Fifth International Conference on
Geometry, Integrability and Quantization
June 5–12, 2003, Varna, Bulgaria
Ivaïlo M. Mladenov and Allen C. Hirshfeld, Editors
SOFTEX, Sofia 2004, pp 11–50

INTEGRABLE DYNAMICS OF KNOTTED VORTEX FILAMENTS

ANNALISA CALINI

Department of Mathematics, College of Charleston
Charleston SC 29424 USA

Abstract. The dynamics of vortex filaments has provided for almost a century one of the most interesting connections between differential geometry and soliton equations, and an example in which knotted curves arise as solutions of differential equations possessing an infinite family of symmetries and a remarkably rich geometrical structure. These lectures discuss several aspects of the integrable dynamics of closed vortex filaments in an Eulerian fluid, including its Hamiltonian formulation, the construction of a large class of special solutions, and the role of the Floquet spectrum in characterizing the geometric and topological properties of the evolving curves.

Introduction

Most of the well known soliton equations in one space dimension have been shown to describe integrable curve evolutions: among them the Vortex Filament Equation (or localized induction equation) [28, 34]; the mKdV equation modelling the dynamics of boundaries of vortex patches [23]; the sine-Gordon equation which describes constant torsion curves generating pseudospherical surfaces [16, 15]; and their higher dimensional generalizations [17, 37].

The understanding of connections between curve geometry and integrability has proceeded along several directions. Many of the fundamental properties of soliton equations have been given a geometrical realization: in the case of the Vortex Filament Equation (VFE), its bihamiltonian formulation and recursion operator, its hierarchy of constants of motion, and its relation to the nonlinear Schrödinger equation possess natural geometric interpretations [36, 11]. For equations, such as the VFE, which describe curve dynamics in three-dimensional space, periodic boundary conditions give rise to closed and, in many cases, knotted curves. It is then natural to ask whether the infinitely many symmetries and the associated