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FUNDAMENTAL PROBLEMS IN THE THEORY OF INFINITE-DIMENSIONAL LIE GROUPS

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Abstract. In a preprint from 1982, John Milnor formulated various fundamental questions concerning infinite-dimensional Lie groups. In this paper, we describe some of the answers (and partial answers) obtained in the preceding years.

1. Introduction

While specific classes of infinite-dimensional Lie groups (like groups of operators, gauge groups, and diffeomorphism groups) have been studied extensively and are well understood, much less is known about general infinite-dimensional Lie groups, and many fundamental problems are still unsolved. Typical problems were recorded in John Milnor's preprint [19], which preceded his well-known survey article [20]. In this note, we recall Milnor's questions and their background and describe some of the answers (or partial answers) obtained so far.

2. Basic Definitions

To define infinite-dimensional Lie groups, Milnor uses the following notion of smooth maps between locally convex spaces (known as "Keller C^∞_c-maps" [15]):

Definition 1. *Let E and F be real locally convex spaces, $U \subseteq E$ be open, and $f\colon U \to F$ be a map. For $x \in U$ and $y \in E$, let $(D_y f)(x) := \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} f(x+ty)$ be the directional derivative (if it exists). Given $k \in \mathbb{N} \cup \{\infty\}$, the map f is called C^k if it is continuous, the iterated directional derivatives*

$$\mathrm{d}^j f(x, y_1, \ldots, y_j) := (D_{y_j} \cdots D_{y_1} f)(x)$$

exist for all $j \in \mathbb{N}$ such that $j \leq k$, $x \in U$ and $y_1, \ldots, y_j \in E$, and all of the maps $\mathrm{d}^j f\colon U \times E^j \to F$ are continuous. As usual, C^∞-maps are also called **smooth**.