Introduction to the Theory of Clifford Algebras

Dmitry Shirokov[1,2]

[1] Department of Mathematics, Faculty of Economic Sciences, National Research University Higher School of Economics, Moscow, Russia, dshirokov@hse.ru

[2] Kharkevich Institute for Information Transmission Problems, Russian Academy of Sciences, Moscow, Russia, shirokov@iitp.ru

Abstract

In these lectures we give an introduction to the theory of Clifford algebras. Clifford algebra was invented by W. Clifford in 1878. In his research he combined Hamiltons quaternions (1843) and Grassmanns exterior algebra (1844). Further development of the theory of Clifford algebras is associated with a number of famous mathematicians and physicists – R. Lipschitz, T. Vahlen, E. Cartan, E. Witt, C. Chevalley, M. Riesz and others. Dirac equation (1928) had a great influence on the development of Clifford algebra. Clifford algebra is used in different branches of modern mathematics and physics.
In the first two lectures we study the most important structures related to Clifford algebras and give different examples. In the third lecture we study relation between matrix formalism and formalism of Clifford algebras. In the forth lecture we will speak about different Lie groups in Clifford algebras and corresponding Lie algebras. The most important of them are spin groups. In the last lecture we will speak about one of the most important applications of Clifford algebras – Dirac equation for the electron.

Outline

Lecture 1: **Clifford Algebras and Related Structures**

Definition of Clifford algebras. Examples in small dimensions: complex numbers, double numbers, quaternions, Pauli's matrices, Dirac's matrices. Grassmann algebra. Z_2-grading, grade involution, reversion, Clifford conjugation. Center of Clifford algebra.

Lecture 2: **Unitary Spaces on Clifford Algebras**

Hermitian scalar product in Clifford algebras. Operation of Hermitian conjugation and unitary groups in Clifford algebras.

Lecture 3: **Matrix Representations of Clifford Algebras.**

Cartan's periodicity of 8 for Clifford algebras. Faithful and irreducible representations. Primitive idempotents and minimal left ideals.

Lecture 4: **Lie Groups and Lie Algebras in Clifford Algebras.**

Spin groups as subgroups of Clifford and Lipschitz groups. Double covers of the orthogonal groups. Cartan-Dieudonne theorem. Spin groups in small dimensions. Lie groups in Clifford algebras and corresponding Lie algebras.

Lecture 5: **Dirac Equation.**

Dirac equation in Clifford algebras. Dirac-Hestenes equation. Spinors in n dimensions.

References

[1] Clifford W. K., *Application of Grassmann's Extensive Algebra*, American Journal of Mathematics, I: 350-358, (1878)

[2] Lounesto P., *Clifford Algebras and Spinors*, Cambridge Univ. Press, Cambridge (1997).

[3] Hestenes D., Sobczyk G., *Clifford Algebra to Geometric Calculus - A Unified Language for Mathematical Physics*, Reidel Publishing Company (1984).

[4] Porteous I.R., *Clifford Algebras and the Classical Groups*, Cambridge University Press, (1995).

[5] Benn I. M., Tucker R. W., *An introduction to Spinors and Geometry with Applications in Physics*, Bristol, (1987).

[6] Chevalley C., *The algebraic theory of Spinors and Clifford algebras*, Springer, (1996).

[7] Lawson H. B., Michelsohn M. L., *Spin Geometry*, Princeton Math. Ser., 38, Princeton Univ. Press, Princeton, NJ, 1989.

[8] Snygg J., *Clifford Algebra - A Computational Tool For Physicists*, Oxford University Press, New York (1997).

[9] Marchuk N. G., Shirokov D. S., *Introduction to the theory of Clifford algebras* (in Russian), Phasis, Moscow, 2012, 590 pp.

[10] Shirokov D. S., *Lectures on Clifford algebras and spinors* (in Russian), Lects. Kursy NOC 19, Steklov Math. Inst., RAS, Moscow, 2012, 180 pp. http://mi.mathnet.ru/eng/book1373

[11] Marchuk N. G., Shirokov D. S., *Unitary spaces on Clifford algebras*, Adv. Appl. Clifford Algebr., 18:2 (2008), 237-254, arXiv: 0705.1641.