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GENERALIZED WILLMORE ENERGIES AND APPLICATIONS

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We discuss some of our recent results regarding Generalized Willmore Energies, their critical surfaces, associated gradient flows and real-world applications.

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1. From the Bending Energy to the Generalized Willmore Energy

Variational calculus has played a very important role in mathematics and physical sciences. Given an energy functional, the critical points of the first variation satisfy the *Euler-Lagrange equations*. Significant variational problems in elasticity theory started from the works of J. Bernoulli, D. Bernoulli and L. Euler about elastic curves, and those of S. Germain and S. Poisson about elastic surfaces.

For a surface immersion $\mathbf{r}\colon \Sigma \to \mathbb{R}^3$, the bending energy may be introduced as

$$\mathcal{W}[\mathbf{r}] := \int_\Sigma H^2 \, \mathrm{d}\Sigma \tag{1}$$

which is also known under the name of Willmore energy. This functional was first considered by S. Germain, see, e.g., [7] and later reintroduced by T. Willmore [20]. For closed surfaces, this energy measures how much the immersion X deviates from a round sphere. This is easy to see if one considers as *bending energy* the *conformal Willmore energy*

$$\mathcal{W}^c[\mathbf{r}] := \int_\Sigma (k_1 - k_2)^2 \, \mathrm{d}\Sigma \; = 4 \int_\Sigma (H^2 - K) \, \mathrm{d}\Sigma \tag{2}$$

which is variationally equivalent to the previous functional, by virtue of the *Gauss-Bonnet theorem*.

Among closed surfaces, this functional is conformally invariant, an essential property that was remarked by W. Blaschke [3], in the early 20th century. Derivatives of this type of functional are generally calculated with respect to arbitrary variations.

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