Chapter 1
Geometry and Variational Calculus

Abstract This chapter contains all necessary information in regard to the differential geometry of curves and surfaces needed for the description of membrane shapes. The main focus is on the geometry of the plane curves to which most of the considerations in the book boil down. As usual, coordinate systems, curvature and the Frenet-Serret equations are discussed in some detail. Then, space curves together with their tangent vectors, normal planes and curvature are introduced. Other important notions like principal normal and binormal, osculating plane and moving frame are introduced, exemplified and discussed as well. The Frenet-Serret equations for the space curves are derived and the principal theorem in the local theory of these curves is proved. The second part of this chapter is devoted to variational calculus, as this is the setting in which the problem of the optimal form of membranes is cast later on. Variational calculus occupies a special place in mathematics. Many of the laws of nature are formulated as variational principals. Hence, there is a great interest in the calculus of variations in many applied areas—from celestial mechanics to mathematical economics and management theory. Specifically in this chapter, the equations named after Euler and Lagrange are reviewed (and in some cases derived) in various settings, e.g., for functionals depending on one or more variables and comprising derivatives of first or higher order, etc. Most of these cases are illustrated by examples, which serve both to assist in understanding and to prepare the reader for the applications of the method to follow.

1.1 Plane Curves

Among many different coordinate systems used for presentation of plane curves, we choose only those two which are appropriate for our considerations. These are the parametric representations using the Cartesian coordinate system and polar coordinates. The coordinate axes of the Cartesian coordinate system are specified by a pair of perpendicular lines, the so-called abscissa (the *OX* axis) and the ordinate (*OZ* axis) (see Fig. 1.1).

The coordinates in the polar coordinate system are composed of a point (pole) and a vector from this point. Further on, we shall assume that the pole and the

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