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# THE WITTEN CONJECTURE

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**Abstract**. Low-dimensional topology has experienced a number of revolutionary upheavals in the past twenty years. For many of these the seeds of the revolution were sown in theoretical physics and, more particularly, in the work of Edward Witten. The most recent such event occurred in 1994 when Witten suggested that the topological information about smooth 4-manifolds contained in the Donaldson invariants should also be contained in the much simpler and now famous Seiberg–Witten invariants. This lecture will provide an informal survey of some of the background behind the conjecture and how it came to be made.

## 1. Donaldson Theory

The first application of gauge-theoretic techniques to the study of smooth 4-manifolds was made by Donaldson [3] who proved that if $M$ is a compact, simply connected, oriented, smooth 4-manifold and the intersection form $q_M\colon H_2(M,\mathbb{Z})\times H_2(M,\mathbb{Z})\to\mathbb{Z}$ is definite, then, in fact, $q_M$ is diagonalizable over $\mathbb{Z}$. It follows that there exists a basis for $H_2(M,\mathbb{Z})$ over $\mathbb{Z}$ relative to which the matrix of $q_M$ is $\pm\,\mathrm{Id}_{b_2(M)}$. Roughly, the proof goes something like this (in the negative definite case):

Consider the principal $SU(2)$-bundle $SU(2)\hookrightarrow P_1\to M$ over $M$ with second Chern class $c_2(P_1)=1$. Choose a Riemannian metric $g$ on $M$. Taubes [11] has shown that the bundle admits connections that are **anti-self-dual** (ASD) relative to the Hodge star operator determined by $g$ and the given orientation of $M$ (such connections are called instantons). Two such connections are said to be gauge equivalent if they differ by an automorphism of $P_1$ and the collection $\mathcal{M}_1(g)$ of gauge equivalence classes of such connections is called the moduli