On Some Deformations of the Cassinian Oval

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Abstract. The work is concerned with the determination of explicit parametric equations of several plane curves whose curvature depends solely on the distance from the origin. Here we suggest and exemplify a simple scheme for reconstruction of a plane curve if its curvature belongs to the above-mentioned class. Explicit parameterizations of generalized Cassinian ovals including also the trajectories of a charged particle in the field of a magnetic dipole are derived in terms of Jacobian elliptic functions and elliptic integrals.

Keywords: classical differential geometry, plane curves, curvature, Cassinian oval, magnetic dipole
PACS: 02.40-k, 02.40.Hw, 46.25-y

INTRODUCTION

Remarkably, the curvature of a lot of the famous plane curves (see [5, 14]), such as conic sections, Bernoulli's lemniscate, Cassinian ovals and many others, depends solely on the distance from a certain point in the Euclidean plane, which may be chosen as its origin.

The most fundamental existence and uniqueness theorem in the theory of plane curves states that a curve is uniquely determined (up to Euclidean motion) by its curvature given as a function of its arc-length (see [3, p. 296] or [9, p. 37]). The simplicity of the situation however is quite elusive because in many cases it is impossible to find the sought-after curve explicitly. Having this in mind, it is clear that if the curvature is given by a function of its position the situation is even more complicated. Viewing the Frenet-Serret equations as a ficticious dynamical system it was proven in [11] that when the curvature is given just as a function of the distance from the origin the problem can always be reduced to quadratures. The cited result should not be considered as entirely new because Singer [10] had already shown that in some cases it is possible that such curvature gets an interpretation of a central potential in the plane and therefore the trajectories could be found by the standard procedures in classical mechanics. The approach which we will follow here, however is entirely different from the group-theoretical [11] or mechanical [10] ones proposed in the aforementioned papers. The method is illustrated on a class of curves whose curvature depend solely on the distance from the origin.

International Workshop on Complex Structures, Integrability and Vector Fields
AIP Conf. Proc. 1340, 81-89 (2011); doi: 10.1063/1.3567127