Appendix A
Elliptical Integrals and Functions

Elliptic integrals and functions are mathematical objects, which nowadays are often omitted in the mathematical curricula of universities. One quite trivial explanation is the presence of plenty of efficient computational programs that can be implemented on modern computers. While the standard integration techniques allow us to obtain explicit expressions (in terms of trigonometric, exponential and logarithmic functions) for every integral of the form

$$\int \mathcal{R}(x, \sqrt{P(x)})\mathrm{d}x,$$

where $\mathcal{R}(x, \sqrt{P(x)})$ is a rational function, and $P(x)$ is a linear or quadratic polynomial, we have to widen our vocabulary of "elementary" functions if we want to work with polynomials of higher degree. In particular, when $P(x)$ is a polynomial of the third or fourth degree, the corresponding function is called **elliptic**. Of course, when teaching calculus, one must stop somewhere, and it is reasonable to stay loyal to well-known linear and quadratic functions, while using numerical methods to calculate integrals of the third and fourth degree. The possibilities of easy-to-use computer systems for symbolic manipulation of the type represented by *Maple*® and `Mathematica`® makes this course of action even more understandable.

The main point in this Appendix is that elliptic functions provide effective means for the description of geometric objects. The second is that the above-mentioned computer programs, through their built-in tools for calculation and visualization, are, in fact, a real motivation for the teaching and using of elliptic functions.

In this Appendix, we will consider a few examples in order to prove that elliptic integrals and functions are necessary to get more interesting geometric and mechanical information than that given by direct numerical calculations.

The history of the development of elliptic functions can be followed in Stillwell (1989). Clear statements of their properties and applications can be found in the books by Greenhill (1959), Hancock (1958), Bowman (1953) and Lawden (1989). A more recent approach to the problem from the viewpoint of dynamical systems is given by Meyer (2001).

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A.1 Jacobian Elliptic Functions

The easiest way to understand elliptic functions is to consider them as analogous to ordinary trigonometric functions. From the calculus, we know that

$$\arcsin(x) = \int_0^x \frac{\mathrm{d}u}{\sqrt{1-u^2}}.$$

Of course, if $x = \sin(t)$, $-\pi/2 \leq t \leq \pi/2$, we will have

$$t = \arcsin(\sin(t)) = \int_0^{\sin(t)} \frac{\mathrm{d}u}{\sqrt{1-u^2}}. \tag{A.1}$$

In this case, we can consider $\sin(t)$ as the inverse function of the integral (A.1). The real understanding of trigonometric functions includes knowledge of their graphs, their connection with other trigonometric functions, such as in $\sin^2(\theta) + \cos^2(\theta) = 1$, and of course, the fundamental geometric and physical parameters in which they are included (i.e., circumferences and periodical movements). We will follow this example for elliptic functions too.

Let us begin by fixing some k, $0 \leq k \leq 1$, which, from now on, will be called an **elliptic modulus** and introduce the following:

Definition A.1 The Jacobi sine function $\mathrm{sn}(u, k)$ is the inverse function of the following integral:

$$u = \int_0^{\mathrm{sn}(u,k)} \frac{\mathrm{d}t}{\sqrt{1-t^2}\sqrt{1-k^2t^2}}. \tag{A.2}$$

More generally, we will call

$$F(z, k) = \int_0^z \frac{\mathrm{d}t}{\sqrt{1-t^2}\sqrt{1-k^2t^2}} \tag{A.3}$$

the elliptic integral of the first kind. The elliptic integrals of the second and third kinds are defined by the equations

$$E(z, k) = \int_0^z \frac{\sqrt{1-k^2t^2}}{\sqrt{1-t^2}}\,\mathrm{d}t \tag{A.4}$$

$$\Pi(n, z, k) = \int_0^z \frac{\mathrm{d}t}{(1+nt^2)\sqrt{(1-t^2)(1-k^2t)}}.$$

When the argument z in $F(z, k)$, $E(z, k)$ and $\Pi(n, z, k)$ is equal to one, these integrals are denoted, respectively, as $K(k)$, $E(k)$ and $\Pi(n, k)$ and called complete elliptic integrals of the first, second and third kinds, respectively.

If we put $t = \sin\phi$, the above integrals are transformed, respectively, into

$$\begin{aligned} F(\phi, k) &= \int_0^\phi \frac{\mathrm{d}\phi}{\sqrt{1 - k^2 \sin^2 \phi}} \\ E(\phi, k) &= \int_0^\phi \sqrt{1 - k^2 \sin^2 \phi}\, \mathrm{d}\phi \\ \Pi(n, \phi, k) &= \int_0^\phi \frac{\mathrm{d}\phi}{(1 + n \sin^2 \phi)\sqrt{1 - k^2 \sin^2 \phi}}. \end{aligned} \tag{A.5}$$

Let us note that when $k \equiv 1$, $E(\phi, 1) = \sin\phi$, and therefore one can consider $E(\phi, k)$ to be a generalization of the function $\sin\phi$.

The Jacobi cosine function $\mathrm{cn}(u, k)$ can be defined in terms of $\mathrm{sn}(u, k)$ by means of the identity

$$\mathrm{sn}^2(u, k) + \mathrm{cn}^2(u, k) = 1. \tag{A.6}$$

The third Jacobi elliptic function $\mathrm{dn}(u, k)$ is defined by the equation

$$\mathrm{dn}^2(u, k) + k^2\, \mathrm{sn}^2(u, k) = 1. \tag{A.7}$$

The integral definition of $\mathrm{sn}(u, k)$ makes it clear that $\mathrm{sn}(u, 0) = \sin(u)$. Of course, $\mathrm{cn}(u, 0) = \cos(u)$ as well.

Besides sn, cn and dn, there are another nine functions that are widely used, and their definitions are given below:

$$\begin{aligned} \mathrm{ns} &= \frac{1}{\mathrm{sn}}, & \mathrm{nc} &= \frac{1}{\mathrm{cn}}, & \mathrm{nd} &= \frac{1}{\mathrm{dn}} \\ \mathrm{sc} &= \frac{\mathrm{sn}}{\mathrm{cn}}, & \mathrm{cd} &= \frac{\mathrm{cn}}{\mathrm{dn}}, & \mathrm{ds} &= \frac{\mathrm{dn}}{\mathrm{sn}} \\ \mathrm{cs} &= \frac{\mathrm{cn}}{\mathrm{sn}}, & \mathrm{dc} &= \frac{\mathrm{dn}}{\mathrm{cn}}, & \mathrm{sd} &= \frac{\mathrm{sn}}{\mathrm{dn}}. \end{aligned}$$

The derivatives of the elliptic functions can be found directly from their definitions (or vice versa, as in Meyer (2001), where the elliptical functions are defined by their derivatives). For instance, the derivative of $\mathrm{sn}(u, k)$ may be computed as follows. In (A.3), suppose that $z = z(u)$. Then,

$$\frac{\mathrm{d}F}{\mathrm{d}u} = \frac{\mathrm{d}F}{\mathrm{d}z}\frac{\mathrm{d}z}{\mathrm{d}u} = \frac{1}{\sqrt{1 - z^2}\sqrt{1 - k^2 z^2}}\frac{\mathrm{d}z}{\mathrm{d}u}.$$

But from (A.2) and (A.3), we know that for $z = \mathrm{sn}(u,k)$, we have $F(z,k) = u$. So, replacing z with $\mathrm{sn}(u,k)$ and using $\mathrm{d}u/\mathrm{d}u = 1$, we obtain

$$1 = \frac{1}{\sqrt{1-\mathrm{sn}(u,k)^2}\sqrt{1-k^2\mathrm{sn}(u,k)^2}}\frac{\mathrm{d\,sn}(u,k)}{\mathrm{d}u}$$
$$\frac{\mathrm{d\,sn}(u,k)}{\mathrm{d}u} = \sqrt{1-\mathrm{sn}(u,k)^2}\sqrt{1-k^2\mathrm{sn}(u,k)^2} \tag{A.8}$$
$$\frac{\mathrm{d\,sn}(u,k)}{\mathrm{d}u} = \mathrm{cn}(u,k)\,\mathrm{dn}(u,k).$$

After differentiation to (A.6) with respect of u and taking into account (A.8), we obtain

$$\frac{\mathrm{d\,cn}(u,k)}{\mathrm{d}u} = -\mathrm{sn}(u,k)\,\mathrm{dn}(u,k), \tag{A.9}$$

Finally, after differentiating (A.7) and using (A.8) once more, we have

$$\frac{\mathrm{d\,dn}(u,k)}{\mathrm{d}u} = -k^2\mathrm{sn}(u,k)\,\mathrm{cn}(u,k). \tag{A.10}$$

Symbolic computational programs such as *Maple*® or `Mathematica`® have embedded modules for working with elliptic functions, so these functions can be easily drawn. Graphs of the elliptic sin function sn, cos function cn and function dn are shown in Fig. A.1. We can see that $\mathrm{sn}(u,k)$ and $\mathrm{cn}(u,k)$ are periodic. We can define their period referring to the definitions above (see A.2)

$$K(k) = \int_0^1 \frac{\mathrm{d}t}{\sqrt{1-t^2}\sqrt{1-k^2t^2}}.$$

We can see that $\mathrm{sn}(K(k),k) = 1$. Obviously, from the graph, we are also convinced that $K(k)$ is $1/4$ of the $\mathrm{sn}(u,k)$ period and that the period of $\mathrm{dn}(u,k)$ is $2K(k)$. Of course, this can be checked analytically (e.g., see Woods (1934), p. 368), but this argument satisfies our objectives.

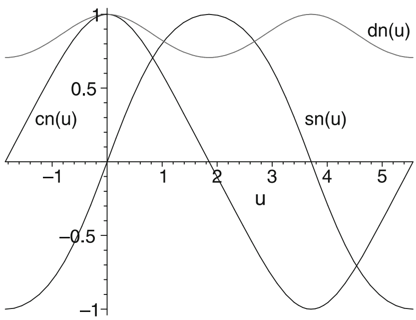

Fig. A.1 Graphs of the elliptic sin function $\mathrm{sn}(u,k)$, elliptic cos function $\mathrm{cn}(u,k)$ and the function $\mathrm{dn}(u,k)$ drawn with $k = \frac{1}{\sqrt{2}}$

Using the computer program, we can look for a numerical solution, from which we can find $K(k)$, i.e., to solve the equation $\mathrm{sn}(u,k)=1$. Note that the equation $\mathrm{sn}^2(u,k)+\mathrm{cn}^2(u,k)=1$ supposes that $\mathrm{cn}(u,k)$ has the same period as $\mathrm{sn}(u,k)$, and therefore $\mathrm{cn}(K(k),k)=0$.

Now we have an idea of the algebraic and graphic properties of the elliptic functions. In order to "complement" our understanding, let us look at two simple examples—one physical and one geometrical.

Example A.1 (Pendulum) Let the angle of a pendulum swing be denoted by x. Then, it is straightforward to derive that the equation of motion is: $\ddot{x}+(g/l)\sin(x)=0$, where g is the acceleration due to gravity and l is the length of the pendulum. If we take such units that give $g/l=1$, the pendulum equation becomes $\ddot{x}+\sin(x)=0$. Then, we can multiply by $\dot{x}$ to obtain

$$\begin{aligned}\dot{x}(\ddot{x}+\sin(x)) &= 0\\ \dot{x}\ddot{x}+2\sin(x)\frac{\dot{x}}{2} &= 0\\ \dot{x}\ddot{x}+4\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)\frac{\dot{x}}{2} &= 0,\end{aligned}$$

and, by integrating the last equation, end up with

$$\frac{1}{2}\dot{x}^2+2\sin^2\left(\frac{x}{2}\right)=c. \tag{A.11}$$

Note also that because of the identity $2\sin^2(x/2)=1-\cos(x)$, the last equation expresses the conservation of energy of the motion of a particle with a unit mass. Now let $z=\sin(x/2)$, and therefore $2\dot{z}=\cos(x/2)\dot{x}=\sqrt{1-z^2}\,\dot{x}$. Then,

$$\begin{aligned}4\dot{z}^2 &= (1-z^2)\dot{x}^2=\dot{x}^2-\sin^2\left(\frac{x}{2}\right)\dot{x}^2=\dot{x}^2\cos^2\left(\frac{x}{2}\right)\\ \dot{z}^2 &= \frac{1}{4}\dot{x}^2\cos^2\left(\frac{x}{2}\right).\end{aligned}$$

By the first part of the calculation, we have

$$\frac{1}{4}\dot{x}^2=\frac{1}{2}c-\sin^2\left(\frac{x}{2}\right)=\frac{1}{2}c-z^2 \qquad \text{and} \qquad \cos^2\left(\frac{x}{2}\right)=1-z^2.$$

Hence, $\dot{z}^2=(A-z^2)(1-z^2)$, where $A=c/2$. Taking a square root and separating the variables gives us

$$t=\int_0^z\frac{\mathrm{d}z}{\sqrt{(A-z^2)(1-z^2)}}=\int_0^{\sqrt{A}u}\frac{\mathrm{d}u}{\sqrt{(1-u^2)(1-Au^2)}}=F(\sqrt{A}u,\sqrt{A}).$$

That is, we see that the elliptic integrals appear even in the most standard of mechanical situations.

Example A.2 (Ellipse) Let us parameterize the ellipse by the polar angle, which we will denote by t, i.e., $\alpha(t) = (x(t), z(t)) = (a\sin(t), c\cos(t))$, where $0 \le t \le 2\pi$, and $a \ge c$. Then, the arclength integral is

$$\begin{aligned} L = \int_0^{2\pi} \sqrt{\dot{x}^2 + \dot{z}^2}\, \mathrm{d}t &= 4\int_0^{\pi/2} \sqrt{a^2\cos^2(t) + c^2\sin^2(t)}\, \mathrm{d}t \\ &= 4a \int_0^{2\pi} \sqrt{1 - \varepsilon^2 \sin^2(t)}\, \mathrm{d}t, \end{aligned}$$

in which $\varepsilon = \sqrt{a^2 - c^2}/a$ is the eccentricity of the ellipse. If we substitute $\sin(t) = u$, then $\mathrm{d}t = \mathrm{d}u/\sqrt{1-u^2}$, and in this way, we obtain

$$L = 4a \int_0^1 \frac{\sqrt{1-\varepsilon^2 u^2}}{\sqrt{1-u^2}}\, \mathrm{d}u = 4a\, E(\varepsilon).$$

So, we have been convinced once more that the elliptic integrals present themselves even in the most natural geometric problems.

All Jacobian elliptic functions have integrals, given by the formulas below, that can be verified by direct differentiations

$$\begin{aligned}
\int \mathrm{sn}u\, \mathrm{d}u &= \frac{1}{k}\ln(\mathrm{dn}u - k\mathrm{cn}u), & \int \mathrm{cn}u\, \mathrm{d}u &= \frac{1}{k}\arcsin(k\mathrm{sn}u) \\
\int \mathrm{dn}u\, \mathrm{d}u &= \arcsin(\mathrm{sn}u), & \int \mathrm{ns}u\, \mathrm{d}u &= -\ln(\mathrm{ds}u + \mathrm{cs}u) \\
\int \mathrm{nc}u\, \mathrm{d}u &= \frac{1}{\tilde{k}}\ln(\mathrm{dc}u + \tilde{k}\mathrm{sc}u), & \int \mathrm{nd}u\, \mathrm{d}u &= \frac{1}{\tilde{k}}\arcsin(\mathrm{cd}u) \\
\int \mathrm{sc}u\, \mathrm{d}u &= \frac{1}{\tilde{k}}\ln(\mathrm{dc}u + \tilde{k}\mathrm{nc}u), & \int \mathrm{cd}u\, \mathrm{d}u &= \frac{1}{k}\ln(\mathrm{nc}u + \mathrm{sc}u) \\
\int \mathrm{ds}u\, \mathrm{d}u &= \ln(\mathrm{ns}u - \mathrm{cs}u), & \int \mathrm{cs}u\, \mathrm{d}u &= -\ln(\mathrm{ns}u + \mathrm{ds}u) \\
\int \mathrm{dc}u\, \mathrm{d}u &= \ln(\mathrm{nc}u + \mathrm{sc}u), & \int \mathrm{sd}u\, \mathrm{d}u &= -\frac{1}{k\tilde{k}}\arcsin(k\mathrm{cd}u).
\end{aligned}$$

A.2 Weierstrassian Elliptic Functions

Let us consider the elliptic integral of the first kind in the Weierstrassian approach

$$u = \int_z^\infty \frac{\mathrm{d}z}{\sqrt{4z^3 - g_2 z - g_3}}, \tag{A.12}$$

in which g_2 and g_3 are arbitrary complex numbers.

It can be proven that the above integral defines z as a unique function of u, denoted as $\wp(u, g_2, g_3)$, in which u is the argument and g_2 and g_3 are the so-called invariants of the function $\wp$.

One should note that if the discriminant $\Delta = g_2^3 - 27g_3^2$ of the cubic polynomial under the square root (A.12) is positive, then it takes real values for real values of z.

Besides $\wp(u)$, Weierstrass introduced another two functions-$\zeta(u)$ and $\sigma(u)$. They are defined by the equalities

$$\zeta(u) = \frac{1}{u} - \int_0^u (\wp(\tilde{u}) - \frac{1}{\tilde{u}^2})\mathrm{d}\tilde{u} \tag{A.13}$$

and

$$\sigma(u) = u \exp(\int_0^u (\zeta(\tilde{u}) - \frac{1}{\tilde{u}})\mathrm{d}\tilde{u}). \tag{A.14}$$

Of the many interesting properties of the Weierstrassian functions, we will mention only those which have some direct relations to the applications used in the present text, namely,

$$\zeta'(u) = -\wp(u), \qquad \frac{\sigma'(u)}{\sigma(u)} = \zeta(u) \tag{A.15}$$
$$\wp(-u) = \wp(u), \qquad \zeta(-u) = -\zeta(u), \qquad \sigma(-u) = -\sigma(u).$$

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