Lectures on Differential Geometry using Clifford's Geometric Algebra

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Abstract

If things were as they should be, Clifford algebra would be a central feature of undergraduate mathematics. Around 1900, there was a debate amongs mathematicians and physicists between those who advocated quaternions and those who advocated vectors for the study of geometry and physics. Quaternions were superior for dealing with rotations. This debate ended in 1916 when Einstein published his general theory of relativity. It was then clear that the study of spaces with dimension higher that three was important both for geometers and physicists. Quaternions would not be sufficient for the task and vectors won.
Absent in this debate was the possibility of using Clifford's geometric algebra which includes both quaternions and vectors as subspaces. William Kingdon Clifford died in 1879 at the age of 33 and was not around to participate in the debate. Clifford was articulate both as a speaker and a writer so his geometric algebra would probably have won out had he lived another 20 or 30 years.
He wrote two papers on the subject before he died. Although Clifford was recognized as one of England's most distinguished mathematicians the two papers were overlooked and ignored. Why? The first paper appeared in the very first issue of the *American Journal of Mathematics.* Now the journal is one of the most widely read math journals in the world but few read that first issue. Why did Clifford choose an obscure journal to publish that paper? In 1876, James Joseph Sylvester was entrusted with the task of constructing the first American doctorial program in mathematics at John Hopkins University in Baltimore. Two years later he founded the *American Journal of Mathematics.* William Kingdon Clifford had a very high regard for Sylvester and presumably decided to give his support to Sylvester's endeavors.
The second paper was published after the death of Clifford in unfinished form as part of his collected papers but this too was largely overlooked.
In 1946 and 1958, Marcel Riesz published some results of Clifford algebra that stimulated David Hestenes to investigate the subject. In 1966, David Hestenes published a thin volume entitled *Space-Time Algebra.* And 18 years later, with his student Garrett Sobczyk, he wrote a more extensive work entitled *Clifford Algebra to Geometric Calculus - A Unified Language for Mathematics and Physics.* Since then, extensive research has been carried out in Clifford algebra with a multitude of applications.ctor
During the twentieth century, the formalism of tangent vectors (directional derivatives) and differential forms has become the preferred framework for the study of Riemannian and quasi-Riemannian spaces. Perhaps there is a necessity for using this cumbersome formalism for spaces without a metric. However for spaces with a metric, the formalism of Clifford algebra is conceptually much simpler and in my opinion, more powerful.
In the following five lectures, I will try to persuade you that indeed, the formalism of Clifford algebra is useful. In the third lecture, I will show how a vector that is parallel transported around an infintesimal loop undergoes an infintesimal rotation that is determined by the Riemann curvature tensor.

Outline

Lecture 1. What is a Clifford algebra?

1.1 Definition of a Clifford algebra with a not necessarily positive definite metric.

1.2 Generalized reflections and rotations.

1.3 Coordinate bases.

Lecture 2. The Geodesic Derivative and Riemann 2-forms.

2.1 The geodesic derivative compared with the covariant derivative.

2.2 Moving frames and Fock-Ivanenko coefficients.

2.3 A formula for the Riemann 2-form in terms of the Fock-Ivanenko coefficients

Lecture 3. The Rotation of an object in n-dimensions when it is parallel transported about an infintesimal coordinate loop.

3.1 Suppose one starts at the North pole, walks south to the equator along a line of constant longitude, walks several degrees east along the equator , and then finally

returns to the North pole by walking directly North. If one holds a spear pointing directly south (or any other fixed direction), one will discover that the spear has undergone

a rotation. I shall demonstrate how this can be generalized in n-dimensional spaces at least for infintesimal coordinate loops. The result will be expressed as a product of

infintesimal rotation operators expressed in terms of Riemannian 2-forms.

Lecture 4. Reflections, Rotations, and a new distance formula for the Poincaré disk.

4.1 This lecture will not use any Clifford algebra. I will discuss the general nature of the Poincaré disk model for non-Euclidean geometry. In the process I will introduce a

distance formula which is much easier to compute than that which is traditionally used.

References

[1] Brannan D., *Geometry*, M. Esplen and J. Gray (Eds), Cambridge University Press, Cambridge 1999

[2] Chisholm M., *Such Silver Currents: The Story of William and Lucy Clifford 1845 - 1929*, Lutterworth Press, Cambridge 2002.

[3] Clifford W., *Applications of Grassmann's extensive algebra*, American Journal of Mathematics, vol. **1** 1878 350-358. (Republished in *Mathematical Papers by William Kingdon Clifford*, R. Tucker (Ed), Macmillan and Co. London 1882, pp 266-276; Reprinted *Mathematical Papers by William Kingdon Clifford* by Chelsea, New York 1968).

[4] Clifford W., *On the Classification of Geometric Algebras* Published posthumously in unfinished form as paper XL.II In: Mathematical Papers by William Kingdon Clifford, R. Tucker (Ed), Macmillan, London 1882 397-401.

[5] Hestenes D., *Space-Time Algebra*, Gordon and Breach, New York 1966.

[6] Hestenes D. and Garret S., *Clifford Algebra to Geometric Calculus - A Unified Language for Mathematical Physics*, Reidel, Dordrecht 1984.

[7] Lounesto P., *Clifford Algebras and Spinors*, Cambridge University Press, Cambridge 1997.

[8] Riesz M., *Clifford Numbers and Spinors*, E. Bolinder and P. Lounesto (Eds), Kluwer, Dordrecht 1993.

[9] Snygg J., *Clifford Algebra - A Computational Tool For Physicists*, Oxford University Press, New York 1997.

[10] Snygg J., *A New Approach to Differential Geometry Using Clifford's Geometric Algebra*, Birkhauser, New York 2010.