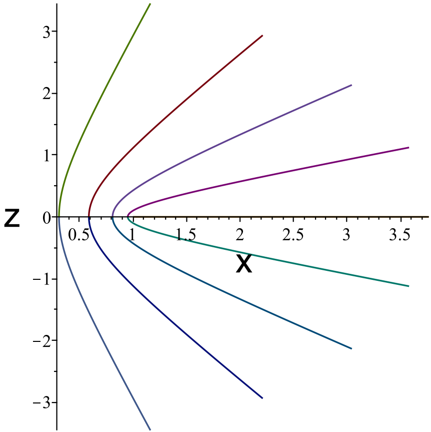
The oblate spheroidal coordinates

The oblate spheroidal coordinates are related to the 3D Cartesian coordinates by the transformation

$$x = a\cosh(\xi)\sin(\eta)\cos(\phi),$$
$$y = a\cosh(\xi)\sin(\eta)\sin(\phi),$$
$$z = a\sinh(\xi)\cos(\eta),$$

metric:

$$g_{11} = a^2(\cosh^2(\xi) - \sin^2(\eta)),$$
$$g_{22} = a^2(\cosh^2(\xi) - \sin^2(\eta)),$$
$$g_{33} = a^2\cosh^2(\xi)\sin^2(\eta),$$
$$g_{ij} = 0, \qquad i \neq j.$$



Associated integrable physical system

The scalar potential is

$$W = \frac{\alpha(\eta) + \beta(\xi)}{2a^2(\cosh^2(\xi) - \sin^2(\eta))} - \frac{1}{8}\left(\frac{f(\eta) - g(\xi)}{a(\cosh^2(\xi) - \sin^2(\eta))}\right)^2.$$

The potential vector can be chosen as

$$A_\xi = A_\eta = 0, \qquad A_\phi = \frac{\sin^2(\eta)g(\xi) - \cosh^2(\xi)f(\eta)}{2(\cosh^2(\xi) - \sin^2(\eta))},$$

such that the magnetic field is

$$B_\xi = \partial_\eta A_\phi, \qquad B_\eta = -\partial_\xi A_\phi, \qquad B_\phi = 0.$$

Associated integrals of motion

$$X_1 = \frac{\sin^2(\eta)(p_\xi^A)^2 + \cosh^2(\xi)(p_\eta^A)^2}{\cosh^2(\xi) - \sin^2(\eta)} + \frac{\cosh^2(\xi) + \sin^2(\eta)}{\cosh^2(\xi)\sin^2(\eta)}(p_\phi^A)^2$$
$$+ \left(\frac{f(\eta) - g(\xi)}{\cosh^2(\xi) - \sin^2(\eta)}\right) p_\phi^A + \frac{\cosh^2(\xi)\alpha(\eta) + \sin^2(\eta)\beta(\xi)}{\cosh^2(\xi) - \sin^2(\eta)},$$

$$X_2 = \left(p_\phi^A + \frac{\cosh^2(\xi) f(\eta) - \sin^2(\eta) g(\xi)}{2(\cosh^2(\xi) - \sin^2(\eta))}\right)^2 .$$

Superintegrability:

Linear integrals

Additional first order integrals of motion

A general first order integral of motion takes the form

$$Y = k_1 p_x^A + k_2 p_y^A + k_3 p_z^A + k_4 L_x^A + k_5 L_y^A + k_6 L_z^A + m_3(x, y, z)$$

and such that

$$\{H, Y\} = 0.$$

The integral Y does not need to be in involution with the other integrals of motion.

For the circular parabolic case, we distinguish the three following systems that are superintegrable with at least one additional linear integral of motion for which the magnetic field does not vanish completely.

Case 1: $p_z^A + m_3(x, y, z)$

$$
\begin{aligned}
H &= \frac{1}{2}\left((p_x^A)^2 + (p_y^A)^2 + (p_z^A)^2\right) + \frac{\omega}{x^2+y^2} - \frac{1}{8} b_z^2 \left(x^2+y^2\right), \\
B &= b_z \mathrm{d}x \wedge \mathrm{d}y, \\
X_1 &= L_x^A p_y^A - L_y^A p_x^A + b_z z L_z^A - \frac{1}{4} b_z^2 z \left(x^2+y^2\right) - \frac{2\omega z}{x^2+y^2}, \\
\tilde{X}_2 &= L_z^A - \frac{1}{2} b_z \left(x^2+y^2\right), \\
Y_3 &= p_z^A.
\end{aligned}
$$

There is free motion along the z-axis.
The magnetic field is constant and oriented along the z-axis.
This system belongs to all previously studied cases.

Case 2: $p_x^A + m_3(x, y, z)$ and $p_y^A + m_4(x, y, z)$

$$
\begin{aligned}
H &= \frac{1}{2}\left((p_x^A)^2 + (p_y^A)^2 + (p_z^A)^2\right) + \frac{b_z^2 z^2}{2}, \\
B &= b_z \mathrm{d}x \wedge \mathrm{d}y, \\
X_1 &= L_x^A p_y^A - L_y^A p_x^A + b_z z L_z^A, \\
\tilde{X}_2 &= L_z^A - \frac{1}{2} b_z \left(x^2 + y^2\right), \\
Y_3 &= p_x^A + b_z y, \\
Y_4 &= p_y^A - b_z x.
\end{aligned}
$$

The magnetic field is constant and oriented along the z-axis. This Hamiltonian is linked to the center of mass of the two-electron quantum dots for special values of its magnetic field and its confinement frequencies.

Case 3: $L_x^A + m_3(x,y,z)$ and $L_y^A + m_4(x,y,z)$

$$
\begin{aligned}
H &= \frac{1}{2}((p_x^A)^2 + (p_y^A)^2 + (p_z^A)^2) + \frac{b_m^2}{2R^2} + \frac{\omega}{2R},\\
B &= \frac{b_m}{R^3}\left(x\mathrm{d}y\wedge\mathrm{d}z + y\mathrm{d}z\wedge\mathrm{d}x + z\mathrm{d}x\wedge\mathrm{d}y\right),\\
X_1 &= L_x^A p_y^A - L_y^A p_x^A - \frac{b_m L_z^A}{R} - \frac{\omega z}{2R},\\
\tilde{X}_2 &= L_z^A + \frac{b_m z}{R},\\
Y_3 &= L_x^A + \frac{b_m x}{R}, \qquad R = \sqrt{x^2+y^2+z^2},\\
Y_4 &= L_y^A + \frac{b_m y}{R}.
\end{aligned}
$$

This system is characterized by the magnetic field of a magnetic monopole together with the (3D) Coulomb potential.

Superintegrability:

The special case:

$$Y = (L^A)^2 + ...$$

Magnetic field

The associated magnetic field is composed of the superposition of three types of magnetic fields, i.e.

$$B = B_z + B_m + B_n.$$

Constant magnetic field:

$$B_z = b_z \mathrm{d}x \wedge \mathrm{d}y.$$

Magnetic monopole:

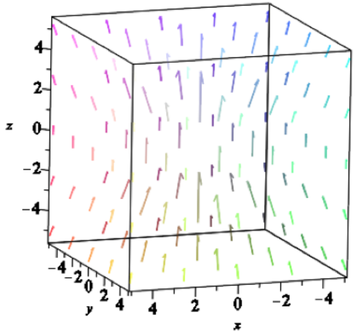
$$B_m = \frac{b_m}{R^3} \left(x \mathrm{d}y \wedge \mathrm{d}z + y \mathrm{d}z \wedge \mathrm{d}x + z \mathrm{d}x \wedge \mathrm{d}y \right).$$

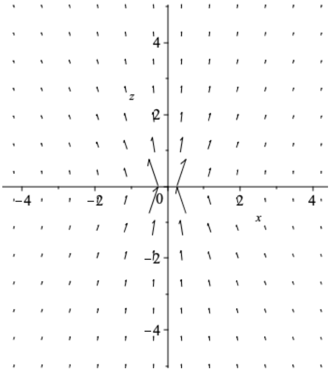
and a magnetic field of the form:

$$B_n = \frac{b_n}{R^3} \left(xz \mathrm{d}y \wedge \mathrm{d}z + yz \mathrm{d}z \wedge \mathrm{d}x + (R^2 + z^2) \mathrm{d}x \wedge \mathrm{d}y \right).$$

"New" magnetic field

$$\left(\frac{xz}{R^3}, \frac{yz}{R^3}, \frac{R^2+z^2}{R^3}\right), \qquad |B| \leq \frac{2|b_n|}{R}$$





(Minimally) superintegrable Hamiltonian

$$
\begin{aligned}
H = \quad & \frac{(p_x^A)^2 + (p_y^A)^2 + (p_z^A)^2}{2} + \frac{u_1}{x^2+y^2} + \frac{u_2}{R} + \frac{u_3 z}{\left(x^2+y^2\right) R} \\
& + \frac{b_m^2}{2R^2} + \frac{b_z b_m z}{2R} - \frac{b_z b_n \left(x^2+y^2\right)}{2R} \\
& + \frac{b_m b_n z}{R^2} - \frac{b_n^2 (x^2+y^2)}{2R^2} - \frac{1}{8} b_z^2 \left(x^2+y^2\right)
\end{aligned}
$$

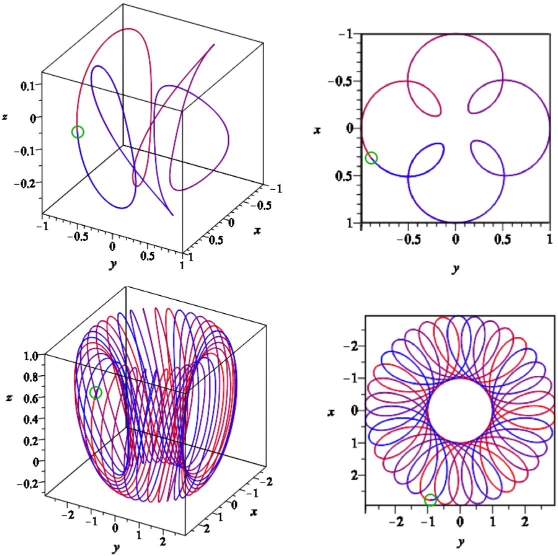
Integrals of motion

$$
\begin{aligned}
X_1 = \; & p_y^A L_x^A - p_x^A L_y^A + \left(\frac{b_m}{R} + \frac{b_n z}{R} + b_z z\right) L_z^A \\
& - \frac{b_m b_z \left(x^2+y^2\right)}{2R} - \frac{b_n b_z z \left(x^2+y^2\right)}{2R} - \frac{b_z^2 z}{4}\left(x^2+y^2\right) \\
& - \frac{2u_1 z}{x^2+y^2} - \frac{u_2 z}{R} - \frac{u_3\left(R^2+z^2\right)}{\left(x^2+y^2\right)R},
\end{aligned}
$$

$$
\tilde{X}_2 = L_z^A + \frac{b_m z}{R} - \frac{b_n\left(x^2+y^2\right)}{R} - \frac{b_z}{2}\left(x^2+y^2\right),
$$

$$
\begin{aligned}
Y_3 = \; & (L^A)^2 - \left(2b_n R + b_z R^2\right) L_z^A + \frac{2u_1 z^2}{x^2+y^2} + \frac{2u_3 z R}{x^2+y^2} \\
& + b_n b_z \left(x^2+y^2\right) R + b_n^2\left(x^2+y^2\right) + \frac{1}{4} b_z^2 \left(x^2+y^2\right) R^2.
\end{aligned}
$$

The algebra of the integrals of motion closes polynomially and there exists no additional first or second order integral.



z
x
y
0.1
0
−0.1
−0.2
−1
−0.5
0.5
1
1.0
0.8
0.6
0.4
0.2
−0.2
−2
2

Conclusions

- We have investigated integrability for the three cases: Circular parabolic, Prolate and Oblate spheroidal.
- In all three cases, the quadratic integral $L_z^2 + ...$ degenerates to a first order integral of motion and the magnetic field is not constrained by the lower order determining equations
- For the circular parabolic case, all additional first order integrals of motion have been found. All these systems already appeared in the literature.
- A particular additional second order integral of motion ($L^2 + ...$) leads to an interesting new superintegrable system.
- Among these results, no purely quantum system exists.

Future perspectives

It would be interesting to:

- establish or disprove the equivalence of integrability and the separability of the Hamilton–Jacobi and the Schrödinger equations admitting magnetic forces.
- study the remaining integrable cases of non-subgroup type. (Work in progress.)
- investigate the superintegrability of the prolate and oblate spheroidal cases.
- develop more efficient techniques to deal with higher order integrals for superintegrability.
- extend such results to a relativistic approach for additional physical applications.

Thank you

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S Bertrand and L Šnobl (2019)
J. Phys. A: Math. Theor. **52** 195201 (25pp).
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