A Generalization of The Quantization of Poisson Manifolds

Akifumi Sako (Tokyo Univ of Science)

with Jumpei Gohara, Yuji Hirota

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Geometry, Integrability and Quantization

1. Intro. & Motivation

- Classical Mechanics $\xrightarrow{\text{Quantization}}$ Quantum Mechanics

  several ways of Quantization
  - canonical quantization (prequantization)
  - path integral

- Classical Field theory → Quantum field theory
- Classical Gravity → ? String theory
  - ? loop-gravity
  - ? Matrix model

How can we generalized and restrict the way of quantization ?

~ Quantizations ~

- Dirac.

$$\hat{\ } : f \in C^\infty(M) \longrightarrow \hat{f} \in \mathrm{End}(\mathcal{H})$$

$$(1)\ \hat{H}_1 + \hat{H}_2 = \widehat{H_1 + H_2}, \quad (2)\ \widehat{\lambda H} = \lambda \hat{H}$$

$$(3)\ [\hat{H}_1, \hat{H}_2] = i\,\widehat{\{H_1, H_2\}}, \quad (4)\ \hat{1} = Id$$

There is no Perfect Quantization

- Deformation Quantization
  De Wilde-Lecomte, Fedosov, Omori-Maeda-Yoshioka, Kontsevich, etc
- Matrix regularization
  Belezin, Toeplitz, Hoppe, de Wit, Nicolai. etc.

  $$\cancel{(3)\ [\hat{H}_1, \hat{H}_2] = \widehat{\{H_1, H_2\}}}$$

- Geometric Quan.
  Weyl, Kostant, Souriau, etc
  ~~Observable~~

It is convinient if there is a perspective unifying these quantizations!

2. Def. of Quantization Category

▷ Preparation.

"limit" is used as the following meaning.

J : index category

$$1 \xrightarrow{f_{12}} 2 \xrightarrow{f_{23}} 3$$

$$J \xrightarrow{F} \mathcal{C}$$

diagram

$\mathcal{C}$: category

$$\begin{tikzcd} & N \arrow[d, dashed, "\text{Unique}"] \arrow[ddl, bend right] \arrow[dd, bend right] \arrow[ddr, bend left] & \\ & L \text{ Limit} \arrow[dl, "\pi_1"'] \arrow[d, "\pi_2"] \arrow[dr, "\pi_3"] & \\ F(1) \arrow[r, "F(f_{12})"'] & F(2) \arrow[r, "F(f_{23})"'] & F(3) \end{tikzcd}$$

(L, π) is called "limit"

Def). Pre-2 category. $\mathcal{C}$

RMod : category of R-module over com. R.

$A(M)$: Poisson alg. $(C^\infty(M), \cdot, \{,\})$

fixed

M is a Poisson mfd.

Not alg.

$Mor(\mathcal{C})$ is linea fun.

$\mathcal{C}$: sub Category of RMod s.t.

1. $\forall M_i \in ob(\mathcal{C})$ is a Lie alg $([,]_i)$ and a norm sp.

2. $A(M) \in ob(\mathcal{C})$

3. $\forall M_i$ there exist $T_i \in \mathcal{C}(A(M), M_i)$ and norm $\|T_i(f)\| < \infty$

Def). Quantization Category $\mathcal{Q}$ of Poisson alg $A(M)$.

J : index category with a $\underset{\text{diagram}}{F} : J \longrightarrow \underset{\text{pre }\mathcal{Q}\text{ category}}{\mathcal{C}(M)}$

$\mathcal{Q}(\mathcal{C}(M), J, F, \chi)$ is a category $\mathcal{C}(M)$ satisfying following conditions

i) $^{\exists}\chi : ob(\mathcal{C}) \longrightarrow \mathbb{R}$. s.t.

$\forall\ M_i, M_j \in ob(\mathcal{C})$ with $i, j \in ob(J)$

If $^{\exists}f_{ij} \in J(i,j)$ $\quad F(i) = M_i,\ F(j) = M_j$

$\Rightarrow\ \chi(M_i) \leq \chi(M_j)$

χ corresponds with inverse of tempreture.

ii) There exist Limit (M_∞, π) of diag. F of J

$$\begin{tikzcd}
 & A(M) \arrow[ddl, "T_i"'] \arrow[ddr, "T_j"] \arrow[d, dashed] & \\
 & M_\infty \arrow[dl] \arrow[dr] & \\
M_i \arrow[rr, "T_{ij}"'] & & M_j
\end{tikzcd}$$

"limit" means classical limit.

iii). $\forall f, g \in A(M)$. $T := T_\infty \in \mathcal{C}(A(M), M_\infty)$

satisfies the following quantization conditions:

Q1.) $\| T(f) \|_\infty < \infty$

Q2.) $\| T(fg) - T(f)T(g) \|_\infty = 0$

Q3.) $\| [T(f), T(g)]_\infty - T(\{f, g\}) \|_\infty = 0$

Q1 ~ Q3. are similar conditions

with them in Berezine-Toeplitz quantization

or Matrix regularization.

3. Matrix Regularization (including B-T quantization)

Def). $\mathcal{C}_{MR}$: pre-2 category for Matrix regularization

$\{N_i\}$: strictly increasing sequence of $\mathbb{N}$

・$\hbar$: strictly decreasing fun. s.t. $\lim_{N\to\infty} N\hbar(N)$ converges

$\mathcal{C}_{MR}(M)$ is defined as follows.

$N_k \times N_k$ Matrix alg.

・$ob(\mathcal{C}_{MR}(M)) = \{A(M), Mat_{N_k} (k=1,2,\cdots), Mat_\infty\}$

・$Mor(\mathcal{C}_{MR}(M))$: set of linear fun.

s.t. $\exists!\, T_i : A(M) \longrightarrow Mat_{N_i}$, $\exists!\, T_{ij} : Mat_{N_i} \longrightarrow Mat_{N_j}$

with $T_i = T_{ij} \circ T_j$

$$\mathcal{J}_{MR} \qquad \begin{tikzcd} 1 & 2 \arrow[l, "f_{12}"'] & 3 \arrow[l, "f_2"'] & \cdots \end{tikzcd}$$

$$j \to i \text{ for } i \leq j$$

$$\xrightarrow{\ F\ }$$

$$\mathcal{C}_{MR} \qquad \begin{tikzcd} & A(M) \arrow[ddl, "T_1"'] \arrow[dd, bend right, "T_2"'] \arrow[d, dashed] \arrow[ddr, "T_3"] & \\ & M_\infty \arrow[dl, "\pi_1"'] \arrow[d, "\pi_2"] \arrow[dr, "\pi_3"] & \\ M_{N_1} & M_{N_2} \arrow[l, "T_{12}"] & M_{N_3} \arrow[l, "T_{23}"] & \cdots \arrow[l] \end{tikzcd}$$

$\chi_{MR} = \dim M_i$ is consistent with $\mathcal{J}$.

<u>Thm.</u> $(\mathcal{C}_{MR}, \mathcal{J}, F, \chi)$ is 2 category $(\mathcal{2}_{MR})$ with the limit (M_∞, π)

4. Deformation Quantization

Let's review. D.Q.

Def). Deformation Quantization $(\mathcal{F}, *)$.

$\mathcal{F} := \{ f \mid f = \sum \hbar^k f_k, \ f_k \in C^\infty(M) \}$ formal P.S.

$f * g = \sum_k \hbar^k C_k(f, g)$

1. $*$ is associative
2. C_k is bidifferential op.
3. $C_0(f, g) = fg$, $C_1(f, g) = i\{f, g\}$
4. $f * 1 = 1 * f = f$

For arbitrary Poisson mfd M.
there exist $(\mathcal{F}, *)$.

Def). pre 2 category $\mathcal{C}_{DQ}$

$ob(\mathcal{C}_{DQ}) = \{A(M), (\mathcal{F}, *)\}$

Lie alg by $[f,g]_* = f*g - g*f$

$$\begin{tikzcd} A(M) \arrow[r, hook] \arrow[r, "T"'] & \mathcal{F} \end{tikzcd}$$

$T = \pi$

$\mathcal{C}_{DQ}$

$$J_{DQ} \; \circ \xrightarrow{F_{DQ}} \begin{matrix} A(M) \\ \downarrow \pi \\ \mathcal{F} \end{matrix}, \quad \chi_{DQ}(A(M)) = 1. \quad \chi_{DQ}(\mathcal{F}) = 0$$

Thm $(\mathcal{C}_{DQ}, J_{DQ}, F_{DQ}, \chi_{DQ})$ is 2-category with limit $(A(M), \pi)$

Thm. $\partial_{MR} \underset{\text{Categorically equiv}}{\simeq} \partial_{DQ}$ for $A(M) \simeq M_\infty$ $\perp$

Ex). Bordemann, Meinrenken, Schlichenmaier, B.T.qu cpt. Kähler mfd

Proof) $\partial_{MR}|_{Mat_{N_i}}$ $\qquad$ ∂_{DR}

$$\begin{tikzcd}
 & A(M) \arrow[dl, "T_\infty"'] \arrow[d, "T_i"] \arrow[rr, shift left, "F"] \arrow[rr, bend left=40, "Id"] & & A(M) \arrow[ll, shift left, "G"] \arrow[d, "T"] \arrow[loop above, "Id"] \\
M_\infty \arrow[r, "\pi_i"] & Mat_{N_i} \arrow[rr, shift left, "F"] & & \mathcal{F} \arrow[ll, shift left, "G"]
\end{tikzcd}$$

Natural transformation

$\theta : FG \to id_{\partial_{DQ}}$

$\theta' : GF \to id_{\partial_{MR}}$

$$\begin{tikzcd}
GF(A(M)) \arrow[r, "\theta'_{A(M)} = Id"] \arrow[d, "id = GF(T)"'] & A(M) \arrow[d, "T_\infty"] \\
GF(M_\infty) \arrow[r, "\theta'_{M_\infty} = T_\infty"'] & M_\infty
\end{tikzcd}$$

$GF(M_\infty) = A(M)$

$$\begin{tikzcd}
GF(M_\infty) \arrow[r, "\theta'_{M_\infty} = T_\infty"] \arrow[d, "T_i = GF(\pi_i)"'] & M_\infty \arrow[d, "\pi_i"] \\
GF(Mat_{N_i}) \arrow[r, "\theta'_{Mat_{N_i}} = Id"'] & Mat_{N_i}
\end{tikzcd}$$

$GF(M_\infty) = A(M)$, $\quad GF(Mat_{N_i}) \simeq Mat_{N_i}$

//

5 Other Quantization

Def. PreQuantization $\qquad$ Set of Op acting on $\mathcal{H}$

$\mathcal{C}_{PQ}$: $ob(\mathcal{C}_{PQ}) = \{A(M), Q(A(M))\}$

$J_{PQ} \bullet \xrightarrow{F_{PQ}} Q(A(M))$

$\chi_{PQ}(A(M)) = 1, \quad \chi_{PQ}(Q(A(M))) = 0$

Thm. $(\mathcal{C}_{PQ}, J_{PQ}, F_{PQ}, \chi_{PQ})$ is $\mathcal{Q}$ category. $\mathcal{Q}_{PQ}$

with limit $(A(M), Q)$

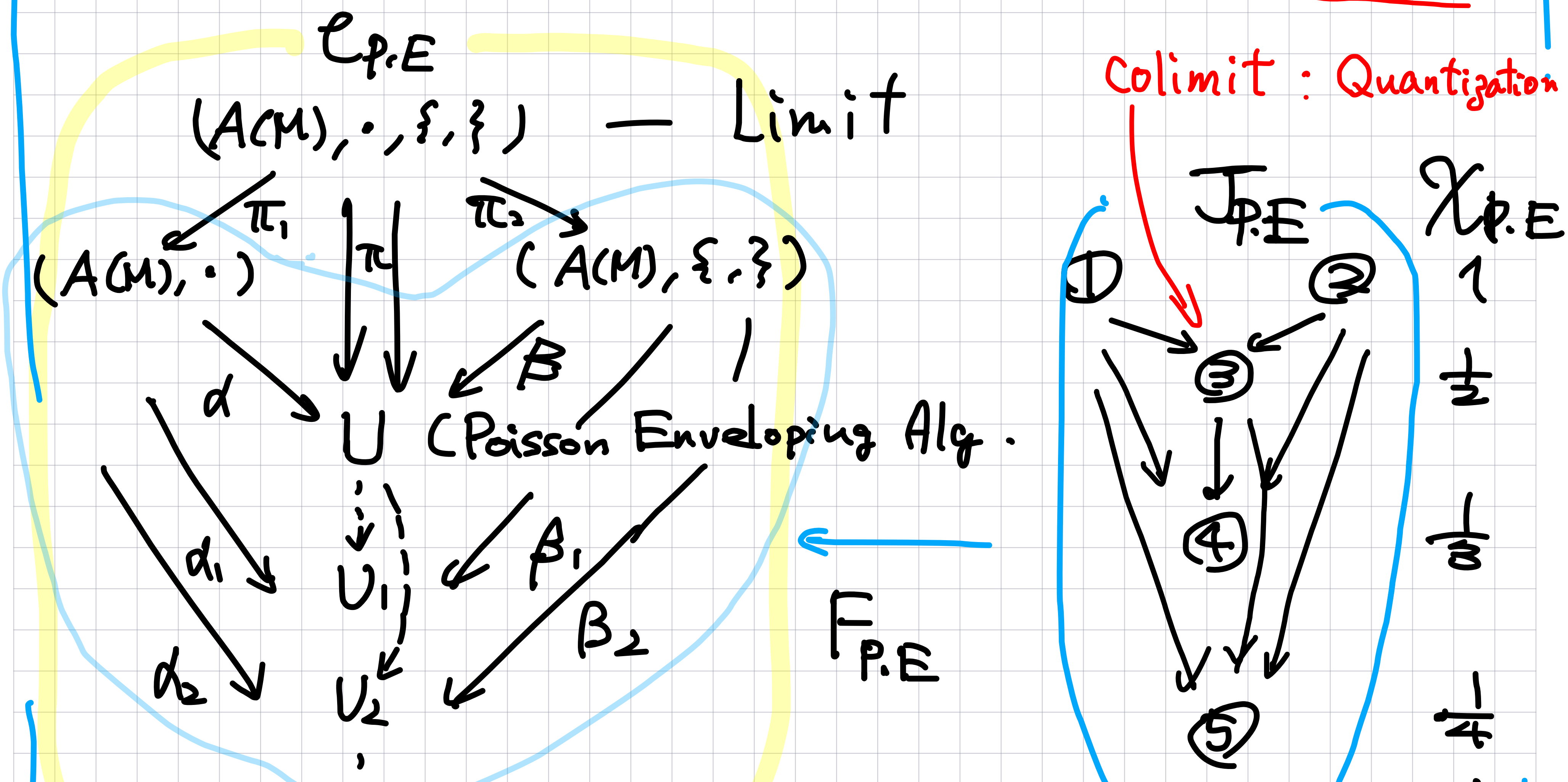
$$\begin{array}{c} \mathcal{Q}_{PQ} \\ \bullet\ A(M) \\ \downarrow Q \\ J_{PQ} \bullet \xrightarrow{F_{PQ}} Q(A(M)) \end{array}$$

Thm. equiv.

$$\mathcal{Q}_{PQ} \simeq \mathcal{Q}_{DQ} \cong \mathcal{Q}_{MR}\big|_{A(M) \simeq M_\omega}$$

Thm. Poisson Enveloping Alg.

$(\mathcal{C}_{P.E}, J_{P.E}, F_{P.E}, \chi_{P.E})$ is $\mathcal{Q}$ with limit $(A(M), \pi)$



α, β is Ring hom

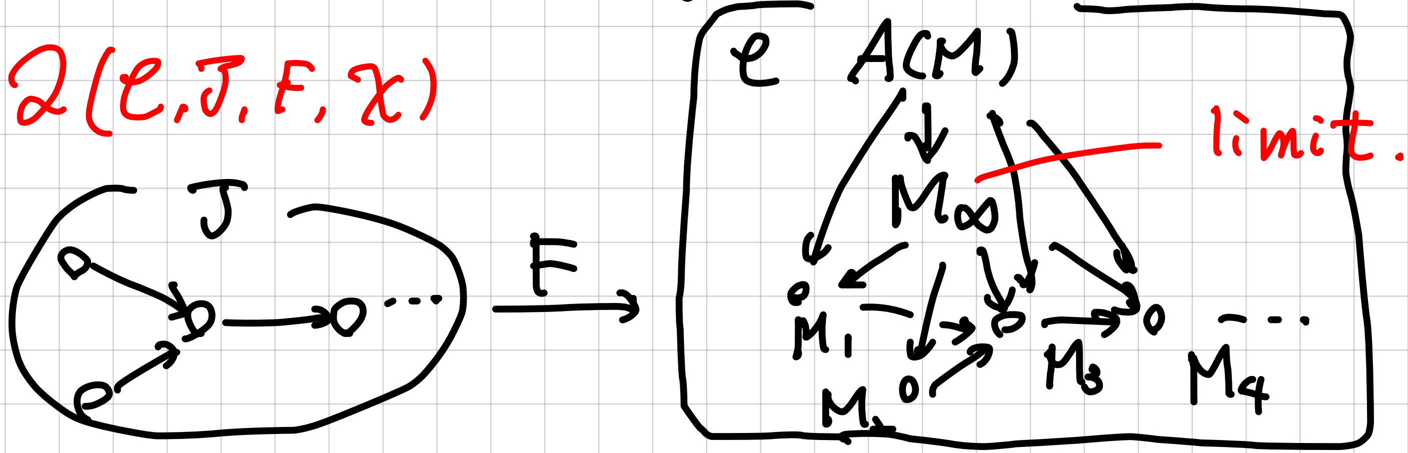
with $\alpha(\{a,b\}) = \beta(a)\alpha(b) - \alpha(b)\beta(a)$

$\beta(a,b) = \alpha(a)\beta(b) + \alpha(b)\beta(a)$

6. Conclusions.

We define $\mathcal{Q}$ as a generalization of Quantization.

$\mathcal{Q}(\mathcal{C},\mathcal{J},F,\chi)$



Thm. Matrix reg, Def. Q, PreQ., Poissn E.A. are $\mathcal{Q}$

Thm. $\mathcal{Q}_{MR} \simeq \mathcal{Q}_{DQ} \simeq \mathcal{Q}_{P.Q}$. (for $A(M)=M_\infty$)

Poisson Enveloping $\mathcal{Q}$ is not equiv. with. the others

$$\mathcal{Q}_{P.E} \not\simeq \mathcal{Q}_{M.R.}|_{A(M)\simeq M_\infty}$$

$$\mathcal{Q}_{DQ} \simeq \mathcal{Q}_{P.Q.}$$

Thank you for your attentions.

議論

○ 行列正則化の際

$A(M)$ = M上のなめらかな関数全体

1) $A(M) = M_\infty$ となる条件 ⇔ T_∞ が同相である条件は何か。

$$A(M) \xrightarrow{T_\infty} M_\infty$$

2) 特にベレジン－トープリッツ量子化で $A(M)$ にプロジェクションで

$$A(M) \longrightarrow \mathcal{H}_k \xrightarrow[k\to\infty]{} \mathcal{H}$$

適当なプロジェクション $\Pi_k = T_k$

$\mathcal{H}$：ヒルベルト空間作る

このときに Mがコンパクト → $A(M) \subset L^2(M) =: H$

2乗可積分　ヒルベルト

これは等号 $A(M) = L^2(M)$ ではない？（不連続ふくまない）なら、$\mathcal{H}$, $\mathcal{H}_k$ はどうとる？

3) $\mathcal{H}_k$ の次元の公式も $A(M)$ からの Map だと変更いるのか？

Def). Minimized $\mathcal{Q}^*_{MR}\big|_{N_i}$

$$\mathrm{ob}(\mathcal{Q}^*_{MR})\big|_{N_i} = \{A(M), \mathrm{Mat}_{N_i}, \mathrm{Mat}_\infty\}$$

$$\begin{tikzcd} & A(M) \arrow[dl, "T_i"'] \arrow[d, dashed, "T"] \\ \mathrm{Mat}_{N_i} & \mathrm{Mat}_\infty \arrow[l, "\pi"] \end{tikzcd}$$