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AN ANALYTIC UNIFYING FORMULA OF OSCILLATORY AND ROTARY MOTION OF A SIMPLE PENDULUM

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Abstract. The motion of a simple pendulum, whether oscillatory or rotary, is explicitly expressed as an analytic elliptic function of time, which lattice is either rhombic, in the first (oscillatory) case, or rectangular, in the second (rotary) case. The corresponding period, whether it is the time required for a swing (in the oscillatory case) or for a revolution (in the rotary case), is most efficiently calculated via Gauss arithmetic-geometric mean method (for evaluating complete elliptic integrals of the first kind). The doubly periodic solution "degenerates" to a singly periodic (equilibrium) solution, which period is either real (for stable equilibrium) or imaginary (for unstable equilibrium). Without the latter (elementary) critical solution, which will be given in this paper, this fundamental problem of classical mechanics had never been entirely solved!

1. Introduction

Calculating the period of a simple pendulum in many "popular" references on elliptic functions, such as [9, p. 59, 77], and "authoritative" references on mechanics, such as [11, p. 73], have relied on a "classical" presentation of the motion of the pendulum, given in [7]. The latter reference included a chapter on Landen transformations. Yet, its leading author Appell was unaware of Gauss most efficient method for calculating complete elliptic integrals of the first kind, outlined in [3,4], and although he did attain a mechanical interpretation of the imaginary period [6], he fell short of finding an analytic expression for the period, convergent for all values of the maximal angle, including complex values (!), corresponding to rotary modes of motion. The ubiquitous use of power series for calculating the period (after Appell) has led to restricting these calculations to "small angles" along with