Variational Principles for Supinf problems with constraints

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$$(P) \qquad \sup_{x\in X} \inf_{y\in Kx} f(x,y),$$

$f : X \times Y \to [-\infty, \infty]$
X, Y - completely regular topological spaces
$K : X \rightrightarrows Y$ - a set-valued mapping with non-empty images

A solution to the problem $(\mathrm{P}\ )$ is every couple $(x_0, y_0) \in X \times Y$, such that:

$$f(x_0, y_0) = \inf_{y\in Kx_0} f(x_0, y) = \sup_{x\in X} \inf_{y\in Kx} f(x,y).$$

Find the conditions under which after a perturabation of the function f by a continuous and bounded function:
1. the problem (P) has a solution;
2. the problem (P) is Tykhonov well-posed?

Existing results for functions of one variable

Ekeland's variational principle

Ekeland I. (1974) On the variational principle, J. Math. Anal. Appl., 47, 324вЂ“353.

Stegall's variational principle

Stegall, C., Optimization of functions on certain subsets of Banach spaces, (1978) *Math. Ann.*, **236**, 171–176.

Borwein-Preiss and Deville-Godefroy-Zizler smooth variational principles

Borwein, J.M., D. Preiss, (1987) A smooth variational principle with applications to subdifferentiability and differentiability of convex functions, *Trans. Amer. Math. Soc.* **303**, 517–527.
Deville, R., G. Godefroy, V. Zizler, (1993) A smooth variational principle with applications to Hamilton–Jacobi equations in infinite dimensions, *J. Funct. Anal.*, **111**, 197–212.

Coban-Kenderov-Revalski smooth variational principles

Čoban, M.M., P.S. Kenderov, J.P. Revalski, (1989) Generic well-posedness of optimization problems in topological spaces, *Mathematika*, **36**, 301–324.

Kenderov. P. and J.P. Revalski, Dense existence of solutions of perturbed optimization problems and topological games, (2010) *Compt. Rend. Acad. Bulg. Sci*, **63**, 937–942.

McLinden

McLinden,L., An application of Ekeland's theorem to minmax problems, (1982) *Nonlinear Analysis, Theory, Methods & Applications*, Vol. 6, No. 2., 189–196.

Kenderov, Revalski

Kenderov P. and J. P. Revalski, Variational principles for supinf problems, (2017) *Compt. rend. Acad. bul Sci*, **70**(12).
$Kx = Y, \forall x \in X$

Definition (Tykhonov well-posed problem)

The problem to minimize $h : Z \to \mathbb{R} \cup \{+\infty\}$, Z - a topological space, is called Tykhoniv well-posed if there is a unique solution $z_0 \in Z$ and every minimizing sequence $z_n \to z_0$.

The problem to maximize $h : Z \to \mathbb{R} \cup \{-\infty\}$ is called Tykhoniv well-posed if $-h$ the problem to minimize $-h : Z \to \mathbb{R} \cup \{+\infty\}$ is Tykhonov well-posed.

Definition (Sup-well-posed)

The problem (P) is called *sup-well-posed* if the problem to maximize the function $v(\cdot) := \inf_{y \in K(\cdot)} f(\cdot, y)$ is well-posed in the sense of Tykhonov (i.e. every maximizing sequence $(x_n)_n$ $(v(x_n) \to \sup_{x \in X} v(x))$ converges to the unique maximizer of v.

Definition (an optimizing sequence for the problem (P))

The sequence $(x_n, y_n) \in X \times Y$ is called optimizing for (P) if:

1. $y_n \in Kx_n$ for every n;
2. $v(x_n) \to v_f := \sup_{x \in X} \inf_{y \in Kx} f(x, y)$;
3. $f(x_n, y_n) \to v_f$.

(P) is well-posed if every optimizing sequence for (P) converges to the unique solution (x_0, y_0) of (P).

Leader-follower games

$v_f = \sup_{x \in X} \inf_{y \in Kx} f(x, y)$ - expresses the guaranteed utility for the first player.

Stackelberg problem

for every $x \in X$:

$$Kx := \{y' : g(x, y') = \inf_{y \in Y} g(x, y)\},$$

$g : X \times Y \to \mathbb{R}$ - a given function;

Lemma 1

Let $h : Z \to \mathbb{R} \cup \{+\infty\}$ be a proper, lower semicontinuous and bounded from below function. Let $z_0 \in dom(h)$ and $\varepsilon > 0$ be such that $h(z_0) < \inf_Z h + \varepsilon$. Then, there exists a continuous and bounded function $g : Z \to \mathbb{R}_+$, $g(z_0) = 0$, $||g||_{Z,\infty} \leq \varepsilon$ and the function $h + g$ attains its minimum in Z at z_0. Moreover, g can be chosen such that $||g||_{Z,\infty} = h(z_0) - \inf_Z h$.

Kenderov P. and J. P. Revalski, 2010

Lemma 2

Let $h : Z \to \mathbb{R} \cup \{+\infty\}$ be a proper, lower semicontinuous and bounded from below function. A is a closed subset of Z and $A \cap dom(h) \neq \emptyset$. Let $z_0 \in A \cap dom(h)$ and $\varepsilon > 0$ be such that $h(z_0) < \inf_A h + \varepsilon$. Then, there exists a continuous and bounded function $g : Z \to \mathbb{R}_+$, $g(z_0) = 0$, $||g||_{Z,\infty} \leq \varepsilon$ and the function $h + g$ attains its minimum in A at z_0. Moreover, g can be chosen such that $||g||_{Z,\infty} = h(z_0) - \inf_A h$.

Lemma 3

Let $f : X \times Y \to [-\infty, +\infty]$ be an upper semicontinuous function. $K : X \rightrightarrows Y$ is a lower semicontinuous set-valued mapping with non-empty images, then $v(\cdot) = \inf\{f(\cdot, y), y \in K(\cdot)\}$ is upper semicontinuous in X.

Assumptions

1. f is upper semicontinuous in $(x, y) \in X \times Y$;
2. $K : X \rightrightarrows Y$ is a lower semicontinuous set-valued mapping with non-empty closed images;
3. the function $v(\cdot) := \inf_{y \in K \cdot} f(\cdot, y)$ is bounded from above in X and proper as a function with values in $\mathbb{R} \cup \{-\infty\}$;
4. for every $x \in X$ the function $f(x, \cdot)$ is lower semicontinuous in Y.

Variational principles for supinf problems with constraints

Theorem 1

Let $f : X \times Y \to [-\infty, +\infty]$ be a real-valued function, that satisfies (1), (2), (3) and (4) together with the set-valed mapping K. Let $\varepsilon > 0$ and $x_0 \in X$ be such that $v(x_0) > sup_{x \in X} v(x) - \varepsilon$ and let $\delta > 0$ and $y_0 \in Kx_0$ be such that $f(x_0, y_0) < \inf_{y \in Kx_0} f(x_0, y) + \delta$. Then, there exist continuous bounded functions $q : X \to \mathbb{R}_+$ and $p : Y \to \mathbb{R}_+$, such that $q(x_0) = p(y_0) = 0$, $||q||_{X,\infty} \leq \varepsilon$, $||p||_{Y,\infty} \leq \delta$ and the supinf problem $\sup_{x \in X} \inf_{y \in Kx} \{f(x, y) - q(x) + p(y)\}$ has a solution at (x_0, y_0).

Apply Lemma 1 for $-v(\cdot)$, point x_0 and $\varepsilon \Rightarrow \exists\ q \in C(X)$:
$q(x_0) = 0$, $||q||_{X,\infty} \leq \varepsilon$ and $v(\cdot) - q(\cdot)$ attains its maximum in X at x_0.

Apply Lemma 2 for $f(x_0, \cdot)$, point y_0 and $\delta \Rightarrow \exists\ p \in C(Y)$:
$p(y_0) = 0$, $||p||_{Y,\infty} \leq \delta$ and $f(x_0, \cdot) + p(\cdot)$ attains its minimum on Kx_0 at y_0 and $||p||_{Y,\infty} = f(x_0, y_0) - \inf_{y \in Kx_0} f(x_0, y) := c \geq 0$.

Take an arbitrary x:

$$\begin{aligned}
&\inf_{y\in Kx}\{f(x,y)-q(x)+p(y)\}-c=\\
&\inf_{y\in Kx}\{f(x,y)+p(y)-c\}-q(x)\\
&\leq \inf_{y\in Kx} f(x,y)-q(x)=v(x)-q(x)\\
&\leq v(x_0)-q(x_0)=v(x_0).
\end{aligned}$$

$$\begin{aligned}
&\inf_{y\in Kx}\{f(x,y)-q(x)+p(y)\}\leq v(x_0)+c=\\
&\inf_{y\in Kx_0} f(x_0,y)+c=f(x_0,y_0)\\
&=f(x_0,y_0)+p(y_0)=\inf_{y\in Kx_0}\{f(x_0,y)+p(y)\}\\
&=\inf_{y\in Kx_0}\{f(x_0,y)-q(x_0)+p(y)\}.
\end{aligned}$$

Theorem 2

Let $f : X \times Y \to [-\infty, +\infty]$ be a real-valued function which satisfies (1), (2), (3) and (4) together with the set-valed mapping K, then:

1. The set $\{(q, p) \in C(X) \times C(Y)$: the supinf problem has a solution for the function $f(x, y) + q(x) + p(y), (x, y) \in X \times Y$ $\}$ is a dense subset of $C(X) \times C(Y)$;
2. The set $\{u \in C(X \times Y)$: the supinf problem has a solution for the function $f(x, y) + u(x, y), (x, y) \in X \times Y$ $\}$ is a dense subset of $C(X \times Y), || \cdot ||_{X \times Y, \infty}$;

Proposition 1

Let the assumptions on $f : X \times Y \to [-\infty, +\infty]$, $K : X \rightrightarrows Y$, x_0 Pë y_0 from Theorem 1 hold and x_0 has a countable local base in X. Then, there exist continuous and bounded functions $q : X \to \mathbb{R}_+$ and $p : Y \to \mathbb{R}_+$, such that $q(x_0) = p(y_0) = 0$, $||q||_{X,\infty} \leq \varepsilon$, $||p||_{Y,\infty} \leq \delta$ and the supinf problem $\sup_{x \in X} \inf_{y \in Kx}\{f(x, y) - q(x) + p(y)\}$ has a solution at (x_0, y_0) and the problem is sup-well-posed with unique sup-solution x_0.

If $Kx = Y$ for every $x \in X$ the above three results were proved by Kenderov and Revalski, (2017).

$S_f : C(X) \times C(Y) \rightrightarrows X \times Y$ - a set-valued mapping which assigns to every functions $q \in C(X)$ and $p \in C(Y)$ the solution set (possibly empty) of the problem
$\sup_{x \in X} \inf_{y \in Kx} \{f(x, y) + q(x) + p(y)\}$.

Theorem 3

Let the assumptions 1)-4) for $f : X \times Y \to [-\infty, +\infty]$ and $K : X \rightrightarrows Y$ hold. Then, the mapping S_f is single-valued and upper semicontinuous at $(q, p) \in C(X) \times C(Y)$, if and only if the supinf problem $\sup_{x \in X} \inf_{y \in Kx} \{f(x, y) + q(x) + p(y)\}$ is well-posed.

S_f is single-valued and upper semicontinuous at $(q, p) \in C(X) \times C(Y)$, $(x_0, y_0) = S_f(q, p)$.
$(x_n, y_n)_n$ - an optimizing sequence for the supinf problem $\sup_{x \in X} \inf_{y \in Kx}\{f(x, y) + q(x) + p(y)\}$:

1. $y_n \in Kx_n$ for every n;
2. $v(x_n) = \inf_{y \in Kx_n}\{f(x_n, y) + q(x_n) + p(y)\} \to v_f = \sup_{x \in X} \inf_{y \in Kx}\{f(x, y) + q(x) + p(y))\}$;
3. $f(x_n, y_n) \to v_f$.

Suppose $(x_n, y_n)_n \nrightarrow (x_0, y_0) \Rightarrow \exists$ open neighbouhoods $U \ni x_0$, $V \ni y_0$ and a subsequence $(x_n, y_n) \notin U \times V$, $\forall n$.

S_f - an upper semicontinuous set-valued mapping $\Rightarrow \exists \varepsilon > 0$: from $||q' - q||_{X,\infty} < \varepsilon$, $||p' - p||_{Y,\infty} < \varepsilon$, $q' \in C(X)$, $p' \in C(Y)$, it follows $S_f(q', p') \subset U \times V$.

Let n be so large that:
$v_f - v(x_n) < \varepsilon/2$ and $|v_f - f(x_n, y_n) - q(x_n) - p(y_n)| < \varepsilon/2$
$\Rightarrow$
$f(x_n, y_n) + q(x_n) + p(y_n) < v(x_n) + \varepsilon =$
$\inf_{y \in Kx_n}\{f(x_n, y) + q(x_n) + p(y)\} + \varepsilon.$

Apply Theorem 1 for the function
$f(x,y) + q(x) + p(y), (x,y) \in X \times Y$, the point x_n, $\varepsilon/2$, $y_n \in Kx_n$ and ε:
$\exists\ q_n \in C(X)$, $p_n \in C(Y)$: $||q_n||_{X,\infty} \leq \varepsilon/2$, $||p_n||_{Y,\infty} \leq \varepsilon$ and the problem $\sup_{x\in X} \inf_{y \in Kx}\{f(x,y) + q(x) - q_n(x) + p(y) + p_n(y)\}$ has a solution at (x_n, y_n).
$||q - q_n - q||_{X,\infty} \leq \varepsilon/2$, $||p + p_n - p||_{Y,\infty} \leq \varepsilon$ and $(x_n, y_n) \in S_f(q - q_n, p + p_n) \subset U \times V$, a contradiction with the assumption that $(x_n, y_n) \notin U \times V$.

The supinf problem for the function $f(x, y) + q(x) + p(y)$ and the mapping K is well-posed with a solution at $(x_0, y_0) \Rightarrow$ $S_f(q, p) = \{(x_0, y_0)\}$.

Suppose S_f is not uppersemicontinuous at $(q, p) \Rightarrow \exists$ open neighbourhoods $U \ni x_0$, $V \ni y_0$: for every n, there exist q_n and p_n, $||q_n - q||_{X,\infty} < 1/n$, $||p_n - p||_{Y,\infty} < 1/n$, $q_n \in C(X)$, $p_n \in C(Y)$, $S_f(q_n, p_n)$ is not contained in $U \times V$, i.e. for every n $\exists$ $(x_n, y_n) \in S_f(q_n, p_n) \setminus (U \times V)$.

$f(x_n, y_n) + q_n(x_n) + p_n(y_n) = \inf_{y \in Kx_n}\{f(x_n, y) + q_n(x_n) + p_n(y)\}$
$= \sup_{x \in X} \inf_{y \in Kx}\{f(x, y) + q_n(x) + p_n(y)\}.$

Denote:
$\alpha_n = f(x_n, y_n) + q_n(x_n) + p_n(y_n),$
$v_n(x_n) = \inf_{y \in Kx_n}\{f(x_n, y) + q_n(x_n) + p_n(y)\},$
$v_n = \sup_{x \in X} \inf_{y \in Kx}\{f(x, y) + q_n(x) + p_n(y)\}.$

$v_n \to v_f = \sup_{x \in X} \inf_{y \in Kx}\{f(x, y) + q(x) + p(y)\};$
$|v_n(x_n) - v(x_n)| < \varepsilon$, for n large enough;
$|\alpha_n - f(x_n, y_n) - q(x_n) - p(y_n)| < \varepsilon$, for n large enough;

(x_n, y_n) - an optimizing sequence for the problem $\sup_{x \in X} \inf_{y \in Kx}\{f(x, y) + q(x) + p(y)\}$, a contradiction with the well-posedness of the problem.

Theorem 4

Let the assumptions 1)-4) for $f : X \times Y \to [-\infty, +\infty]$ and $K : X \rightrightarrows Y$ hold. Then, the mapping $\tilde{S}_f : (C(X \times Y), || \cdot ||_\infty) :\rightrightarrows X \times Y$ which assigns to every $u \in C(X \times Y)$ the solution set of the problem $\sup_{x \in X} \inf_{y \in Kx} \{f(x, y) + u(x, y)\}$ is single-valued and upper semicontinuous at $(u \in C(X \times Y)$, if and only if the supinf problem $\sup_{x \in X} \inf_{y \in Kx} \{f(x, y) + u(x, y)\}$ is well-posed.

Than you for your attention!

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