On a problem of David A. Singer about prescribing curvature for curves

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Fundamental Theorem for plane curves

THEOREM

Prescribe $\kappa = \kappa(s)$ (continuous):
$\theta(s) = \int \kappa(s)\, ds, \quad x(s) = \int \cos\theta(s)\, ds, \quad y(s) = \int \sin\theta(s)\, ds$
$\Rightarrow (x(s), y(s))$ unique up to rigid motions

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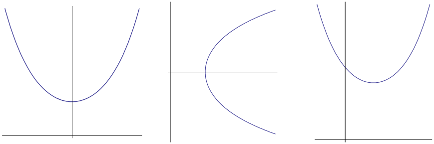
Example (Catenary)

$$\kappa(s) = \frac{1}{1+s^2} \Rightarrow \theta(s) = \arctan s$$

$$x(s) = \log\left(s + \sqrt{s^2+1}\right),\ y(s) = \sqrt{1+s^2} \leftrightarrow y = \cosh x,\ x \in \mathbb{R}$$



Singer's Problem

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Can a plane curve be determined if
its curvature is given in terms of its position?

$$\kappa = \kappa(x, y), \qquad \frac{x'(t)y''(t) - y'(t)x''(t)}{\left(x'(t)^2 + y'(t)^2\right)^{3/2}} = \kappa(x(t), y(t))$$

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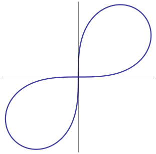
Bernoulli lemniscate: $r^2 = 3\sin 2\theta$



Euler's elastic curves

Elastica under *tension* $\sigma \in \mathbb{R}$:
Critical points of $\int (\kappa^2 + \sigma) ds$: $2\ddot{\kappa} + \kappa^3 - \sigma\kappa = 0$

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$$\int \kappa(y)dy = \lambda y^2 + c$$

$$\text{Tension } \sigma = -4\lambda c$$

$$\text{Maximum curvature } k_0 = 2\sqrt{\lambda}\sqrt{1-c},\ c < 1$$

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- $c > -1$, *wavelike*:

$\kappa(s) = k_0 \operatorname{cn}\left(\frac{k_0 s}{2p}, p\right)$,
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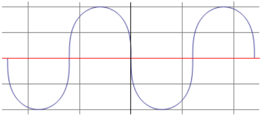

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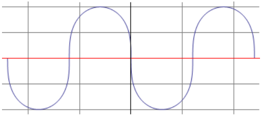

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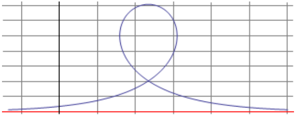

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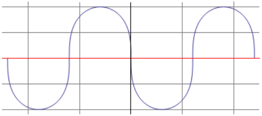

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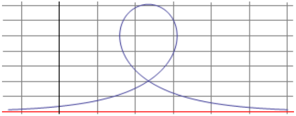

- $c < -1$, *orbitlike*:

  $\kappa(s) = k_0 \ \mathrm{dn}\left(\frac{k_0 s}{2}, p\right)$,

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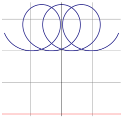

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Theorem $\kappa = \kappa(r)$

Prescribe $\kappa = \kappa(r)$ such that $r\kappa(r)$ continuous.
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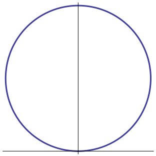
Example (Circles)

$\kappa \equiv k_0 > 0, \quad \mathcal{K}(r) = k_0 r^2/2 + c$

$s = \int \frac{r\,dr}{\sqrt{r^2 - (k_0 r^2/2 + c)^2}} \overset{(c=0)}{=} (2/k_0) \arcsin(k_0 r/2)$

$r(s) = (2/k_0)\sin(k_0 s/2), \ \theta(s) = k_0 s/2$



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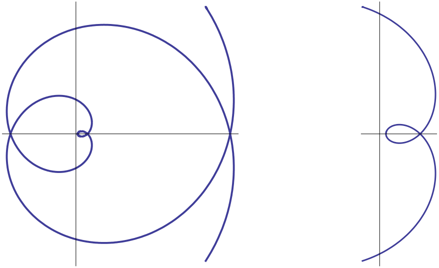
Sturm or Norwich spiral



Plane curves such that $\boxed{\kappa(r) = \lambda\, r^{n-1}\ (\lambda > 0, n \neq -1, 0)}$

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$$d\theta = \frac{\frac{\lambda}{n+1}\, r^{n-1}}{\sqrt{1 - \left(\frac{\lambda}{n+1}\right)^2 r^{2n}}} dr$$

Sinusoidal spirals $r^n = \frac{n+1}{\lambda} \sin\left(n\theta\right)$

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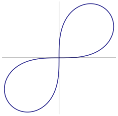

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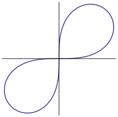
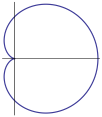


- $n = 1/2$: *Cardioid* $r = \frac{9}{4\lambda^2} \sin^2 \frac{\theta}{2}$



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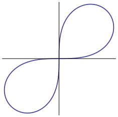

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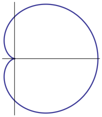


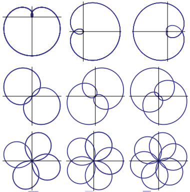
- $n \in \mathbb{Q}$:

Algebraic curves

$n = 1/3, 1/4, 1/6$

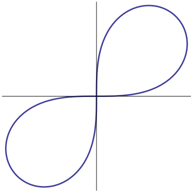
$n = 2/3, 2/5, 2/7$

$n = 4/3, 5/4, 6/5$

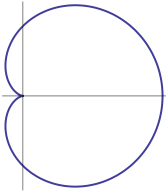


Uniqueness results for plane curves

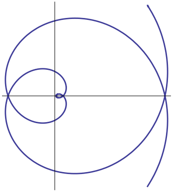
The **Bernoulli lemniscate** $r^2 = 3\sin 2\theta$
is the only plane curve (up to rotations)
with geometric angular momentum $\mathcal{K}(r) = r^3/3$
(and curvature $\boxed{\kappa(r) = r}$).



The **cardioid** $r = \frac{1}{2}(1 + \cos\theta)$
is the only plane curve (up to rotations)
with geometric angular momentum $\mathcal{K}(r) = r\sqrt{r}$
(and curvature $\boxed{\kappa(r) = \frac{3}{2\sqrt{r}}}$).



The **Norwich spiral** is the only (non circular)
plane curve (up to rotations) with curvature
$\boxed{\kappa(r) = 1/r}$.



Plane curves such that $\boxed{\kappa(r) = \frac{\lambda}{r^3} + 3\mu r \ (\lambda \in \mathbb{R}, \mu > 0)}$

$\mathcal{K}(r) = \mu r^3 - \lambda / r$

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$$d\theta = \frac{\mu r^4 - \lambda}{r\sqrt{r^4 - (\mu r^4 - \lambda)^2}} dr$$

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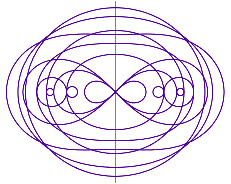
Cassini ovals $r^4 - 2a^2 r^2 \cos 2\theta + a^4 = b^4$



$a \in \{1, 2, 3\}, b \in \{1, 2, 3, 4\}$

Plane curves such that $\boxed{\kappa(r) = 2\lambda + \mu/r \ (\lambda = 1, \mu \neq 0)}$

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- $\underline{\mu \in (-1, 1)}$: $\mu = \cos\gamma$, $0 < \gamma < \pi$
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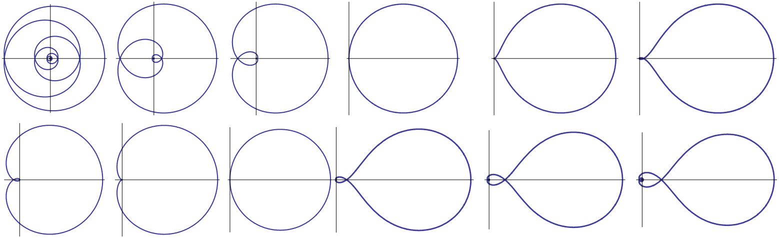
$$r_\gamma(s) = \cos s - c_\gamma, \ \theta_\gamma(s) = s + \frac{2}{t_\gamma} \operatorname{arctanh}\left(\frac{s_\gamma}{1 - c_\gamma} \tan\frac{s}{2}\right)$$



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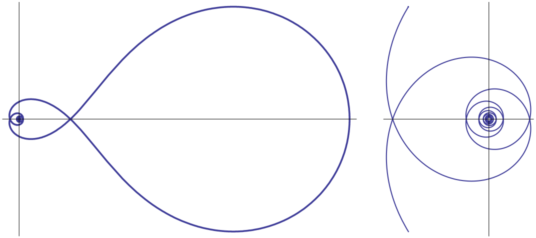
- $\underline{\mu = -1:}$

Inverse Norwich spiral

$r(s) = \cos s + 1, \ \theta(s) = s - \tan \frac{s}{2}$



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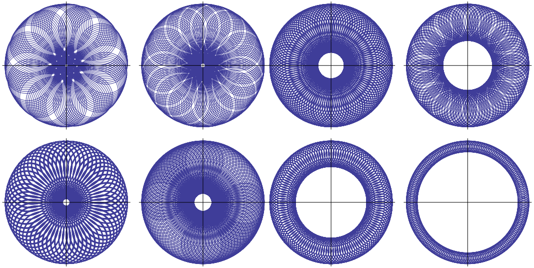

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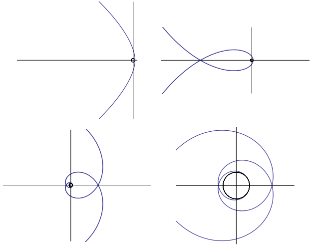

Plane curves such that $\boxed{\kappa(r) = \lambda/\sqrt{r^2-1}\ (\lambda > 0)}$
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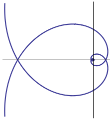

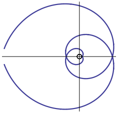

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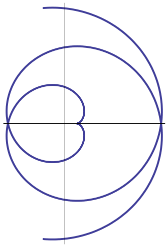

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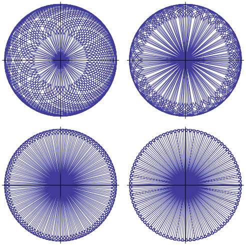

Index

Spherical version of Singer's Problem

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Can a spherical curve be determined
if its (geodesic) curvature is given in terms of its position?

$$\begin{vmatrix} x(s) & y(s) & z(s) \\ \dot{x}(s) & \dot{y}(s) & \dot{z}(s) \\ \ddot{x}(s) & \ddot{y}(s) & \ddot{z}(s) \end{vmatrix} = \kappa(x(s), y(s), z(s))$$

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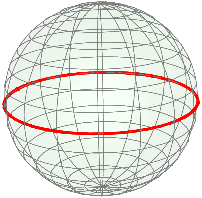
[I. Castro, I. Castro-Infantes and J. Castro-Infantes. *Spherical curves whose curvature depends on distance to a great circle*. Preprint.]

$\kappa(x, y, z) = \kappa(z)$, $z = \sin\varphi$, φ latitude



Spherical curves whose curvature depends on distance to a geodesic (or from a point)

Spherical curves whose curvature depends on distance to a geodesic (or from a point)

Theorem

Prescribe $\kappa = \kappa(z)$ *continuous.*
The problem of determining a spherical curve
$\xi(s) = (x(s), y(s), z(s))$ *-s arc parameter- whose curvature is* $\kappa(z)$*,*
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- ξ *is uniquely determined (up to rotations around the z-axis) by* $\mathcal{K}(z)$

Examples

Example (Great circles)

$\kappa \equiv 0$: $\int \kappa(z)dz = c$,

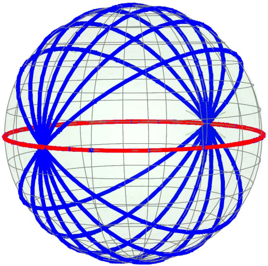
$s = \int \frac{dz}{\sqrt{1-c^2-z^2}} = \arcsin \frac{z}{\sqrt{1-c^2}}$, $|c| < 1$,

$z(s) = \sqrt{1-c^2} \sin s$,

$\lambda(s) = -\arctan(c \tan s)$,

$\xi(s) = (\cos s, -c \sin s, \sqrt{1-c^2} \sin s)$,

$\mathbb{S}^2 \cap \{\sqrt{1-c^2}\, y + c\, z = 0\}$, $\mathcal{K} \equiv c$



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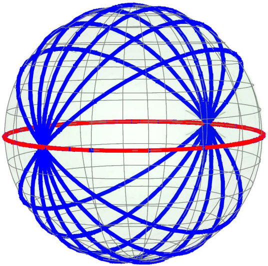


Example (Small circles)

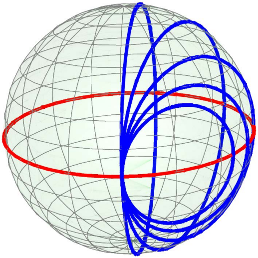
$\kappa \equiv k_0 \geq 0:$ $\int \kappa(z)\,dz = k_0 z + c$

$z(s) =$

$\frac{1}{1+k_0^2}\left(\sqrt{1-c^2+k_0^2}\,\sin(\sqrt{1+k_0^2}\,s) - c\,k_0\right)$,

$|c| < \sqrt{1+k_0^2}$.

$c = 0$: $\mathbb{S}^2 \cap \{y = \frac{k_0}{\sqrt{1+k_0^2}}\}$, $\mathcal{K}(z) = k_0 z$



Spherical elasticae: characterization and generalization

Elasticae under tension $\sigma \in \mathbb{R}$:
critical points of $\mathcal{F}_\sigma(\xi) := \int_\xi (\kappa^2 + \sigma) ds$ ($\sigma = 0$ *free* elasticae)

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$\Rightarrow$ *ξ critical points of* $\int_\xi (\kappa^2 - 2b\kappa + b^2 - 4ac) ds$

- $b = 0$ ($\mathcal{K}(z) = az^2 + c$): *$\xi$ elastica under tension* $\sigma = -4ac$

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Theorem

(i) *ξ spherical curve,* $\boxed{\kappa(z) = 2az + b}$, $a \neq 0$, $b \in \mathbb{R}$

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$\Rightarrow$ *ξ critical points of* $\int_\xi (\kappa^2 - 2b\kappa + b^2 - 4ac) ds$

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Spherical elasticae: characterization and generalization

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(ii) *Conversely, ξ critical point of*
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$\Rightarrow \exists a \neq 0, b \in \mathbb{R}$: $\kappa(z) = 2az + b$

Spherical *borderline* elastic curves.

- $a > 1/2$, $b = 0$, $c = 1$: $\boxed{\kappa(z) = 2az}$, $a > 0$, $\mathcal{K}(z) = az^2 + 1$

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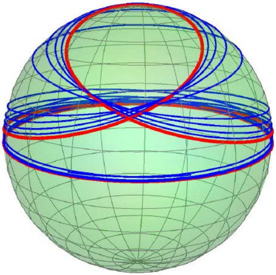

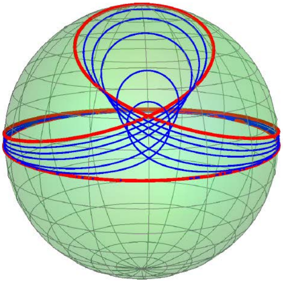

Seiffert's spherical elastic *spirals*

$r = \mathrm{sn}(s,k),\ \theta = k\,s,\ z = \mathrm{cn}(s,k),\ (k>0)$ [Erdös, 2000]

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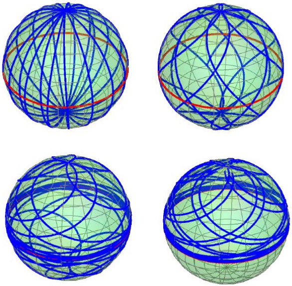


New spherical curves I:

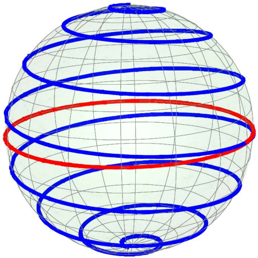
$$\boxed{\kappa(z) = \frac{z}{\sqrt{a-z^2}}}, \quad 0<a=\sin^2\alpha<1 \ (0<\alpha<\pi/2)$$

$$\mathcal{K}(z) = -\sqrt{\sin^2\alpha - z^2}$$

$$\varphi(s) = \arcsin(\cos\alpha\, s)$$

$$\kappa(s) = \frac{c_\alpha s}{\sqrt{s_\alpha^2 - c_\alpha^2 s^2}}, \quad |s| < \tan\alpha$$

$$\lambda(s) = \frac{1}{c_\alpha}\arctan\left(\frac{c_\alpha s}{\sqrt{s_\alpha^2-c_\alpha^2 s^2}}\right) - \frac{1}{2}\arctan\left(\frac{c_\alpha s + s_\alpha^2}{c_\alpha\sqrt{s_\alpha^2-c_\alpha^2 s^2}}\right) - \frac{1}{2}\arctan\left(\frac{c_\alpha s - s_\alpha^2}{c_\alpha\sqrt{s_\alpha^2-c_\alpha^2 s^2}}\right)$$



New spherical curves II:

$$\boxed{\kappa(z) = \frac{az}{\sqrt{1-az^2}}}, \quad a=\cosh^2\delta > 1, \ (\delta > 0)$$

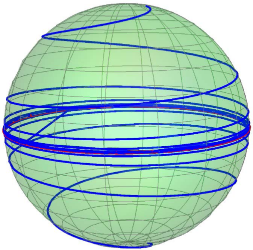
$$\mathcal{K}(z) = -\sqrt{1-\cosh^2\delta z^2}$$

$$\varphi(s) = \arcsin(e^{\sinh\delta\, s})$$

$$\kappa(s) = \frac{\cosh^2\delta\, e^{\sinh\delta\, s}}{\sqrt{1-\cosh^2\delta\, e^{2\sinh\delta\, s}}}, \quad s < -\log\cosh\delta/\sinh\delta$$

$$\lambda(s) =$$

$$-\frac{1}{\sinh\delta}\operatorname{arctanh}\left(\sqrt{1-\cosh^2\delta\, e^{2\sinh\delta\, s}}\right) + \arctan\left(\frac{\sqrt{1-\cosh^2\delta\, e^{2\sinh\delta\, s}}}{\sinh\delta}\right)$$



New spherical curves III:

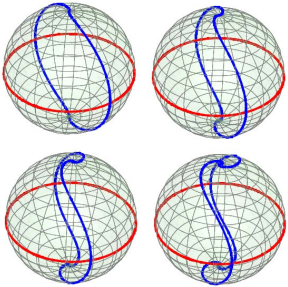
$$\boxed{\kappa(z) = \frac{p(1-2z^2)}{\sqrt{1-z^2}} = \frac{p\cos 2\varphi}{\cos\varphi} = \kappa(\varphi)}, \quad 0 < p < 1$$

$$\mathcal{K}(z) = p\,z\sqrt{1-z^2} = \tfrac{p}{2}\sin 2\varphi = \mathcal{K}(\varphi)$$

$$\varphi(s) = \mathrm{am}(s,p)$$

$$\kappa(s) = p(2\,\mathrm{cn}(s,p) - 1/\,\mathrm{cn}(s,p))$$

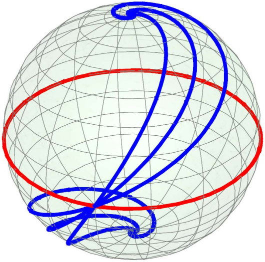
$$\lambda(s) = -\tfrac{p}{2p'}\log\left(\tfrac{\mathrm{dn}(s,p)+p'}{\mathrm{dn}(s,p)-p'}\right), \quad p' = \sqrt{1-p^2}$$



Uniqueness results on classical spherical curves

Uniqueness results on classical spherical curves

Loxodromes



Uniqueness results on classical spherical curves

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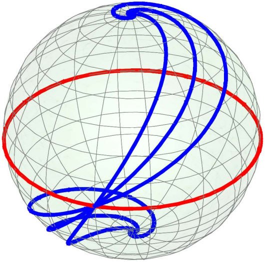

The loxodromes, $d\lambda = \cot\alpha \frac{d\varphi}{\cos\varphi}$, $\alpha \in (0, \pi/2)$, are the only spherical curves (up to rotations around z-axis) with spherical angular momentum
$\mathcal{K}(\varphi) = -\cos\alpha\cos\varphi$
(and curvature $\boxed{\kappa(\varphi) = \cos\alpha\tan\varphi}$).

$$\kappa(s) = \cos\alpha\ \tan(\sin\alpha\, s),\ \alpha \in (0, \pi/2)$$

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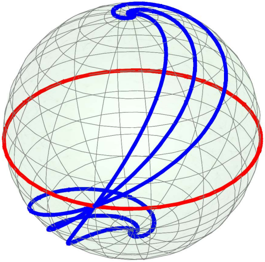

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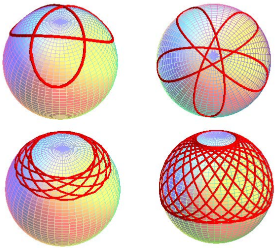
Spherical catenaries



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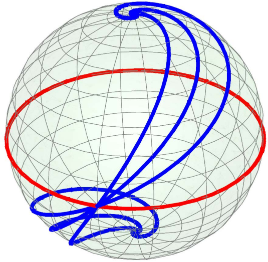

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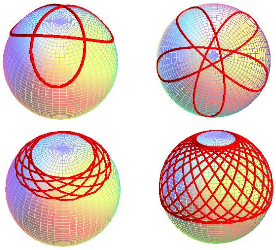

The spherical catenaries, $\sin\varphi\cos^2\varphi\,\frac{d\lambda}{ds} = a$,
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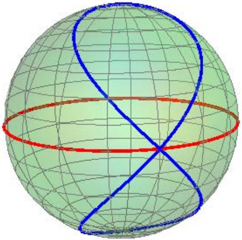
$$\kappa(s) = \frac{2a}{1+\sqrt{1-4a^2}\sin 2s}$$

Uniqueness results on classical spherical curves

Viviani's curve



Uniqueness results on classical spherical curves

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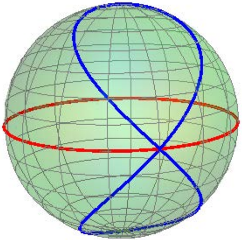

Viviani's curve, $\lambda = \varphi$, is the only spherical curve (up to rotations around z-axis)
with spherical angular momentum $\mathcal{K}(z) = \frac{z^2-1}{\sqrt{2-z^2}}$

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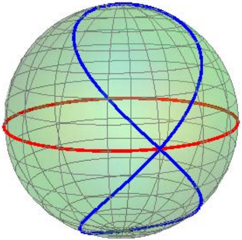
Uniqueness results on classical spherical curves

Viviani's curve



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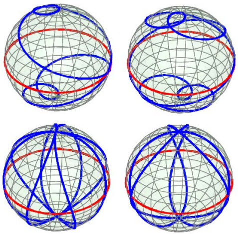
Archimedean spherical spirals



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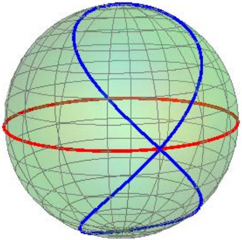

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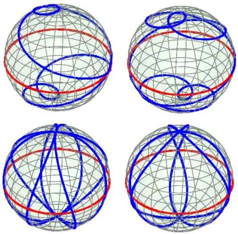

Archimedean spherical spirals, $\varphi = n\lambda$, $n > 0$, are the only spherical curves (up to rotations around z-axis) with spherical angular momentum

$\mathcal{K}(z) = \frac{z^2-1}{\sqrt{1+n^2-z^2}}$

(and curvature $\boxed{\kappa(z) = \frac{z(2n^2+1-z^2)}{(n^2+1-z^2)^{3/2}}}$).

On a problem of David A. Singer about prescribing curvature for curves

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(Ildefonso Castro-Infantes and Jesús Castro-Infantes)
Universidad de Jaén
Universidad de Granada

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