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CONFORMAL IMMERSIONS OF DELAUNAY SURFACES AND THEIR DUALS

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Abstract. A few explicit formulas providing conformal coordinates of the axially symmetric constant mean curvature surfaces introduced by Delaunay and their duals are derived. These results give also new examples in a long line of research connected with finding isothermic immersions of surfaces and their duals.

1. Introduction

Let us assume that the parametrized surface $\mathcal{S}$ is (locally) an image of the immersion

$$(u, v) \longrightarrow \mathbf{x}[u, v] = (\mathrm{x}(u, v), \mathrm{y}(u, v), \mathrm{z}(u, v)) \tag{1}$$

defined on an open set $\mathcal{D} \subset \mathbb{R}^2$. In these coordinates the pullback of the Riemannian metric on $\mathcal{S}$ can be expressed (using the standard notation) in the form

$$I = E\,\mathrm{d}u^2 + 2F\,\mathrm{d}u\,\mathrm{d}v + G\,\mathrm{d}v^2 \tag{2}$$

which is known as the first fundamental form of $\mathcal{S}$. The coefficients in I are given by

$$E = \mathbf{x}_u\,.\,\mathbf{x}_u, \quad F = \mathbf{x}_u\,.\,\mathbf{x}_v, \quad G = \mathbf{x}_v\,.\,\mathbf{x}_v.$$

One has to notice that these three functions determine completely the Riemannian structure of $\mathcal{S}$, but that they are not determined by it. For we can apply a diffeomorphic change of coordinates $u = u(\tilde{u}, \tilde{v})$, $v = v(\tilde{u}, \tilde{v})$ in order to obtain an isometric structure which is actually the same. Then the new coefficients $\tilde{E}$, $\tilde{F}$, $\tilde{G}$ can be easily found by plugging in the expressions for

$$\mathrm{d}u = u_{\tilde{u}}\,\mathrm{d}\tilde{u} + u_{\tilde{v}}\,\mathrm{d}\tilde{v} \qquad \text{and} \qquad \mathrm{d}v = v_{\tilde{u}}\,\mathrm{d}\tilde{u} + v_{\tilde{v}}\,\mathrm{d}\tilde{v}$$