N^{th}-order superintegrable systems separating in polar coordinates

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A standard way to gain insight into the behaviour of a physical system is:

$$\left\{ \begin{array}{c} \text{construct a mathematical model} \\ \text{analyze the model} \\ \text{use it to make predictions} \\ \text{compare with experiment} \end{array} \right\}$$

The models provided by classical and quantum mechanics are spectacularly successful in this regard.

A relatively few systems, however, can be solved exactly with explicit analytic expressions: integrable Hamiltonian systems. A special subclass is: superintegrable systems.

- a fundamental procedure for the positioning of satellites and orbital maneuvering of interplanetary spacecraft is based on the superintegrability of the Kepler system.
- the periodic table is based on perturbations of the superintegrable hydrogen atom.

Definition

In classical mechanics a Hamiltonian system with n degrees of freedom is called integrable if it allows n functionally independent integrals of motion

$$\{\mathbf{X_1}, \ldots, \mathbf{X_n}\}.$$

A superintegrable system is one that allows some additional integrals of motion

$$\{\mathbf{Y_1}, \ldots, \mathbf{Y_k}\},$$

such that the set $\{\mathbf{X_1}, \ldots, \mathbf{X_n}, \mathbf{Y_1}, \ldots, \mathbf{Y_k}\}$ is functionally independent.

$$\text{If } k = \begin{cases} n-1 & \text{then the system} \quad \text{maximally} \\ 1 & \text{is called} \quad \text{minimally} \end{cases} \text{superintegrable}.$$

The concepts of integrability and superintegrability are also introduced in quantum mechanics.

Integrals of motion $\Longrightarrow$ **well defined linear QM operators algebraically independent**

The best known superintegrable systems are:

- Kepler, or Coulomb system
- Harmonic oscillator

They are characterized by the fact that all finite classical trajectories in these systems are periodic.

Kepler system

$$H = \frac{1}{2}\mathbf{p}^2 + V(r)\,, \qquad \begin{array}{c} \mathbf{L} = \mathbf{r} \times \mathbf{p} \\ L_i = \epsilon_{ikl} x_k p_l \end{array}$$

$$\{H, L_{\rm i}\} = 0 \quad \Longleftrightarrow \quad \frac{dL_{\rm i}}{dt} = 0\,.$$

$$V(r) = -\frac{k}{r} \qquad \textbf{Kepler potential}$$

Another conserved vector Laplace-Runge-Lenz

$$\mathbf{A} = \mathbf{p} \times \mathbf{L} - \mathrm{mk}\frac{\mathbf{r}}{r}\,.$$

H, $\mathbf{A}$, $\mathbf{L}$: $6 + 1 = 7$ conserved quantities.

However, we have 2 relations

$(\mathbf{A}, \mathbf{L}) = 0$ and H can be written in terms of $\mathbf{A}$ and $\mathbf{L}$.

Derivation of Kepler orbits

In particular we have $A^2 = m^2k^2 + 2mEl^2$

$$\mathbf{A} \cdot \mathbf{r} = \mathbf{r} \cdot (\mathbf{p} \times \mathbf{L}) - mkr$$

$$Ar \cos\theta = l^2 - mkr$$

$$\frac{1}{r} = \frac{mk}{l^2}\left(1 + \frac{A}{mk}\cos\theta\right), \qquad e = \frac{A}{mk} = \sqrt{1 + \frac{2El^2}{mk^2}}\,.$$

- ellipse ($e < 1$ or $-\frac{mk^2}{2l^2} < E < 0$)
- circle ($e = 0$ or $E = -\frac{mk^2}{2l^2}$)
- hyperbola ($e > 1$ or $E > 0$)
- parabola ($e = 1$ or $E = 0$)

Quantum Coulomb problem in E_3

$$H = \mathbf{p}^2 - \frac{\alpha}{r}\,, \qquad \alpha > 0\,, \qquad \mathbf{L} = \mathbf{r} \times \mathbf{p}$$

Quantum version of Laplace-Runge-Lenz

$$\mathbf{A} = \mathbf{p} \times \mathbf{L} - \mathbf{L} \times \mathbf{p} - \frac{\alpha}{r}\mathbf{r}\,.$$

The commutation relations between these operators are

$$[\mathbf{L}, H] = [\mathbf{A}, H] = 0\,,$$

$$[L_j, L_k] = i\epsilon_{jkl}L_l\,, \quad [L_j, A_k] = i\epsilon_{jkl}A_l\,, \quad [A_j, A_k] = -i\epsilon_{jkl}L_l H\,.$$

The famous Balmer formula, $E = -\frac{2me^4}{\hbar^2}\frac{1}{n}$.

W. Pauli in his remarkable 1926 paper uses precisely the above formulas to derive the Balmer formula. He obtained this result before the Schrödinger equation was known using only the algebra, no calculus.

Harmonic oscillator

Also maximally superintegrable:

$$H = \frac{1}{2}\left(\mathbf{p}^2 + \omega^2 r^2\right) , \qquad \mathbf{L} = \mathbf{r} \times \mathbf{p} ,$$

$$Q_{\mathrm{ik}} = p_{\mathrm{i}} p_{\mathrm{k}} + \omega^2 x_{\mathrm{i}} x_{\mathrm{k}}$$

10 conserved quantities, 5 functionally independent ones.

Bertrand's theorem (1873)

The only spherically symmetric potentials in E_3 for which all bounded trajectories are closed are

$$V(r) = -\frac{k}{r} , \quad \text{and} \quad V(r) = \omega^2 r^2 .$$

A systematic search for SIS and their properties was started some time ago (1965 Winternitz *et. al*). Originally the approach concentrated on Hamiltonians of the type

$$H = -\frac{1}{2}\Delta + V(\mathbf{r})$$

in 2- and 3- dimensional Euclidean spaces with the restriction that all integrals of motion should be first- or second-order polynomials in the momenta.

For Hamiltonians of the given type with second-order integrals of motion there is a close relation between integrability and the separation of variables in the Schrödinger and Hamilton-Jacobi equations. Superintegrable systems of this type are multiseparable.

This relationship between integrability and separability breaks down in other cases. For example, for natural Hamiltonians, the existence of third-order integrals of motion does not lead to the separation of variables (1935 Drach, 2002 Gravel-Winternitz, 2007 Marquette-Winternitz, 2009 Marquette, 2010Tremblay-Winternitz). Furthermore, if we consider velocity dependent potentials

$$H = -\frac{1}{2}\Delta + V(\mathbf{r}) + (\mathbf{A}, \mathbf{p}) \, ,$$

then quadratic integrability no longer implies the separation of variables (1985 Dorizzi *et. al*, 2004 Bérubé-Winternitz).
With second-order integrals of motion for the Hamiltonians having velocity dependent potentials and higher-order integrals of motion with natural Hamiltonians, CM and QM potentials do not necessarily coincide (1984 Hietarinta, pure QI).

In 2006 together with P. Winternitz we initiated the study of integrability and superintegrability for systems involving particles with spin. More specifically, we considered two nonrelativistic quantum particles, one with $s = \frac{1}{2}$ and the other with $s = 0$. From the physical point of view, the most interesting Hamiltonian to consider would be

$$H = -\frac{\hbar^2}{2\mu}\Delta + V_0(\mathbf{r}) + \frac{1}{2}\left\{ V_1(\mathbf{r}), (\boldsymbol{\sigma}, \mathbf{L}) \right\}.$$

As integrals of motion we considered first- and second-order matrix differential operators and classified the superintegrable systems in E_2 and E_3.

After the publication of the 2009 paper "An infinite family of solvable and integrable quantum systems on a plane" (Tremblay-Turbiner-Winternitz) the direction of the research has been shifted to higher-order integrability/superintegrability (2011, 2012 Kalnins-Kress-Miller, 2015 Post-Winternitz, Marquette). Higher-order integrable and SIS can now be more easily tractable (2012, 2013 Ranada, 2013 Ballesteros *et. al*).
More recently, exotic and standard potentials appearing in Quantum SIS has been studied for $N = 4$ both in Cartesian (2017 Marquette-Sajedi-Winternitz) and Polar (2017, 2018 Escobar-Ruiz–Lopez Vieyra-Winternitz-Yurduşen) coordinates.
Also doubly exotic potentials for $N = 5$ in Cartesian coordinates has been studied (2017 Abouamal-Winternitz).

Superintegrable systems are interesting both from the mathematical and physical point of view.

In CM they have the following:

- In the case of **maximal superintegrability** all bounded trajectories are closed and the motion is periodic.
- The Poisson algebra of integrals of motion has an interesting non-Abelian structure. It may be a finite dimensional Lie algebra, a Kac-Moody algebra or a polynomial algebra.
- A special case is quadratic superintegrability when integrals are second-order polynomials in the momenta. This is related to multiseparability: the H-J equation allows separation of variables in more than one coordinate system.

In QM the SIS have similar distinguishing properties:

- The quantum energy levels display "*accidental*" degeneracy, *i.e.* a degeneracy explained by higher symmetries rather than the geometrical ones.
- Integrability provides a complete set of quantum numbers, characterizing the system. Superintegrability, in all cases studied so far, entails exact solvability.
  - the energy levels can be calculated algebraically.
  - the wave functions expressed in terms of polynomials.
- Integrals form a non-Abelian algebra under Lie commutation. Again it can be a finite dimensional Lie algebra, a Kac-Moody algebra or a polynomial algebra.
- For quadratic superintegrability Schrödinger equation separates in more than one system of coordinates. Moreover, the QM and CM potentials coincide for quadratic superintegrability.

In this talk we restrict ourselves to the plane E_2. The Hamiltonian, in Cartesian coordinates, has the form

$$H = \frac{1}{2}(p_x^2 + p_y^2) + V(x, y) .$$

In CM: p_x and p_y are the conjugate momenta.
In QM: they are the corresponding operators

$$p_x = -i\,\hbar\,\frac{\partial}{\partial x} , \qquad p_y = -i\,\hbar\,\frac{\partial}{\partial y} .$$

The classical Hamiltonian in polar coordinates reads

$$H = \frac{1}{2}\left(p_r^2 + \frac{p_\theta^2}{r^2}\right) + V(r, \theta) ,$$

whereas the corresponding quantum operator takes the form

$$\mathcal{H} = -\frac{\hbar^2}{2}\left(\partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\theta^2\right) + V(r, \theta) .$$

INTEGRABILITY: EXISTENCE OF 2^{nd}-ORDER IOM

Let us consider

$$X = L_z^2 + 2S(\theta)$$

For X to be an integral of motion, it must

$$\{H, X\}_{PB} = 0\,, \qquad [H, X] = 0\,.$$

This condition specifies the form of the potential

$$V(r,\theta) \;=\; R(r) \;+\; \frac{S(\theta)}{r^2}\,,$$

where $R(r)$, $S(\theta)$ are arbitrary functions.

Radial Component $R(r)$

Having the 2^{nd}-order IOM X (which guarantees the separability of H-J or S) we showed that the $R(r)$ part of the potential must satisfy

$$R(r) = 0\,, \qquad R(r) = \frac{a}{r}\,, \qquad R(r) = b\,r^2\,.$$

By canonical transformation we prepare our problem to resemble the one, for which the Bertrand's theorem will apply. In both problems we have

$$\dot{r} = \pm\sqrt{2(H - R_{eff})}\,, \qquad R_{eff}(r) = R(r) + \frac{X}{2\,r^2}\,,$$

however, we have to modify

$$\dot{\theta} = \frac{\ell}{r^2} \qquad \Longrightarrow \qquad \dot{\Omega}(\theta) = \frac{\sqrt{X}}{r^2}\,,$$

i,e,; $(\theta, p_\theta) \Longrightarrow (\Omega, P_\Omega)$

For this we define

$$\Omega(\theta) = \int^{\theta} \sqrt{\frac{X_0}{X_0 - 2\,S(\omega)}}\,d\omega\,, \qquad X_0 > 0\,,$$

whose time evolution is given by

$$\dot{\Omega} = \{\Omega,\, H\}_{PB} = \frac{1}{r^2}\sqrt{\frac{X_0(X - 2\,S(\theta))}{X_0 - 2\,S(\theta)}}\,.$$

Substituting the values of the integrals $H = E$ and $X = X_0$, we obtain

$$\frac{dr}{d\Omega} = \pm\frac{r^2}{\sqrt{X_0}}\sqrt{2(E - R_{eff})}\,.$$

Upon introducing $u = \frac{1}{r}$ and squaring both sides we get

$$E = \frac{1}{2}\,X_0\left(\frac{du}{d\Omega}\right)^2 + \tilde{R}_{eff}(u)\,,$$

after which the Bertrand's theorem will follow.

SUPERINTEGRABILITY: EXISTENCE OF N^{th}-ORDER IOM

General IOM (Classically)

$$Y = Y^{(N)} + \sum_{\ell=1}^{\left[\frac{N}{2}\right]} \sum_{j=0}^{N-2\ell} F_{j,2\ell}\, p_x^j\, p_y^{N-j-2\ell}$$

where $Y^{(N)}$ contains only the N^{th}-order terms

$$Y^{(N)} = \sum_{0 \leq m+n \leq N} A_{N-m-n,m,n}\, L_z^{N-m-n}\, p_x^m\, p_y^n\, .$$

Here $A_{N-m-n,m,n}$ are $\frac{(N+1)(N+2)}{2}$ constants. These leading order terms will define the form of the potential.

In Polar case, by putting

$$p_x = (\cos\theta\, p_r - \frac{\sin\theta}{r} L_z)\,, \qquad p_y = (\sin\theta\, p_r + \frac{\cos\theta}{r} L_z)\,,$$

we obtain the corresponding expressions in polar coordinates. Also, the N^{th}-order terms can be written more suitably

$$Y^{(N)} = \sum_{0\le s+2k\le N} L_z^{N-s-2k}\, P^{s+2k} \left[B^{(1)}_{N-s-2k,s,k} \cos s\,\Theta + B^{(2)}_{N-s-2k,s,k} \sin s\,\Theta \right]$$

$$P^2 \equiv p_x^2 + p_y^2\,, \qquad \tan\Theta \equiv \frac{p_y}{p_x}$$

For fixed s, each pair

$$\left(B^{(1)}_{N-s-2k,s,k} \cos s\,\Theta,\ B^{(2)}_{N-s-2k,s,k} \sin s\,\Theta \right)$$

($s \neq 0$) forms a doublet under $O(2)$ rotations. At $s = 0$, the pair reduces to a singlet. Under rotations through the angle θ around the z-axis, the doublets rotate through $s\,\theta$.

Similarly, in the quantum case we introduce the Hermitian N^{th}-order operator $Y^{(N)}$

General IOM (Quantum case)

$$Y = Y^{(N)} + \sum_{\ell=1}^{\left[\frac{N}{2}\right]} \sum_{j=0}^{N-2\ell} \left\{ F_{j,2\ell}\,,\ p_x^j\ p_y^{N-j-2\ell} \right\}$$

with $Y^{(N)}$

$$Y^{(N)} = \sum_{0 \le s+2k \le N} \left\{ L_z^{N-s-2k}\,, (p_x^2 + p_y^2)^k \right.$$

$$\left. \left(B^{(1)}_{N-s-2k,s,k} \left[(p_x + ip_y)^s \right]_{\mathrm{Re}} + B^{(2)}_{N-s-2k,s,k} \left[(p_x + ip_y)^s \right]_{\mathrm{Im}} \right) \right\}$$

Determining equations: $[H,\ Y] = 0$.

In the quantum case, $[H,\ Y]$ is an operator of order $(N+1)$, *i.e.*

$$[H,\ Y] \;=\; \sum_{k+l=0}^{N+1} Z_{k,l}(r,\,\theta)\,\frac{\partial^{k+l}}{\partial r^k\,\partial\theta^l}\ .$$

We require $Z_{k,l} = 0$ for all k and l and obtain the determining equations. The terms of order $k+l = N+1$ and $k+l = N$ vanish automatically. Moreover, only the terms with $k+l$ having the opposite parity than N provide independent determining equations $(Z_{k,l} = 0)$. Those with the same parity provide equations that are differential consequences of the first ones. For the classical case, the determining equations are obtained from the quantum case by taking the limit $\hbar \to 0$.

Vanishing of the $[H, Y] = 0$ implies that the potential V must satisfy a linear PDE of order N. For arbitrary potential this linear PDE takes the form

Linear Compatibility Condition (LCC)

$$\sum_{j=0}^{N-1} (-1)^j \, \partial_x^{N-1-j} \, \partial_y^j \left[(j+1) \, f_{j+1,0} \, \partial_x V \; + \; (N-j) \, f_{j,0} \, \partial_y V \right] \; = \; 0$$

This is a necessary (not sufficient) condition for the existence of the integral Y. The functions $f_{j,0}$ do not depend on the potential

$$f_{j,0} \; = \sum_{n=0}^{N-j} \sum_{m=0}^{j} \binom{N-m-n}{j-m} A_{N-m-n,m,n} \; x^{N-j-n} \; (-y)^{j-m}$$

and they are completely determined by the coefficients $A_{N-m-n,m,n}$ of $Y^{(N)}$.

- For $R(r)=0$, certain constants among the $B^{(1)}_{N-s-2k,s,k}$ and $B^{(2)}_{N-s-2k,s,k}$ that define the N^{th}-order terms $Y^{(N)}$ of the integral Y do not appear in the LCC.
- Hence, the classical N^{th}-order terms $Y^{(N)}$ in the integral Y can be decomposed into two parts

$$Y^{(N)} = Y_I^{(N)} + Y_{II}^{(N)}$$

$$Y_I^{(N)}=\sum_{N-1\leq s+2k\leq N} L_z^{N-s-2k}\, P^{s+2k}\left[B^{(1)}_{N-s-2k,s,k}\cos s\,\Theta + B^{(2)}_{N-s-2k,s,k}\sin s\,\Theta\right],$$

$$Y_{II}^{(N)}=\sum_{0\leq s+2k\leq N-2} L_z^{N-s-2k}\, P^{s+2k}\left[B^{(1)}_{N-s-2k,s,k}\cos s\,\Theta + B^{(2)}_{N-s-2k,s,k}\sin s\,\Theta\right],$$

We define two cases:

Standard potentials

$Y_I^{(N)} \neq 0$: this case corresponds to Standard potentials for which the angular component $S(\theta)$ satisfies the LCC.

Exotic potentials

$Y_{II}^{(N)} \neq 0$: this situation corresponds to the Exotic potentials where the function $S(\theta)$ satisfies non-linear equations rather than linear ones. Unlike the previous case, here the LCC is trivially satisfied.

Quantum case

$$Y_I^{(N)} = \sum_{N-1 \leq s+2k \leq N} \left\{ L_z^{N-s-2k},\ P^{2k} \left(B^{(1)}_{N-s-2k,s,k} \left[(p_x + i\, p_y)^s \right]_{Re} + B^{(2)}_{N-s-2k,s,k} \left[(p_x + i\, p_y)^s \right]_{Im} \right) \right\},$$

$$Y_{II}^{(N)} = \sum_{0 \leq s+2k \leq N-2} \left\{ L_z^{N-s-2k},\ P^{2k} \left(B^{(1)}_{N-s-2k,s,k} \left[(p_x + i\, p_y)^s \right]_{Re} + B^{(2)}_{N-s-2k,s,k} \left[(p_x + i\, p_y)^s \right]_{Im} \right) \right\}.$$

$Y^{(3)}$ is composed of the following terms:

$$\begin{matrix} C_1 \{L_z^2,\ p_x\} \\ C_2 \{L_z^2,\ p_y\} \end{matrix}\ ,\quad D_0\, L_z^3,\quad B_0\, L_z\,(p_x^2 + p_y^2),$$

$$\begin{matrix} A_3\, p_x(p_x^2 + p_y^2) \\ A_4\, p_y(p_x^2 + p_y^2) \end{matrix}\ ,\quad \begin{matrix} B_1 \{L_z, (p_x^2 - p_y^2)\} \\ B_2 \{L_z,\ 2p_x\, p_y\} \end{matrix}\ ,\quad \begin{matrix} A_1\,(p_x^3 - 3p_x p_y^2) \\ A_2\,(3p_x^2 p_y - p_y^3) \end{matrix}\ ,$$

Notation:

Exotic constants

$$B^{(1)}_{3,0,0} = B^{(2)}_{3,0,0} \equiv D_0 \equiv A_{3,0,0}\,, \qquad (s = 0, k = 0)$$

$$B^{(1)}_{2,1,0} \equiv C_1 \equiv A_{2,1,0}\,, \quad B^{(2)}_{2,1,0} \equiv C_2 \equiv A_{2,0,1}\,, \qquad (s = 1, k = 0)$$

Standard constants

$$B^{(1)}_{1,0,1} = B^{(2)}_{1,0,1} \equiv B_0 \equiv \frac{A_{1,2,0} + A_{1,0,2}}{2}, \qquad (s = 0, k = 1)$$

$$\left.\begin{array}{l} B^{(1)}_{0,1,1} \equiv A_3 \equiv \frac{3A_{0,3,0}+A_{0,1,2}}{4}, \\ B^{(2)}_{0,1,1} \equiv A_4 \equiv \frac{3A_{0,0,3}+A_{0,2,1}}{4}, \end{array}\right\} \qquad (s = 1, k = 1)$$

$$\left.\begin{array}{l} B^{(1)}_{1,2,0} \equiv B_1 \equiv \frac{A_{1,2,0}-A_{1,0,2}}{2}, \\ B^{(2)}_{1,2,0} \equiv B_2 \equiv \frac{A_{1,1,1}}{2}, \end{array}\right\} \qquad (s = 2, k = 0)$$

$$\left.\begin{array}{l} B^{(1)}_{0,3,0} \equiv A_1 \equiv \frac{A_{0,3,0}-A_{0,1,2}}{4}, \\ B^{(2)}_{0,3,0} \equiv A_2 \equiv \frac{A_{0,2,1}-A_{0,0,3}}{4}, \end{array}\right\} \qquad (s = 3, k = 0)$$

- $[H, Y^3]$ is a 4^{th}-oder operator in general.
- relevant information is coming from 2^{nd} and 0^{th}-order terms. (*i.e.;* there exist 4 determining equations)
- depending on the fact that which constants appear in the LCC we obtain and classify various potentials.
- most of the doublets do not mixed in the determining equations

Standard Potentials

$$T_1(\theta) = \frac{s_1 \sin(\theta) + s_2 \cos(\theta) + s_3 \sin(3\theta) + s_4 \cos(3\theta)}{A_4 \cos(\theta) - A_3 \sin(\theta) + 3(A_2 \cos(3\theta) - A_1 \sin(3\theta))},$$

$$T_2(\theta) = \frac{s_1 + s_2 \cos(2\theta) + s_3 \sin(2\theta)}{B_1 \cos(2\theta) + B_2 \sin(2\theta)} + s_4,$$

Only one nonlinear determining equation is left and that fixes the parameters $s_1, s_2, \ldots$.

In many cases we recover the TTW systems but for some of the solutions we obtain pure quantum potentials (proportional to $\hbar^2$) which cannot be reduced or transformed to TTW systems.

Exotic Potentials

- for D_0 we obtain a potential expressible in terms of Weierstrass elliptic function, however,

$$\left(\frac{Y}{2}\right)^2 = 8\left(\frac{X}{2}\right)^3 - c_1\frac{\hbar^4}{4}X + c_2\frac{\hbar^6}{4}.$$

- We have a 4^{th}-order nonlinear equation for (C_1, C_2).
- After making the change of variables $z = \tan(\theta)$ and with the help of transformation $(z, T(z)) \longrightarrow (x, W(x))$ where $z = \frac{2\sqrt{x}\sqrt{1-x}}{1-2x}$, we arrived the derivative of the first canonical subcase of the Cosgrove's master Painlevé equation

$$T(x) = \hbar^2\left[\frac{W(x)}{\sqrt{x}\sqrt{1-x}} + \gamma\frac{(1-2x)}{4\sqrt{x}\sqrt{1-x}}\right], x \equiv \begin{cases} \cos^2[\frac{\theta}{2}] \\ \sin^2[\frac{\theta}{2}], \end{cases}$$

with $\gamma = (\gamma_2 + \gamma_4) - (\gamma_1 + \gamma_3) + \sqrt{2\gamma_1} - \frac{3}{4}$ and $(\gamma_2 + \gamma_3)(\gamma_1 + \gamma_4 - \sqrt{2\gamma_1}) = 0$.

In the above potential $W(x)$ is expressed in terms of Painlevé transcendent P_6 as

$$W(x\,;\gamma_1,\gamma_2,\gamma_3,\gamma_4) = \frac{x^2(x-1)^2}{4P_6(P_6-1)(P_6-x)}\left[P_6' - \frac{P_6(P_6-1)}{x(x-1)}\right]^2 + \frac{1}{8}(1-\sqrt{2\gamma_1})^2(1-2P_6) - \frac{1}{4}\gamma_2\left(1-\frac{2x}{P_6}\right) - \frac{1}{4}\gamma_3\left(1-\frac{2(x-1)}{P_6-1}\right) + \left(\frac{1}{8}-\frac{\gamma_4}{4}\right)\left(1-\frac{2x(P_6-1)}{P_6-x}\right),$$

with $\gamma_1, \gamma_2, \gamma_3$ and γ_4 are the parameters that define the sixth Painlevé transcendent P_6 which satisfies the well known second order differential equation:

$$P_6'' = \frac{1}{2}\left[\frac{1}{P_6}+\frac{1}{P_6-1}+\frac{1}{P_6-x}\right](P_6')^2 - \left[\frac{1}{x}+\frac{1}{x-1}+\frac{1}{P_6-x}\right]P_6' + \frac{P_6(P_6-1)(P_6-x)}{x^2(x-1)^2}\left[\gamma_1 + \frac{\gamma_2\, x}{P_6^2} + \frac{\gamma_3\,(x-1)}{(P_6-1)^2} + \frac{\gamma_4\, x(x-1)}{(P_6-x)^2}\right].$$

This potential expressed in terms of Painlevé transcendent is the most interesting one.

- It provides one of the relationships between the theory of quantum superintegrable systems and soliton theory *i.e.* the theory of infinite dimensional integrable systems, usually described by nonlinear partial differential equations that are compatibility conditions for certain linear equations obtained from Lax pairs.
- The "Painlevé conjecture" states that all reductions of soliton type equations to ordinary differential equations should have the Painlevé property *i.e.* should be single-valued about movable singularities.

- The 6 Painlevé transcendents were discovered in a study of second order ODEs with the Painlevé property and they are the only equations of the studied class that can not be expressed in terms of elliptic functions, or solutions of linear differential equations.
- The Painlevé equations come up as solutions of many of the nonlinear equations of mathematical physics, such as the Korteweg-de-Vries, Boussinesq or Kadomtsev-Petviashvili to name just a few examples.
- Here it appears as superintegrable potentials in the linear Schrödinger equation in quantum mechanics.

$Y^{(4)}$ is composed of the following terms:

$$\begin{matrix} A_1\,\{L_z^3,\ p_x\} \\ A_2\,\{L_z^3,\ p_y\} \end{matrix},\quad \begin{matrix} B_3\,\{L_z^2,(p_x^2-p_y^2)\} \\ B_4\,\{L_z^2,,2p_x\,p_y\} \end{matrix},\quad \begin{matrix} L_z^4, \\ \{L_z^2,(p_x^2+p_y^2)\}, \end{matrix}\quad (p_x^2+p_y^2)^2,$$

$$\begin{matrix} A_3\,\{L_z,p_x(p_x^2+p_y^2)\} \\ A_4\,\{L_z,p_y(p_x^2+p_y^2)\} \end{matrix},\quad \begin{matrix} B_1\,(p_x^2-p_y^2)(p_x^2+p_y^2) \\ B_2\,2p_x\,p_y(p_x^2+p_y^2) \end{matrix},$$

$$\begin{matrix} C_1\,\{L_z,(3p_x^2p_y-p_y^3)\} \\ C_2\,\{L_z,(p_x^3-3p_xp_y^2)\} \end{matrix},\quad \begin{matrix} D_1\,(p_x^4+p_y^4-6p_x^2p_y^2) \\ D_2\,4p_xp_y(p_x^2-p_y^2) \end{matrix},$$

$$\begin{pmatrix} s & k \\ 0 & 0 \\ 0 & 1 \\ 1 & 0 \\ 2 & 0 \end{pmatrix},\qquad \begin{pmatrix} s & k \\ 0 & 2 \\ 1 & 1 \\ 2 & 1 \\ 3 & 0 \\ 4 & 0 \end{pmatrix},$$

- $[H, Y^4]$ is a 5^{th}-oder operator in general.
- relevant information is coming from 3^{rd} and 1^{st}-order terms. (*i.e.;* there exist 6 determining equations)
- depending on the fact that which constants appear in the LCC we obtain and classify various potentials.
- most of the doublets do not mixed in the determining equations

Standard Potentials

$$T_1(\theta) = \frac{s_1 + s_2\cos(2\theta) + s_3\sin(2\theta) + s_4\cos(4\theta) + s_5\sin(4\theta)}{B_2\cos(2\theta) - B_1\sin(2\theta) + 2(D_2\cos(4\theta) - D_1\sin(4\theta))},$$

$$T_2(\theta) = \frac{s_1 + s_2\cos(\theta) + s_3\sin(\theta) + s_4\cos(3\theta) + s_5\sin(3\theta)}{A_3\cos(\theta) + A_4\sin(\theta) + C_2\cos(3\theta) + C_1\sin(3\theta)},$$

Only one nonlinear determining equation is left and that fixes the parameters $s_1, s_2, \ldots$.

In many cases we recover the TTW systems but for some of the solutions we obtain pure quantum potentials (proportional to $\hbar^2$) which cannot be reduced or transformed to TTW systems.

As an example let us investigate $T_1(\theta)$ in detail. Introducing it into the nonlinear determining equation we obtain the following solutions

$$s_1^{(\ell)} = \frac{q_\ell\,\hbar^2}{4\,D_1^2\,(B_2^2-8D_1^2)^{\,2}D_2^2}\Big[B_2D_2\,(8D_1^2-B_2^2)\,(B_2^2\,(D_1^2+D_2^2)+8D_1^2\,(4D_1^2+3D_2^2))$$

$$+\;(8B_2^2\,(D_1^5-6D_1^3D_2^2)-B_2^4\,(D_1^3+3D_2^2D_1)+64\,(2D_1^2+D_2^2)\,D_1^5)\;q_\ell$$

$$B_2D_1^2D_2\,(3B_2^2+40D_1^2)\,q_\ell^2\;-\;D_1^3\,(B_2^2+8D_1^2)\;q_\ell^3\Big]\;,$$

$$s_2^{(\ell)} = q_\ell\,\hbar^2\;,$$

$$s_3^{(\ell)} = \frac{q_\ell\,\hbar^2}{D_1\,D_2^2\,(B_2^2-8D_1^2)^{\,2}}\Big[D_2\,(B_2^2-8D_1^2)\,(3B_2^2\,(D_1^2+D_2^2)+8D_1^2\,(2D_1^2+D_2^2))$$

$$+\;(2B_2^3\,(D_1^3+4D_2^2D_1)-32B_2D_1^5)\;q_\ell\;+\;D_1^2D_2\,(7B_2^2+8D_1^2)\;q_\ell^2\;+\;2\,B_2\,D_1^3\,q_\ell^3\Big]\;,$$

$$s_4^{(\ell)} = \frac{q_\ell\,\hbar^2}{D_2^2\,(B_2^2-8D_1^2)^{\,2}}\Big[4\,B_2D_2\,(D_1^2+2D_2^2)\,(B_2^2-8D_1^2)$$

$$+\;2\,D_1\,(B_2^2\,(D_1^2+6D_2^2)-16D_1^2\,(D_1^2+D_2^2))\;q_\ell\;+\;8\,B_2\,D_1^2\,D_2\,q_\ell^2\;+\;2\,D_1^3\,q_\ell^3\Big]\;,$$

$$s_5 = 0\;,$$

where q_ℓ, $\ell = 1, 2, 3, 4$, are the four roots of the quartic equation

$$\hbar^8 \left[D_1^4\, q^4 \;+\; 4\, B_2\, D_2\, D_1^3\, q^3 + \left(B_2^2 \left(D_1^4 + 6D_2^2 D_1^2\right) \;-\; 16 D_1^4 \left(D_1^2 + D_2^2\right)\right) q^2 \right.$$
$$\left. -\; 2\, B_2\, D_1 \,\left(8D_1^2 - B_2^2\right) D_2 \left(D_1^2 + 2D_2^2\right)\, q \;+\; \left(B_2^2 - 8D_1^2\right)^2 D_2^2 \left(D_1^2 + D_2^2\right)\right] \;=\; 0\,,$$

whose discriminant is

$$\Gamma \;=\; -256\,\hbar^{48}\, D_1^{24}\, D_2^2\, \left(B_2^2 - 8D_1^2\right)^2 \Big[B_2^4 \left(60 D_2^2 - 48 D_1^2\right)$$
$$+\; 768 B_2^2 \left(D_1^2 + D_2^2\right)^2 + B_2^6 - 4096 \left(D_1^2 + D_2^2\right)^3\Big]\,.$$

The above solutions are obtained for $\Gamma \neq 0$. Such discriminant is zero if and only if at least two roots are equal. If the discriminant is negative there are two real roots and two complex conjugate roots. If it is positive the roots are either all real or all non-real. From a physical point of view we consider only real solutions. In general, we obtain an angular component $S_I(\theta)$ proportional to $\hbar^2$ with no classical analog, it cannot be transformed or reduced to that of the TTW model.

In particular, the discriminant vanishes for $\hbar=0$. The highest order terms in the nonlinear determining equation are proportional to $\hbar^2$, therefore the limit $\hbar\to 0$ is singular and the above solutions are no longer valid for $\hbar=0$.
Now, let us analyze the zeros of the discriminant
Case I: $\hbar=0$
For $\hbar=0$, non-trivial solutions exist only for $B_2=0$. The corresponding coefficients take the values

$$s_1=s_1\ ,\qquad s_2=0\ ,\qquad s_3=0\ ,\qquad s_4=s_4\ ,\qquad s_5=s_5\ ,$$

which yields the potential

$$S_I(\theta)=\frac{4\,(D_1\cos 4\,\theta+\ D_2\sin 4\,\theta)s_1+4\,(D_1s_5+\ D_2s_4)}{({D_1}^2-{D_2}^2)\cos\,8\,\theta+2\,D_1D_2\sin\,8\,\theta-({D_1}^2+{D_2}^2)}\,.$$

We know that the angular component of the TTW potential is

$$S_{\mathrm{TTW}}(\theta) = \frac{\alpha\, k^2}{\cos^2(k\theta)} + \frac{\beta\, k^2}{\sin^2(k\theta)} = \frac{4\,k^2\,(\alpha\,-\,\beta)\cos 2\,k\,\theta \;-\; 4\,k^2(\alpha\,+\,\beta\,)}{\cos 4\,k\,\theta \;-\; 1}\,.$$

In this potential, it is easy to check that for

$$\theta \to \theta + \frac{1}{4}\arctan(-D_2/D_1)\,, \qquad \alpha = -\frac{-\sqrt{D_1^2 + D_2^2}\,s_1 + D_1 s_4 + D_2 s_5}{8\,(D_1^2 + D_2^2)}\,,$$

$$\beta = -\frac{D_1^2 s_1 + \sqrt{D_1^2 + D_2^2}\,D_1 s_4 + D_2\left(D_2 s_1 + \sqrt{D_1^2 + D_2^2}\,s_5\right)}{8\,(D_1^2 + D_2^2)^{3/2}}\,, \qquad k = 2$$

we recover $S_I(\theta)$. Therefore, $S_I(\theta)$ corresponds to a rotated TTW model (with no radial component $R(r) = 0$) which is a superintegrable system both in the classical and quantum cases.

Case II: $D_1 = 0$

The corresponding coefficients vanish, $s_1 = s_2 = s_3 = s_4 = s_5 = 0$, which gives the trivial solution $S_I(\theta) = 0$.

Case III: $D_2 = 0$

The corresponding coefficients are given by

$$s_1 = s_1 \,, \quad s_2 = 0 \,, \quad s_3 = \frac{B_2^2\, s_4 - 8\, D_1^2\, (s_1 + s_4)}{2\, B_2\, D_1} \,, \quad s_4 = s_4 \,, \quad s_5 = 0 \,,$$

thus

$$S_I(\theta) \;=\; -\frac{2(B_2\, s_4 + 2\, D_1\, s_1\, \sin 2\,\theta + 2\, D_1\, s_4\, \sin 2\,\theta)}{B_2\, D_1\, (1 + \cos 4\,\theta)} \,.$$

This solution corresponds to the angular component of the TTW model with $k = 1$.

Case IV: $B_2^2 - 8\,D_1^2 = 0$

For simplicity, we put $D_1 = D_2 = 1$ thus $B_2 = \sqrt{8}$. In this case, the coefficients

$$s_1 = 0\,,\quad s_2 = -2\,\sqrt{2}\,\hbar^2\,,\quad s_3 = 2\,\sqrt{2}\,\hbar^2\,,\quad s_4 = -4\,\hbar^2\,,\quad s_5 = 0\,,$$

lead to

$$S_I(\theta) \;=\; 2\,\hbar^2\,\frac{\sqrt{2}\cos 6\theta - \sqrt{2}\sin 6\theta + 2}{\left[\cos 4\theta - \sin 4\theta + \sqrt{2}\cos 2\theta\right]^2}\,,$$

which is a pure quantum potential. It can not be reduced to that of the TTW model.
A similar analysis can be done for the last factor of the discriminant.

Exotic Potentials

- we have 2 separate 5^{th}-order nonlinear equations for the doublets (A_1, A_2) and (B_3, B_4).
- Each can be integrated once and after making the change of variables $z = \tan(\theta)$ or $z = \tan(2\theta)$ can be integrated once more.
- making one more transformation $(z, T(z)) \longrightarrow (x, W(x))$ where

$$z = \frac{2\sqrt{x}\sqrt{1-x}}{1-2x},$$

we arrived the derivative of the first canonical subcase of the Cosgrove's master Painlevé equation.

We have the following exotic potentials

$$T_1(x) \;=\; \hbar^2 \left[\frac{W(x)}{\sqrt{x}\sqrt{1-x}} \;+\; \gamma\,\frac{(1-2\,x)}{4\,\sqrt{x}\,\sqrt{1-x}}\right]$$

where

$$x \;\equiv\; \begin{cases} \cos^2[\frac{\theta}{2}] \\ \\ \sin^2[\frac{\theta}{2}]\;, \end{cases}$$

and

$$T_2(x) \;=\; \hbar^2\,2 \left[\frac{W(x)}{\sqrt{x}\sqrt{1-x}} \;+\; \gamma\,\frac{(1-2\,x)}{4\,\sqrt{x}\,\sqrt{1-x}}\right]$$

where

$$x \;\equiv\; \begin{cases} \cos^2[\theta] \\ \sin^2[\theta]\;, \end{cases}$$

with

$$\gamma = (\gamma_2+\gamma_4)-(\gamma_1+\gamma_3)+\sqrt{2\,\gamma_1}-\frac{3}{4}$$

$Y^{(5)}$ is composed of the following terms:

$$\begin{matrix} A_1 \, \{L_z^4, \, p_x\} \\ A_2 \, \{L_z^4, \, p_y\} \end{matrix}, \quad \begin{matrix} B_1 \, \{L_z^3, (p_x^2 - p_y^2)\} \\ B_2 \, \{L_z^3, , 2p_x \, p_y\} \end{matrix}, \quad \begin{matrix} C_1 \, \{L_z^2, (p_x^3 - 3p_x p_y^2)\} \\ C_2 \, \{L_z^2, (3p_x^2 p_y - p_y^3)\} \end{matrix},$$

$$\begin{matrix} F_1 \, \{L_z^2, p_x(p_x^2 + p_y^2)\} \\ F_2 \, \{L_z^2, p_y(p_x^2 + p_y^2)\} \end{matrix}, \quad \begin{matrix} A_0 L_z^5, \\ M_1\{L_z^3, (p_x^2 + p_y^2)\}, \end{matrix} \quad M_2\{L_z, (p_x^2 + p_y^2)^2\},$$

$$\begin{matrix} K_1 \, p_x(p_x^2 + p_y^2)^2 \\ K_2 \, p_y(p_x^2 + p_y^2)^2 \end{matrix}, \quad \begin{matrix} G_1 \, \{L_z, (p_x^2 - p_y^2)(p_x^2 + p_y^2)\} \\ G_2 \, \{L_z, 2p_x \, p_y(p_x^2 + p_y^2)\} \end{matrix},$$

$$\begin{matrix} H_1 \, (p_x^3 - 3p_x p_y^2)(p_x^2 + p_y^2) \\ H_2 \, (3p_x^2 p_y - p_y^3)(p_x^2 + p_y^2) \end{matrix}, \quad \begin{matrix} D_1 \, \{L_z, (p_x^4 + p_y^4 - 6p_x^2 p_y^2)\} \\ D_2 \, \{L_z, 4p_x p_y(p_x^2 - p_y^2)\} \end{matrix},$$

$$\begin{matrix} E_1 \, (p_x^5 - 10p_x^3 p_y^2 + 5p_x p_y^4) \\ E_2 \, (5p_x^4 p_y - 10p_x^2 p_y^3 + p_y^5) \end{matrix},$$

$N = 5$

$$\begin{pmatrix} s & k \\ 0 & 0 \\ 0 & 1 \\ 1 & 0 \\ 1 & 1 \\ 2 & 0 \\ 3 & 0 \end{pmatrix}, \qquad \begin{pmatrix} s & k \\ 0 & 2 \\ 1 & 2 \\ 2 & 1 \\ 3 & 1 \\ 4 & 0 \\ 5 & 0 \end{pmatrix},$$

$N = 5$

- $[H, Y^5]$ is a 6^{th}-oder operator in general.
- relevant information is coming from 4^{th}, 2^{nd} and 0^{th}-order terms. (*i.e.;* there exist 9 determining equations)
- depending on the fact that which constants appear in the LCC we obtain and classify various potentials.
- most of the doublets do not mixed in the determining equations

Standard Potentials

$$T_1(\theta) = \frac{s_1 \sin\theta + s_2 \cos\theta + s_3 \sin 3\theta + s_4 \cos 3\theta + s_5 \sin 5\theta + s_6 \cos 5\theta}{K_\theta + H_\theta + E_\theta},$$

where $K_\theta = K_2 \cos\theta - K_1 \sin\theta$, $H_\theta = 3(H_2 \cos 3\theta - H_1 \sin 3\theta)$ and $E_\theta = 5(E_2 \cos 5\theta - E_1 \sin 5\theta)$.

$$T_2(\theta) = \frac{s_1 + s_2 \cos(2\theta) + s_3 \sin(2\theta) + s_4 \cos(4\theta) + s_5 \sin(4\theta)}{G_1 \cos(2\theta) + G_2 \sin(2\theta) + D_1 \cos(4\theta) + D_2 \sin(4\theta)} + s_6,$$

The parameters $s_1, s_2, \ldots$ are determined by the remaining nonlinear determining equations

In many cases we recover the TTW systems but for some of the solutions we obtain pure quantum potentials (proportional to $\hbar^2$) which cannot be reduced or transformed to TTW systems.

Exotic Potentials (Singlets)

- Exotic Singlet M_1 $(L_z^3(p_x^2 + p_y^2))$
  This constant appears in two nonlinear differential equations of order 6 and 8. We find a compatibility condition which factors into

$$\left(16(T')^2 + 3(T'')^2 - 2T'T^{(3)}\right)\left(\hbar^2 T^{(4)} - 12T'T''\right) = 0$$

$$\left[L_z^3(p_x^2 + p_y^2)\right]^2 = X^3H^2, \quad \left[L_z^3(p_x^2 + p_y^2)\right]^2 = (L_z^3)^2H^2,$$

- Exotic Singlet A_0 L_z^5
  We have only one nonlinear differential equation and it can easily be verified that Weierstrass zeta function is a solution.

$$L_z^5 = L_z^3X.$$

Exotic Potentials (Doublets)

- We have 3 sets of nonlinear differential equations each of which involves 2 equations
  - for (A_1, A_2)
  - for (B_1, B_2)
  - for (C_1, C_2) and (F_1, F_2)

$$[L_z^4 p_x] = Y^{(3)} X, \qquad Y^{(3)} = [L_z^2 p_x],$$

$$[Y^{(5)}]^2 = [Y^{(4)}]^2 X, \; Y^{(5)} = [L_z^3 (p_x^2 - p_y^2)], \; Y^{(4)} = [L_z^2 (p_x^2 - p_y^2)],$$

$$Y^{(5)} = [Y^{(3)}] H, \qquad Y^{(5)} = [L_z^2 p_x (p_x^2 + p_y^2)],$$

New Exotic Potentials for (C_1, C_2).

- There are 2 nonlinear equations of order 6 and 8.
- We find a compatibility condition which factors into

$$\left(4(T')^2 + 3(T'')^2 - 2T'T^{(3)}\right)\Phi(\theta) = 0$$

- $\Phi(\theta)$ is a 4^{th}-order nonlinear equation
- After making the change of variables $z = \tan(3\theta)$ and with the help of transformation $(z, T(z)) \longrightarrow (x, W(x))$ where $z = \frac{2\sqrt{x}\sqrt{1-x}}{1-2x}$, we arrived the derivative of the first canonical subcase of the Cosgrove's master Painlevé equation

$$T(x) = 3\hbar^2\left[\frac{W(x)}{\sqrt{x}\sqrt{1-x}} + \gamma\frac{(1-2x)}{4\sqrt{x}\sqrt{1-x}}\right], x \equiv \begin{cases} \cos^2[\frac{3\theta}{2}] \\ \sin^2[\frac{3\theta}{2}], \end{cases}$$

with $\gamma = (\gamma_2 + \gamma_4) - (\gamma_1 + \gamma_3) + \sqrt{2\gamma_1} - \frac{3}{4}$ and $(\gamma_2 + \gamma_3)(\gamma_1 + \gamma_4 - \sqrt{2\gamma_1}) = 0$.

- Superintegrable Hamiltonians in classical and quantum mechanics differ. Terms depending on $\hbar$ appear in the quantum case. The classical limit $\hbar \to 0$ is singular and must be taken in the determining equations, not in the solutions.
- Two types of potentials occur which we call standard and exotic. Standard ones are solutions of a linear compatibility condition for the determining equations. For exotic potentials the linear compatibility condition is satisfied trivially so the potentials satisfy nonlinear equations. In quantum mechanics the nonlinear equations pass the Painlevé test, in the classical case they do not.
- The integrals of motion H, X and Y satisfy $[H, X] = [H, Y] = 0$, $[X, Y] = C \neq 0$, where $[\cdot, \cdot]$ denotes a Lie bracket in quantum mechanics and a Poisson bracket in the classical case. Further commutations like $[X, C], [Y, C], \ldots,$ yield a finite dimensional polynomial Lie or Poisson algebra. In many cases for $N = 2, \ldots, 5$ it turns out that the commutators $[X, C] = D_1$, $[Y, C] = D_2$ are polynomials in X, Y and H with constant coefficients.

	EXOTIC POTENTIALS	STANDARD POTENTIALS
	${Y_I}^{(N)} = 0$	${Y_{II}}^{(N)} = 0$
CLASSICAL	$T(\theta)$ *satisfies an algebraic polynomial equation*	*TTW System*
QUANTUM	$T(\theta)$ *satisfies an ODE with the Painlevé property*	*TTW System + Q System*

For the standard potentials $Y_{II}^{(N)} = 0$, the TTW and PW systems are fully contained in the angular component $S(\theta) \equiv T'(\theta)$ both in classical and quantum cases. Moreover, in the quantum case there exist more potentials, proportional to $\hbar^2$, which cannot be reduced to TTW (PW). For the exotic potentials $Y_I^{(N)} = 0$, the angular component $S(\theta) \equiv T'(\theta)$ satisfies an algebraic equation in the classical case and in the quantum case it is given by the solution of a non-linear ODE that has the Painlevé property.