Minimizers of Hartree Type Functionals and Related Interpolation Inequalities

Hristo Genev* and George Venkov†

*Faculty of Mathematics and Informatics, Sofia University "St. Kliment Ohridski"
1164 Sofia, Bulgaria
†Faculty of Applied Mathematics and Informatics, Technical University of Sofia
1000 Sofia, Bulgaria

Abstract. In this work we are concerned with the minimization problem associated to the best constant $C_{n,\alpha}$ in the following interpolation inequality

$$\iint \frac{|u(x)|^\alpha |u(y)|^\alpha}{|x-y|^{n-2}} dx\, dy \leq C_{n,\alpha} \|\nabla \psi\|_{L^2}^{n\alpha-(n+2)} \|\psi\|_{L^2}^{(n+2)-(n-2)\alpha}, \qquad u \in H^1(\mathbb{R}^n)$$

for $2 \leq \alpha < 1+4/(n-2)$ and $n \geq 3$. The corresponding variational problem is equivalent to the problem of existence of a ground state solution to the nonlinear non-local elliptic equation

$$-\Delta u + \omega u - (|x|^{-(n-2)} * |u|^\alpha)|u|^{\alpha-2}u = 0.$$

We show that the ground state solution is unique and its mass $\|u\|_{L^2}^2$ determines the best constant $C_{n,\alpha}$. As a consequence of the above results, we obtain sharp sufficient condition for global existence for the L^2-critical Hartree equation.

Keywords: Hartree potential, Weinstein functional, minimizers, interpolation inequalities
PACS: 02.30.Xx, 02.30.Sa

INTRODUCTION

We consider the functional

$$J^{n,\alpha}(u) = \frac{\|u\|_{L^2}^{(n+2)-(n-2)\alpha} \|\nabla u\|_{L^2}^{n\alpha-(n+2)}}{\int (|x|^{-(n-2)} * |u|^\alpha)|u|^\alpha dx} \tag{1}$$

on the space $H^1(\mathbb{R}^n)$, $n \geq 3$, for $2 \leq \alpha < 1+4/(n-2)$. This functional is naturally associated with the interpolation inequality

$$\int (|x|^{-(n-2)} * |u|^\alpha)|u|^\alpha dx \leq C_{n,\alpha} \|\nabla u\|_{L^2}^{n\alpha-(n+2)} \|u\|_{L^2}^{(n+2)-(n-2)\alpha}. \tag{2}$$

Indeed, if we denote by

$$J = \inf\{J^{n,\alpha}(u)\,;\, u \in H^1(\mathbb{R}^n), u \neq 0\} \tag{3}$$

International Workshop on Complex Structures, Integrability and Vector Fields
AIP Conf. Proc. 1340, 203-211 (2011); doi: 10.1063/1.3567138