Symmetries and Conservation Laws of Timoshenko-Type beam Equations

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XXIst International Conference on Geometry, Integrability and Quantization June 3-8, 2019, Sts. Constantine and Elena, Varna, Bulgaria

Abstract

• In this paper, we consider Timoshenko-Type beam equations. In the particular case, when we choose the coeficients to be a positive constants, we yield a system describing the dynamical behaviour of double-wall carbon nanotubes.

• We formulate the problem in terms of calculus of variations, finding the lagrangian density.

• The main focus of the present paper is on investigating both equations and functional local point Lie group of symmetries. Also, we expand the group of symmetries of the functional, adding the so-called divergence symmetries. Knowing, these symmetries, we encounter the corresponding conservation laws, by showing explicitly the densities and the fluxes.

Overview

1. Description of the model
2. Equations and variational statement
3. Local Lie group of point transformations, admitted by the problem
4. Conservation laws
5. Future intensions

Description of the model

• In 2004, Yoon, Ru and Mioduchowski [Ru, 2000] studied a model describing the Timoshenko-beam effects on transverse wave propagation in multi-wall carbon nanotubes regarded as a system of separate nested tubes.In the case of a double-wall carbon nanotube, the suggested system of equations reads:

$$\begin{aligned}
&\rho I_1 \frac{\partial^2 w_1}{\partial t^2} + \kappa G A_1 \left(\frac{\partial \phi_1}{\partial x} - \frac{\partial^2 w_1}{\partial x^2} \right) - c\,(w_2 - w_1) = 0, \\
&\rho I_2 \frac{\partial^2 w_2}{\partial t^2} + \kappa G A_2 \left(\frac{\partial \phi_2}{\partial x} - \frac{\partial^2 w_2}{\partial x^2} \right) + c\,(w_2 - w_1) + K w_2 = 0, \\
&\rho I_1 \frac{\partial^2 \phi_1}{\partial t^2} - E I_1 \frac{\partial^2 \phi_1}{\partial x^2} + \kappa G A_1 \left(\phi_1 - \frac{\partial w_1}{\partial x} \right) = 0, \\
&\rho I_2 \frac{\partial^2 \phi_2}{\partial t^2} - E I_2 \frac{\partial^2 \phi_2}{\partial x^2} + \kappa G A_1 \left(\phi_2 - \frac{\partial w_2}{\partial x} \right) = 0,
\end{aligned} \tag{1}$$

Description of the model

- where x is the axial coordinate; t is the time variable;
- E and ρ are Young's modulus and the mass per unit axial length of carbon nanotube,
- I, A_j and w_j, ϕ_j $(j = 1, 2)$ are the moments of inertia, the cross-section areas , the total deflection and the slope due to bending of the j-th nanotube;
- K is a Winkler-like constant determined by the material properties of the surrounding elastic medium;
- c is the intertube interaction coefficient due to the van der Waals interaction between.

Basic equations

We try to find the local point Lie group of transformations, admitted by the system:

$$\rho(x)A_1(x)\frac{\partial^2 w_1}{\partial t^2}+\frac{\partial}{\partial x}\left(\kappa G A_1(x)\left(\phi_1-\frac{\partial w_1}{\partial x}\right)\right)-c(x)\left(w_2-w_1\right)=0$$

$$\rho(x)A_2(x)\frac{\partial^2 w_2}{\partial t^2}+\frac{\partial}{\partial x}\left(\kappa G A_2(x)\left(\phi_2-\frac{\partial w_2}{\partial x}\right)\right)+c(x)\left(w_2-w_1\right)$$

$$+K(x)w_2=0, \tag{2}$$

$$\rho(x)I_1(x)\frac{\partial^2 \phi_1}{\partial t^2}-\frac{\partial}{\partial x}\left(E I_1(x)\frac{\partial \phi_1}{\partial x}\right)+\kappa G A_1(x)\left(\phi_1-\frac{\partial w_1}{\partial x}\right)=0,$$

$$\rho(x)I_2(x)\frac{\partial^2 \phi_2}{\partial t^2}-\frac{\partial}{\partial x}\left(E I_2(x)\frac{\partial \phi_2}{\partial x}\right)+\kappa G A_2(x)\left(\phi_2-\frac{\partial w_2}{\partial x}\right)=0,$$

Equations and variational statement

We shall introduce total derivative operators and Euler operators:

$$D_\alpha = \frac{\partial}{\partial\, x_\alpha} + w^i_\alpha \frac{\partial}{\partial\, w^1} + w^i_{\alpha\mu} \frac{\partial}{\partial\, w^i_\mu} + w^i_{\alpha\mu\nu} \frac{\partial}{\partial\, w^i_{\mu\nu}} + w^i_{\alpha\mu\nu\sigma} \frac{\partial}{\partial\, w^i_{\mu\nu\sigma}} + \cdots\,,$$

$$E^i = \frac{\partial}{\partial\, w^i} - D_\alpha \frac{\partial}{\partial\, w^i_\alpha} + D_\alpha D_\mu \frac{\partial}{\partial\, w^i_{\alpha\mu}} - D_\alpha D_\mu D_\nu \frac{\partial}{\partial\, w^i_{\alpha\mu\nu}} + \\ + D_\alpha D_\mu D_\nu D_\sigma \frac{\partial}{\partial\, w^i_{\alpha\mu\nu\sigma}} - \cdots$$

Equations and variational statement

Then it is easy to be proved that

$$L = K - P,$$

where K means the kinetic energy density of Eqs.(2) and P expresses the potential energy density of Eqs. (2) i.e:

$$K = \frac{1}{2}\rho(x)\left(A_2(x)\frac{\partial w_2}{\partial t}^2 + I_1(x)\frac{\partial \phi_1}{\partial t}^2 + I_2(x)\frac{\partial \phi_2}{\partial t}^2 + A_1(x)\frac{\partial w_1}{\partial t}^2\right), \tag{3}$$

$$P = \frac{1}{2}\left[E\left(I_1(x)\frac{\partial \phi_1}{\partial x}^2 + I_2(x)\frac{\partial \phi_2}{\partial x}^2\right) + c(x)\left(w_1 - w_2\right)^2 + K(x)w_2^2\right]$$
$$+\left[\kappa G\left(A_1(x)\left(\phi_1 - \frac{\partial w_1}{\partial x}\right)^2 + A_2(x)\left(\phi_2 - \frac{\partial w_2}{\partial x}\right)^2\right)\right].$$

Equations and variational statement

is lagrangian density of the aforementioned system. Indeed if we use Euler operators we will discover that: $E^i[L] = 0$, $i = 1, 2, 3, 4$ coincides with the given equation.
Formally, the expression

$$A[w_1, w_2, \phi_1, \phi_2] = \int_\Omega L \, dx \, dt$$

is the corresponding functional.

Local Lie group of point transformations, admitted by the problem

Firstly, we begin with a connected local one-parameter Lie group of point transformations acting on some open subset Ω of the space $\mathbf{R}^6$ representing the independent and dependent variables x_1, x_2 and w_1, w_2, w_3, w_4 involved in the basic equations. The infinitesimal generator of such a group is a vector field X on the space $\mathbf{R}^6$,

$$X = \xi_1 \left(x_1, x_2, w_1, w_2, w_3, w_4\right) \frac{\partial}{\partial x^1} + \xi_2 \left(x_1, x_2, w_1, w_2, w_3, w_4\right) \frac{\partial}{\partial x^2} +$$

$$\sum_{i=1}^{4} \eta_i \left(x_1, x_2, w_1, w_2, w_3, w_4\right) \frac{\partial}{\partial w_i},$$

whose components ξ_1, ξ_2 and $\eta_i, (i = 1, 2, 3, 4)$ are supposed to be functions of class C^∞ on Ω.

Local Lie group of point transformations, admitted by the problem

By virtue of Theorem 2.31 [P. J. Olver], a vector field X of the aforementioned form generates a connected point Lie symmetry group of a system (1) (or, in other words, the system admits this vector field) if and only if the following infinitesimal criterions of invariance,

$$\mathrm{pr}^{(2)} X \left[\mathcal{E}^j [w_1, w_2, w_3, w_4] \right] = 0,$$

hold, whenever $\mathcal{E}^j[w_1, w_2, w_3, w_4] = 0$ $(j = 1, 2, 3, 4)$ (2); $pr^{(2)}X$ denote the 2-th prolongation of the vector field X [P. J. Olver].

Local Lie group of point transformations, admitted by the problem

The invariance criterions lead, through the standard computational procedure [P. J. Olver], to the following general generator of a local point Lie group of point transformations of the system:

$$X = 0 * \frac{\partial}{\partial x^1} + a_2 \frac{\partial}{\partial x^2} + a_0 \sum_{i=1}^{4} w_i \frac{\partial}{\partial w_i} + \mathbf{I}$$

where a_2, $a_0 \in \mathbf{R}$. This means that system (2), with $K = 0$ is an invariant one under the two-parameter group of a time translation and a dilatation. If $K(x) \neq 0$, then $a_0 = 0$ and the dilatation is not admitted symmetry. Furthermore, as we could verify it is unvariant under the whole set $\mathbf{I}$. The linear space spanned by these vector fields is also a Lie algebra in respect with the Lie bracket. Letting $\mathbf{F}$ to be a whole, admitted local point Lie algebra, then

Local Lie group of point transformations, admitted by the problem

The set **I** has the following form:

$$\sum_{i=1}^{4} u_i[x_1, x_2]\frac{\partial}{\partial w_i},$$

where $u_i[x_1, x_2]$ is an arbitrary solution (smooth) of the system (2).

Theorem. [L. V. Ovsyannikov-(1978)] Every linear system admits a vector field of the type **I**, iff $u_i[x_1, x_2]$ is a solution of this system.

Corollary. The set **I** is an infinite-dimensional abelian ideal of the admitted Lie algebra **F**.

Local Lie group of point transformations, admitted by the problem

Having analyzed earlier the invariance properties of the system, it is convenient to base the determination of the variational symmetries of the system. observation. By virtue of Theorem 4.12 [P. J. Olver], a vector field X_{var} of the same form as the above generates a point Lie symmetry group of the action functional of the problem if and only if the following infinitesimal criterion of invariance,

$$\mathrm{pr}^{(1)} X_{var}[L] + L Div(\xi) = 0,$$

holds. Where ξ is 2-tuple (ξ_1, ξ_2), which are the first two components of the generator. The notation L was introduced and means the lagrangian density of the system. Here, $Div(\xi)$ denotes the total divergence of ξ.

Local Lie group of point transformations, admitted by the problem

we define divergence operator Div of a vector field X, defined on some open subset Ω of the space $\mathbf{R}^n$, whose components are $\xi_1(x_1, x_2, x_3, ..., x_n), \xi_2(x_1, x_2, x_3, ..., x_n) ..., \xi_n(x_1, x_2, x_3, ..., x_n)$ as:

$$\sum_{i=1}^{n} D_i[\xi_i] = Div[X]$$

In this context,

$$X_{var} = a_2 \frac{\partial}{\partial x^2}$$

This is exactly the one dimensional Lie algebra of translation by time variable.

Local Lie group of point transformations, admitted by the problem

When the algebra $\mathbf{F}$ is divided by $\mathbf{I}$, the result is exactly three-dimensional quotient algebra isomorpfic to the variational subalgebra, spanned by X_{var}.
Finally, we have found and so-called divergence symmetries of the functional. By virtue of Theorem 4.33 [P. J. Olver], a vector field X_{div} of the same form as the aforementioned generates a divergence local Lie symmetry group of the functional if and only if there exists a smooth vector field $B := (b_1(x_1, x_2), b_2(x_1, x_2))$ such that the following infinitesimal criterion of invariance,

$$\mathrm{pr}^{(1)} X_{div}[L] + L Div(\xi) = Div[B],$$

holds.

Local Lie group of point transformations, admitted by the problem

Then, in our particular case, the generator has the following form:

$$X_{div} = a_2 \frac{\partial}{\partial x^2} + \sum_{i=1}^{4} u_i[x_1, x_2] \frac{\partial}{\partial w_i},$$

• As we can see variational symmetries form an infinite dimensional Lie subalgebra of **F**, because it contains the ideal **I**, in general, this follows by virtue of Theorem 4.14 and Preposition 4.16 [P. J. Olver].

Local Lie group of point transformations, admitted by the problem

An important problem naturally arises in the light of the above note. It may be placed in the category of the so-called group-classification problems and consists in determining all those equations of the type considered that admit a larger group together with this group itself. Here, in this presentation we are not going to study this problem, in general. However, in the subsequent papers we examine the group-classification problems for this and probably for the non-linear problem, arising naturally when will be included Casimir forces.

Conservation laws

Let Ω is connect open set of the space $\mathbf{R}^6$, representig as before both the independent variables and dependent variables w_1, w_2, w_3, w_4, then if there exists a vector field $P := (p_1(x_1, x_2, w_1, w_2, w_3, w_4), p_2(x_1, x_2, w_1, w_2, w_3, w_4)$, whose components p_1, p_2 are supposed to be functions of class $C^\infty(\Omega)$.
Then in terms of [P. J. Olver], we define conservation laws in divergence form:

$$Div[P] = 0,$$

where this condition is completed on all solutions of the regarded system.

Conservation laws

It is logical that trivial conservation laws do not provide any specific information about the nature of the system (1). That is why, we are interested only in non-trivial conservation laws. In other words we easy can see that conservation laws as divergence expressions form linear space and the kernel of the operator Div generates equivalence clases of conservation laws i.e every two equivalent conservation laws differ by a trivial equation law (we mean by exact total **rot** expression)[P. J. Olver].
We recall that x_1 is a space variable and x_2 is the time variable. Then:

$$D_1[p_1(x_1, x_2, w_1, w_2)] + D_2[p_2(x_1, x_2, w_1, w_2)] = 0$$

Then by definition, see [P. J. Olver], p_1 is called flux of the conservation law and p_2 is called density of the conservation law.

Conservation laws

There is an injective correspondence between variational symmetries and conservation laws - This is a steatment of the first Noether's theorem, however there is a more general result, known as generalised Noether's theorem [P. J. Olver], Theorem 5.58, that states that not only the variational (classical) symmetries correspond to conservation law of the system, but also the divergence symmetries which are natural generalization of the variational symmetries, indeed generalised Noether's theorem [P. J. Olver] is much more deeper it proves bijectivity of the correspondence between every generalised variational symmetries (if variational statement exists) and every consrvation law.

Conservation laws

- Translation by time variable, this means

"Energy conservation law"

$$-kG\left(\frac{\partial w_1}{\partial t}A_1(x)\left(\phi_1-\frac{\partial w_1}{\partial x}\right)+\frac{\partial w_2}{\partial t}A_2(x)\left(\phi_2-\frac{\partial w_2}{\partial x}\right)\right)+$$

$$E\left(I_1(x)\frac{\partial \phi_1}{\partial x}\frac{\partial \phi_1}{\partial t}+I_2(x)\frac{\partial \phi_2}{\partial x}\frac{\partial \phi_2}{\partial t}\right)$$

Density:

$$K+P$$

Conservation laws

An infinite dimensional space of conservation laws

Flux:

$$
\begin{aligned}
&- kGA_1(x)u_1\left(\phi_1 - \frac{\partial w_1}{\partial x}\right) - kGA_2(x)u_2\left(\phi_2 - \frac{\partial w_2}{\partial x}\right)\\
&+ kGA_1(x)w_1\left(u_3 - \frac{\partial u_1}{\partial x}\right) + kGA_2(x)w_2\left(u_4 - \frac{\partial u_2}{\partial x}\right)\\
&+ EI_1(x)u_3\frac{\partial \phi_1}{\partial x} + EI_2(x)u_4\frac{\partial \phi_2}{\partial x} - EI_1(x)\phi_1\frac{\partial u_3}{\partial x} - EI_2(x)\phi_2\frac{\partial u_4}{\partial x}
\end{aligned}
$$

Conservation laws

Density:

$$
\begin{aligned}
&-\rho(x)A_1(x)u_1\frac{\partial w_1}{\partial t}-\rho(x)A_1(x)u_2\frac{\partial w_2}{\partial t}-\rho(x)I_1(x)u_3\frac{\partial \phi_1}{\partial t}\\
&-\rho(x)I_2(x)u_4\frac{\partial \phi_2}{\partial t}+\rho(x)A_1(x)w_1\frac{\partial u_1}{\partial t}+\rho(x)A_1(x)w_2\frac{\partial u_2}{\partial t}\\
&-\rho(x)I_1(x)\phi_1\frac{\partial u_3}{\partial t}+\rho(x)I_2(x)\phi_2\frac{\partial u_4}{\partial t}
\end{aligned}
$$

Future intension

Thank you for your attention!

C.Ru,
Column buckling of multiwalled carbon nanotubes with interlayer radial displacements, *Physical Review B*, 16962 (2000).

J. Yoon, C. Q. Ru, and A. Mioduchowski,
Vibration of an embedded multiwall carbon nanotube, *Composites Science and Technology*, 1533-1542 (2003).

J. Yoon, C. Q. Ru, and A. Mioduchowski,
Vibration and instability of carbon nanotubes conveying fluid, *Composites Science and Technology*, 1326-1336 (2005).

L. Wang, and Q. Ni ,
On vibration and instability of carbon nanotubes conveying fluid, *Computational Materials Science*, 399-402 (2008).

H. Askari,
Nonlocal effect in carbon nanotube resonators: A comprehensive review, *Advances in Mechanical Engineering*, 1-24 (2017).

P. J. Olver,
Applications of Lie Groups to Differential Equations, 2nd ed, *In Graduate Texts in Mathematics*, Springer-Verlag, New York, (1993).

L.V Ovsyannikov,
Group analysis for Differential equations, *Science*, (1978).